Intersection Type System with de Bruijn Indices

Daniel Lima Ventura[1]*and Mauricio Ayala-Rincón[1]†and
Fairouz Kamareddine[2]

[1]Grupo de Teoria da Computação, Dep. de Matemática
Universidade de Brasília, Brasília D.F., Brasil

[2] School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh, Scotland

{ventura,ayala}@mat.unb.br, fairouz@macs.hw.ac.uk

February 15, 2008

Abstract

λ-calculus in de Bruijn notation is relevant because it avoids variable names using instead indices which makes it more adequate computationally; in fact, several calculi of explicit substitutions are written in de Bruijn notation because it simplifies the formalization of the atomic operations involved in β-reductions. Intersection types provide finitary type polymorphism which is of principal interest. Moreover, intersection types characterize normalizable λ-terms, that is a term is normalizable if and only if it is typable. Versions of explicit substitutions calculi without types and with simple type systems are well investigated in contrast to versions with more elaborated type systems such as intersection types. In this paper λ-calculus in de Bruijn notation with an intersection type system is introduced and it is proved that this system satisfies the basic property of subject reduction, that is λ-terms preserve theirs types under β-reduction.

1 Introduction

λ-calculus à la de Bruijn [dB72] was introduced by the Dutch mathematician N.G. de Bruijn in the context of the project Automath [NGdV94], one of the leading projects on automated deduction which still influences modern proof assistants [Kam03]. Instead of names, indices represent variables in this notation assembling each α-class of terms in the λ-calculus with names in a unique term in de Bruijn notation. Despite there is a common sense that de Bruijn notation is unreadable for humans, it is useful for machines and has been adopted for several calculi of explicit substitutions (e.g. [dB78], [ACCL91], [KR95]) in which operations related to β-reductions are atomized in order to create calculi closer to actual implementations of the λ-calculus. Type free and simply typed versions of the λ-calculus as well as of these calculi of explicit substitutions have

*Corresponding author, currently supported by a PhD sandwich scholarship of the CNPq at the Heriot-Watt University.

†Partially supported by the CNPq.

been investigated, but to the best of our knowledge there is no work on more elaborated type systems for these calculi in de Bruijn notation.

In this paper a version of the λ-calculus in de Bruijn notation with an intersection types system is introduced. Intersection types were introduced to provide a characterization of strongly normalizing λ-terms [CDC78, CDC80, Pot80]. In programming, the intersection type discipline is of interest because λ-terms not typable in the standard Curry type assignment system ([CF58]) or in extensions allowing some polymorphism, present in programming languages such as ML ([Mil78]), are typable with intersection types. For instance, $\lambda x.(x\ x)$ is typable, assigning two different types to x ($x : \sigma \rightarrow \varphi\ \cap\ \sigma$). The intersection type system presented in [BCDC83] is closed under β-equality, which is another property that simple type systems do not have. Although, the problem of typability (Given a λ-term t, is there a context Γ and a type σ such that $\Gamma \vdash t\ :\ \sigma$?), which is decidable in the Curry type assignment system, is undecidable in [BCDC83]. This is a consequence of the fact that all terms having normal form can be characterized by their assignable types. In [CW04] Carlier and Wells presented the exact correspondence between the inference mechanism for their intersection type system and β-reduction. They introduce *expansion variables* to perform *Expansion*, a operation used during type inference (see [CW04.2]).

The type system in this paper is based on the one given in [KN07]. The version in de Bruijn notation is proved to preserve subject reduction, that is the property of preserving types under β-reduction: whenever $\Gamma \vdash t\ :\ \sigma$ and t β-reduces into s, $\Gamma \vdash s\ :\ \sigma$.

Section 2 presents the λ-calculus in de Bruijn notation, giving some lemmas about syntactic properties regarding update of free indices (free variables), substitution and β-reduction. In section 3 the intersection type system is introduced and properties about shape of type and contexts (an ordered environment) are presented, analogue to the ones given in [KN07]. Section 4 proves the property of subject reduction, following the standard sketch proving a generation and substitution lemmas. Finally, we conclude talking about future work.

2 The type free calculi

2.1 λ-calculus in de Bruijn notation

Definition 1 (Set Λ_{dB}). *The syntax of the λ-calculus in de Bruijn notation, **the λdB-calculus**, is defined inductively by:*

Terms $M ::= \underline{n} \mid (M\ M) \mid \lambda.M$ *where* $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$

Definition 2. *1. We define $FI(M)$, the set of free indices of $M \in \Lambda_{dB}$, by:*

$$\begin{array}{rcl} FI(\underline{n}) & = & \{\underline{n}\} \\ FI(\lambda.M) & = & \{\underline{n-1}, \forall\, \underline{n} \in FI(M), n > 1\} \\ FI(M_1\ M_2) & = & FI(M_1) \cup FI(M_2) \end{array}$$

2. A term M is called closed if $FI(M) \equiv \emptyset$.

3. The greatest value of a free index in M, denoted by $sup(M)$, is defined by:

$$sup(M) = \begin{cases} 0 & \text{if } FI(M) \equiv \emptyset \\ n \text{ where } \underline{n} \in FI(M) \text{ and } n \geq i,\ \forall\, \underline{i} \in FI(M) & \text{otherwise} \end{cases}$$

Lemma 1. *1. $sup(M_1\ M_2) = max(sup(M_1), sup(M_2))$.*

2. If $sup(M) = 0$, then $sup(\lambda.M) = 0$. Otherwise, $sup(\lambda.M) = sup(M) - 1$.

Proof. 1. If $sup(M_1\ M_2)=0$, nothing to prove. Otherwise, $sup(M_1\ M_2)=n$, where $n \geq i$, $\forall\, \underline{i} \in FI(M_1\ M_2) = FI(M_1) \cup FI(M_2)$ and $\underline{n} \in FI(M_1)$ or $\underline{n} \in FI(M_2)$. Suppose, w.l.o.g., that $\underline{n} \in FI(M_1)$. Hence, $n \geq sup(M_1)$ and $sup(M_1) \geq n$, thus, $n = sup(M_1)$ and $n \geq sup(M_2)$.

2. If $sup(M) = 0$, then $FI(\lambda.M) = FI(M) = \emptyset$, hence, $sup(\lambda.M) = 0$. Let $sup(M) = m > 0$. Hence, $m \geq i$, $\forall\, \underline{i} \in FI(M)$ and $\underline{m} \in FI(M)$. If $m=1$, then $FI(M) = \{\,\underline{1}\}$, thus, $FI(\lambda.M) = \emptyset$ and $sup(\lambda.M) = 0$. Otherwise, $FI(\lambda.M) = \{\,\underline{n{-}1}, \forall\, \underline{n} \in FI(M), n > 1\}$. Thus, $\underline{m{-}1} \in FI(\lambda.M)$ and $m - 1 \geq i - 1$, $\forall\, \underline{i{-}1} \in FI(\lambda.M)$.

□

Terms like $((\dots((M_1\ M_2)\ M_3)\dots)\ M_n)$ are written as $(M_1\, M_2\ \dots\ M_n)$, as usual. The β-contraction definition in this notation needs a mechanism which detects and updates free indices of terms. It follows an operator similar to the one presented in [ARK01].

Definition 3. *Let $M \in \Lambda_{dB}$ and $i \in \mathbb{N}$. The **i-lift** of M, denoted as M^{+i}, is defined inductively by:*

1. $(M_1\, M_2)^{+i} = (M_1^{+i}\, M_2^{+i})$
2. $(\lambda.M_1)^{+i} = \lambda.M_1^{+(i+1)}$
3. $\underline{n}^{+i} = \begin{cases} \underline{n+1}, & \text{if } n > i \\ \underline{n}, & \text{if } n \leq i. \end{cases}$

The **lift** of a term M is its 0-lift, denoted by M^{+}. Intuitively, the lift of M corresponds to an increment by 1 of all free indices occurring in M. The next lemma states general relations between the i-lift and the free indices of M.

Lemma 2. *1. If $i \geq sup(M)$, then $M^{+i} \equiv M$.*

2. $FI(M^{+i}) = \{\,\underline{n} \,|\, \underline{n} \in FI(M), n \leq i\} \cup \{\,\underline{n{+}1} \,|\, \underline{n} \in FI(M), n > i\}$.

3. If $sup(M) > i$, then $sup(M^{+i}) = sup(M)+1$.

4. If $sup(M) \leq i$, then $sup(M^{+i}) = sup(M)$.

Proof. 1 and 2: By induction on the structure of M.
3: If $sup(M) = m$, then $m \geq n$, $\forall\, \underline{n} \in FI(M)$ and $\underline{m} \in FI(M)$. Since $m > i$, by lemma 2.2, $\underline{m{+}1} \in FI(M^{+i})$ and $\forall\, \underline{j} \in FI(M^{+i})$, either $j = n$ or $j = n{+}1$, where $\underline{n} \in FI(M)$. One has $m{+}1 \geq n{+}1 > n, \forall\, \underline{n} \in FI(M)$, thus, $m{+}1 \geq j, \forall\, \underline{j} \in FI(M^{+i})$.
4: From lemma 2.1, $M^{+i} \equiv M$, thus, $sup(M^{+i}) = sup(M)$. □

Using the i-lift, we are able to present the definition of the substitution used by β-contractions, similarly to the one presented in [ARK01].

Definition 4. *Let $m, n \in \mathbb{N}^*$. The **β-substitution** for free occurrences of $\underline{n}$ in $M \in \Lambda_{dB}$ by term N, denoted as $\{\underline{n}/N\}M$, is defined inductively by*

1. $\{\underline{n}/N\}(M_1\ M_2) = (\{\underline{n}/N\}M_1\ \{\underline{n}/N\}M_2)$
2. $\{\underline{n}/N\}\lambda.M_1 = \lambda.\{\underline{n+1}/N^{+}\}M_1$
3. $\{\underline{n}/N\}\underline{m} = \begin{cases} \underline{m-1}, & \text{if } m > n \\ N, & \text{if } m = n \\ \underline{m}, & \text{if } m < n \end{cases}$

Observe that in item 2 of Def. 4, the lift operator is used to avoid captures of free indices in N. We present the β-contraction as defined in [ARK01].

Definition 5. ***β-contraction** in λdB is defined by $(\lambda.M\ N) \rightarrow_\beta \{\underline{1}/N\}M$.*

Notice that item 3 in Definition 4, for $n = 1$, is the mechanism which does the substitution and updates the free indices in M as consequence of the lead abstractor elimination.

Lemma 3. *1. If* $\underline{i} \notin FI(M)$*, then*
$FI(\{\underline{i}/N\}M) = \{\underline{n} \,|\, \underline{n} \in FI(M), n<i\} \cup \{\underline{n-1} \,|\, \underline{n} \in FI(M), n>i\}$.

2. Otherwise,
$FI(\{\underline{i}/N\}M) = FI(N) \cup \{\underline{n} \,|\, \underline{n} \in FI(M), n<i\} \cup \{\underline{n-1} \,|\, \underline{n} \in FI(M), n>i\}$.

3. If $i > sup(M)$*, then* $\{\underline{i}/N\}M \equiv M$.

Proof. By induction on the structure of M. □

In particular, if $FI(M) = \{\underline{i}\}$, then $\{\underline{n} \,|\, \underline{n} \in FI(M), n<i\} \equiv \emptyset$ and $\{\underline{n-1} \,|\, \underline{n} \in FI(M), n>i\} \equiv \emptyset$, thus, $FI(\{\underline{i}/N\}M) = FI(N)$.

Corollary 1. *If* $\underline{1} \in FI(M)$*, then* $FI(\{\underline{1}/N\}M) = FI(\lambda.M\ N)$*. Otherwise,* $FI(\{\underline{1}/N\}M) = FI(\lambda.M)$.

Lemma 4. *Let* M *be a term such that* $sup(M) = m$*:*

1. *If* $i<m$ *and* $\underline{i} \notin FI(M)$*, then* $sup(\{\underline{i}/N\}M) = m-1$.

2. *If* $i>m$*, then* $sup(\{\underline{i}/N\}M) = m$.

3. *Suppose* $\underline{i} \in FI(M)$*. If* $FI(M) = \{\underline{i}\}$*, then* $sup(\{\underline{i}/N\}M) = sup(N)$*. Otherwise,* $sup(\{\underline{i}/N\}M) = max(sup(N), m-1)$.

Proof. 1. One has that $m \geq n, \forall \underline{n} \in FI(M)$ and $\underline{m} \in FI(M)$. Since $m>i$, by lemma 3.1, $\underline{m-1} \in FI(\{\underline{i}/N\}M)$ and $\forall \underline{j} \in FI(\{\underline{i}/N\}M)$, either $j=n<i$ or $j=n-1$, where $\underline{n} \in FI(M)$. Thus, $m-1 \geq n-1 \geq i, \forall \underline{n} \in FI(M)$ such that $n > i$, hence, $m-1 \geq j, \forall \underline{j} \in FI(\{\underline{i}/N\}M)$.

2. If $i > m$, then, by lemma 3.3, $\{\underline{i}/N\}M \equiv M$, thus, $sup(\{\underline{i}/N\}M) = sup(M)$.

3. By lemma 3.2 one has $FI(\{\underline{i}/N\}M) = FI(N) \cup A$, where $A \equiv \{\underline{n} \,|\, \underline{n} \in FI(M), n<i\} \cup \{\underline{n-1} \,|\, \underline{n} \in FI(M), n>i\}$. If $FI(M) = \{\underline{i}\}$, then $A \equiv \emptyset$, thus $FI(\{\underline{i}/N\}M) = FI(N)$. Otherwise, A is not empty and, similarly to case 1, one has that $m-1 \geq j$, $\forall \underline{j} \in A$.

□

Lemma 5. $sup(\{\underline{1}/N\}M) \leq sup(\lambda.M\ N)$.

Proof. If $\underline{1} \in FI(M)$, then $sup(\{\underline{1}/N\}M) = sup(\lambda.M\ N)$. Otherwise, one has two possibilities. If $sup(M) = 0$, then, by lemma 4.2, $sup(\{\underline{1}/N\}M) = 0 \leq max(0, sup(N)) = sup(\lambda.M\ N)$. If $sup(M) > 1$, then, by lemma 4.1, $sup(\{\underline{1}/N\}M) = sup(M)-1 = sup(\lambda.M) \leq max(sup(\lambda.M), sup(N))$. □

Definition 6. ***β-reduction*** *in* λdB *is defined by:*

$$\frac{(\lambda.M\,N) \to_\beta \{\underline{1}/N\}M}{(\lambda.M\,N) \longrightarrow_\beta \{\underline{1}/N\}M} \qquad \frac{M \longrightarrow_\beta N}{\lambda.M \longrightarrow_\beta \lambda.N}$$

$$\frac{M_1 \longrightarrow_\beta N_1}{(M_1\ M_2) \longrightarrow_\beta (N_1\ M_2)} \qquad \frac{M_2 \longrightarrow_\beta N_2}{(M_1\ M_2) \longrightarrow_\beta (M_1\ N_2)}$$

Theorem 1. *If* $M \longrightarrow_\beta N$ *then* $FI(N) \subseteq FI(M)$ *and* $sup(N) \leq sup(M)$.

Proof. By induction on the derivation $M \longrightarrow_\beta N$.

- If $M \equiv (\lambda.M_1\ M_2)$, then $N \equiv \{\underline{1}/M_2\}M_1$ and, by corollary 1, $FI(\{\underline{1}/N\}M_1) \subseteq FI(\lambda.M_1\ M_2)$.

- Let $M \equiv (M_1\ M_2)$ and $N \equiv (M_1\ N_2)$, where $M_2 \longrightarrow_\beta N_2$, then, by IH, $FI(N_2) \subseteq FI(M_2)$. Thus, $FI(N) = FI(M_1) \cup FI(N_2) \subseteq FI(M_1) \cup FI(M_2) = FI(M)$.
- Case $M \equiv (M_1\ M_2)$ and $N \equiv (N_1\ M_2)$, where $M_1 \longrightarrow_\beta N_1$, is similar.
- If $M \equiv \lambda.M'$, then $N \equiv \lambda.N'$, where $M' \longrightarrow_\beta N'$. By IH, $FI(N') \subseteq FI(M')$, hence, $\forall\, \underline{n} \in FI(N')$, $\underline{n} \in FI(M')$. Thus, $\forall\, \underline{n-1} \in FI(\lambda.N')$, $\underline{n-1} \in FI(\lambda.M')$.

□

3 The Type System

Definition 7. *1.* **Intersection types** *are defined by:*

$$\mathbb{T} ::= \mathcal{A} \,|\, \mathbb{U} \to \mathbb{T} \qquad\qquad \mathbb{U} ::= \omega \,|\, \mathbb{U} \sqcap \mathbb{U} \,|\, \mathbb{T}$$

The types are quotiented by taking $\sqcap$ to be commutative, associative, idempotent and to have ω as neutral.

2. Contexts are ordered lists of types $U \in \mathbb{U}$, defined by: $\Gamma ::= nil \,|\, U.\Gamma$

Let Γ be some context and $n \in \mathbb{N}$. Then $\Gamma_{<n}$ denotes the first $n-1$ types of Γ. Similarly we define $\Gamma_{>n}$, $\Gamma_{\leq n}$ and $\Gamma_{\geq n}$. Note that, for $\Gamma_{>n}$ and $\Gamma_{\geq n}$ the final nil element is included. For n=0, $\Gamma_{\leq 0}.\Gamma = \Gamma_{<0}.\Gamma = \Gamma$. The i-th element of Γ is denoted by Γ_i. The length of Γ is defined as $|nil|$=0 and, if Γ is not nil, $|\Gamma| = 1 + |\Gamma_{>1}|$. For any $i > m = |\Gamma|$, let $\Gamma_{\geq i} = \Gamma_{>i} = \Gamma_{>m}$ and $\Gamma_{\leq i} = \Gamma_{<i} = \Gamma_{\leq m}$.

For a term M, we denote env_ω^M the context Γ such that $|\Gamma| = sup(M)$ and $\Gamma = \omega.\omega.\cdots.\omega.nil$.

The extension of $\sqcap$ for contexts is done by $nil \sqcap \Gamma = \Gamma \sqcap nil = \Gamma$ and $(U_1.\Gamma) \sqcap (U_2.\Delta) = (U_1 \sqcap U_2).(\Gamma \sqcap \Delta)$. Hence, $\sqcap$ is commutative, associative and idempotent on contexts.

Some properties over contexts follow from the above definitions.

Lemma 6. *Let Γ and Δ be contexts, where neither Γ nor Δ are nil:*

1. *If $|\Gamma| \geq sup(M)$, then $\Gamma \sqcap env_\omega^M = \Gamma$*
2. *$\Gamma \sqcap \Delta = (\Gamma_1 \sqcap \Delta_1).(\Gamma_{>1} \sqcap \Delta_{>1})$*
3. *If $i \leq |\Gamma|, |\Delta|$, then $(\Gamma \sqcap \Delta)_i = \Gamma_i \sqcap \Delta_i$.*
4. *$(\Gamma \sqcap \Delta)_{<i} = \Gamma_{<i} \sqcap \Delta_{<i}$ and $(\Gamma \sqcap \Delta)_{>i} = \Gamma_{>i} \sqcap \Delta_{>i}$. The same for $(\Gamma \sqcap \Delta)_{\leq i}$ and $(\Gamma \sqcap \Delta)_{\geq i}$.*
5. *$|\Gamma \sqcap \Delta| = max(|\Gamma|, |\Delta|)$.*

Definition 8. *The typing rules are given as follows:*

$$\frac{}{\underline{1}:\langle T.nil \vdash T\rangle}\ \text{var} \qquad \frac{M:\langle nil \vdash T\rangle}{\lambda.M:\langle nil \vdash \omega \to T\rangle}\ \to_i'$$

$$\frac{\underline{n}:\langle \Gamma \vdash U\rangle}{\underline{n+1}:\langle \omega.\Gamma \vdash U\rangle}\ \text{varn} \qquad \frac{M_1:\langle \Gamma \vdash U \to T\rangle \quad M_2:\langle \Gamma' \vdash U\rangle}{M_1\ M_2:\langle \Gamma \sqcap \Gamma' \vdash T\rangle}\ \to_e$$

$$\frac{}{M:\langle env_\omega^M \vdash \omega\rangle}\ \omega \qquad \frac{M:\langle \Gamma \vdash U_1\rangle \quad M:\langle \Gamma \vdash U_2\rangle}{M:\langle \Gamma \vdash U_1 \sqcap U_2\rangle}\ \sqcap_i$$

$$\frac{M:\langle U.\Gamma \vdash T\rangle}{\lambda.M:\langle \Gamma \vdash U \to T\rangle}\ \to_i \qquad \frac{M:\langle \Gamma \vdash U\rangle \quad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M:\langle \Gamma' \vdash U'\rangle}\ \sqsubseteq$$

where the binary relation $\sqsubseteq$ is defined by the following rules:

$$\frac{}{\Phi \sqsubseteq \Phi}\ \text{ref} \qquad \frac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ \text{tr}$$

$$\frac{}{U_1 \sqcap U_2 \sqsubseteq U_1}\ \sqcap_e \qquad \frac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ \sqcap$$

$$\frac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ \to \qquad \frac{U_1 \sqsubseteq U_2}{\Gamma_{\leq i}.U_1.\Gamma_{>i} \sqsubseteq \Gamma_{\leq i}.U_2.\Gamma_{>i}}\ \sqsubseteq_c$$

$$\frac{U_1 \sqsubseteq U_2 \quad \Gamma' \sqsubseteq \Gamma}{\langle \Gamma \vdash U_1\rangle \sqsubseteq \langle \Gamma' \vdash U_2\rangle}\ \sqsubseteq_{\langle\rangle}$$

Φ, Φ', $\Phi_1, \ldots$ are used to denote $U \in \mathbb{U}$, contexts Γ or typings $\langle \Gamma \vdash U\rangle$. Note that in $\Phi \sqsubseteq \Phi'$, Φ and Φ' belong to the same sort.

Type judgements will be of the form $M:\langle \Gamma \vdash U\rangle$, meaning term M has type U provided Γ for $FI(M)$. Briefly, M has type U in Γ.

The next lemmas states some properties about the shape of types and contexts, and their link with the subtyping relation defined by $\sqsubseteq$.

Lemma 7. *1. If $U \in \mathbb{U}$, then $U = \omega$ or $U = \sqcap_{i=1}^n T_i$ where $n \geq 1$ and $\forall\, 1 \leq i \leq n$, $T_i \in \mathbb{T}$.*

2. $U \sqsubseteq \omega$.

3. If $\omega \sqsubseteq U$, then $U = \omega$.

Proof. See [KN07] □

Observe that, from $\underline{2}:\langle \omega.T.nil \vdash T\rangle$ and the $\sqsubseteq$ relation we have that $\underline{2}:\langle U.T.nil \vdash T\rangle$, for any U. This allows some sort of weakening in the type system, which is not allowed in the type system given in [KN07]. This happens because ω's are needed in the context first positions to give the proper type for some free index $\underline{i}$. Although, in lemma 10 we prove this weakening is limited by the term itself.

Lemma 8. *Let $V \neq \omega$.*

1. If $U \sqsubseteq V$, then $U = \sqcap_{j=1}^k T_j$, $V = \sqcap_{i=1}^p T_i'$ where $p, k \geq 1$, $\forall 1 \leq j \leq k$, $1 \leq i \leq p$, $T_j, T_i' \in \mathbb{T}$, and $\forall\, 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ such that $T_j \sqsubseteq T_i'$.

2. *If* $U \sqsubseteq V' \sqcap a$, *then* $U = U' \sqcap a$ *and* $U' \sqsubseteq V'$.

3. *Let* $p, k \geq 1$. *If* $\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq \sqcap_{i=1}^{p}(U'_i \to T'_i)$, *then* $\forall 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ *such that* $U'_i \sqsubseteq U_j$ *and* $T_j \sqsubseteq T'_i$.

4. *If* $U \to T \sqsubseteq V$, *then* $V = \sqcap_{i=1}^{p}(U_i \to T_i)$ *where* $p \geq 1$ *and* $\forall 1 \leq i \leq p$, $U_i \sqsubseteq U$ *and* $T \sqsubseteq T_i$.

5. *If* $\sqcap_{j=1}^{k}(U_j \to T_j) \sqsubseteq V$ *where* $k \geq 1$, *then* $V = \sqcap_{i=1}^{p}(U'_i \to T'_i)$ *where* $p \geq 1$ *and* $\forall 1 \leq i \leq p$, $\exists 1 \leq j \leq k$ *such that* $U'_i \sqsubseteq U_j$ *and* $T_j \sqsubseteq T'_i$.

Proof. See [KN07] □

Lemma 9. 1. *If* $\Gamma \sqsubseteq \Gamma'$ *and* $U \sqsubseteq U'$, *then* $U.\Gamma \sqsubseteq U'.\Gamma'$.

2. $\Gamma \sqsubseteq \Gamma'$ *iff* $|\Gamma| = |\Gamma'| = m$ *and, if* $m > 0$ *then* $\forall 1 \leq i \leq m$, $\Gamma_i \sqsubseteq \Gamma'_i$.

3. *If* $|\Gamma| = sup(M)$, *then* $\Gamma \sqsubseteq env_{\omega}^{M}$.

4. *If* $env_{\omega}^{M} \sqsubseteq \Gamma$, *then* $\Gamma = env_{\omega}^{M}$.

5. $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$ *iff* $\Gamma' \sqsubseteq \Gamma$ *and* $U \sqsubseteq U'$.

6. *If* $\Gamma \sqsubseteq \Gamma'$ *and* $\Delta \sqsubseteq \Delta'$, *then* $\Gamma \sqcap \Delta \sqsubseteq \Gamma' \sqcap \Delta'$.

Proof. 1. By induction on the derivation $\Gamma \sqsubseteq \Gamma'$ we have that if $\Gamma \sqsubseteq \Gamma'$, then $V.\Gamma \sqsubseteq V.\Gamma'$. Using tr we have the result.

2. Only if) By induction on the derivation $\Gamma \sqsubseteq \Gamma'$. If) By induction on m using 1.

3. By lemma 7.2 and 2.

4. By 2, $|\Gamma| = sup(M) = m$. If $m = 0$, them $env_{\omega}^{M} = \Gamma = nil$. Otherwise, for every $1 \leq i \leq m$, $\omega \sqsubseteq \Gamma_i$. Hence, by lemma 7.3, $\forall 1 \leq i \leq m$, $\Gamma_i = \omega$.

5. Only if) By induction on the derivation $\langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle$. If) By $\sqsubseteq_{\langle\rangle}$.

6. This is a corollary of 2.

□

The following lemma shows the strict relation in a type judgement between the length of a context Γ and the free indices of term M, where $M : \langle \Gamma \vdash U \rangle$ for some type U.

Lemma 10. 1. *If* $M : \langle \Gamma \vdash U \rangle$, *then* $|\Gamma| = sup(M)$.

2. *For every* Γ *and* M *such that* $|\Gamma| = sup(M)$, *we have* $M : \langle \Gamma \vdash \omega \rangle$.

Proof. 1. By induction on the derivation $M : \langle \Gamma \vdash U \rangle$.

2. By ω, $M : \langle env_{\omega}^{M} \vdash \omega \rangle$. By lemma 9.3, $\Gamma \sqsubseteq env_{\omega}^{M}$. Hence, by $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $M : \langle \Gamma \vdash \omega \rangle$.

□

Consequently, the weakening allowed in the system is limited by the maximum value of a free index occurring in a term.

The following lemma shows that another version of the var and $\sqcap_i$ rules, axiom and intersection introduction respectively, are derivable from the typing rules and subtyping relation, presented in definition 8.

Lemma 11. *1. The rule* $\dfrac{M:\langle\Gamma\vdash U_1\rangle \quad M:\langle\Delta\vdash U_2\rangle}{M:\langle\Gamma\sqcap\Delta\vdash U_1\sqcap U_2\rangle}\ \sqcap'_i$ *is derivable.*

2. The rule $\dfrac{}{\underline{1}:\langle U.nil\vdash U\rangle}\ \mathrm{var}'$ *is derivable.*

Proof. 1. Let $M:\langle\Gamma\vdash U_1\rangle$ and $M:\langle\Delta\vdash U_2\rangle$. By lemma 10.1, $|\Gamma|=|\Delta|=m$. Thus, $|\Gamma\sqcap\Delta|=m$ and $(\Gamma\sqcap\Delta)_i=\Gamma_i\sqcap\Delta_i$, $\forall 1\le i\le m$. By rule $\sqcap_e$ and lemma 9.2, $\Gamma\sqcap\Delta\sqsubseteq\Gamma$ and $\Gamma\sqcap\Delta\sqsubseteq\Delta$. Hence, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $M:\langle\Gamma\sqcap\Delta\vdash U_1\rangle$ and $M:\langle\Gamma\sqcap\Delta\vdash U_2\rangle$. Thus, by rule $\sqcap_i$, $M:\langle\Gamma\sqcap\Delta\vdash U_1\sqcap U_2\rangle$.

2. By lemma 7.1:

- Either $U=\omega$, then by rule ω the result holds.
- Or $U=\sqcap_{i=1}^k T_i$ where $\forall 1\le i\le k$, $T_i\in\mathbb{T}$, then, by rule var, $\underline{1}:\langle T_i.nil\vdash T_i\rangle$ and, by $k-1$ applications of rule $\sqcap'_i$, $\underline{1}:\langle U.nil\vdash U\rangle$.

□

4 The subject reduction property

4.1 Subject reduction for β

The subject reduction property is proved in the standard way, with a generation and substitutions lemmas (lemmas 12 and 14, respectively) as the properties to be proved at first.

Lemma 12 (Generation). *1. If* $\underline{n}:\langle\Gamma\vdash U\rangle$*, then* $\Gamma_n=V$ *where* $V\sqsubseteq U$*.*

2. If $\lambda.M:\langle\Gamma\vdash U\rangle$ *and* $sup(M)>0$*, then* $U=\omega$ *or* $U=\sqcap_{i=1}^k(V_i\to T_i)$ *where* $k\ge 1$ *and* $\forall 1\le i\le k$*,* $M:\langle V_i.\Gamma\vdash T_i\rangle$*.*

3. If $\lambda.M:\langle\Gamma\vdash U\rangle$ *and* $sup(M)=0$*, then* $\Gamma=nil$*,* $U=\omega$ *or* $U=\sqcap_{i=1}^k(V_i\to T_i)$ *where* $k\ge 1$ *and* $\forall 1\le i\le k$*,* $M:\langle nil\vdash T_i\rangle$*.*

Proof. 1. By induction on the derivation $\underline{n}:\langle\Gamma\vdash U\rangle$. By lemma 10.1, $|\Gamma|=n$.

- If $\dfrac{}{\underline{1}:\langle T.nil\vdash T\rangle}$, nothing to prove.
- If $\dfrac{}{\underline{n}:\langle env_\omega^{\underline{n}}\vdash\omega\rangle}$, nothing to prove.
- Let $\dfrac{\underline{n}:\langle\Gamma\vdash U\rangle}{\underline{n+1}:\langle\omega.\Gamma\vdash U\rangle}$. One has that $(\omega.\Gamma)_{n+1}=\Gamma_n$ and, by IH, $\Gamma_n=V$ where $V\sqsubseteq U$.
- Let $\dfrac{\underline{n}:\langle\Gamma\vdash U_1\rangle \quad \underline{n}:\langle\Gamma\vdash U_2\rangle}{\underline{n}:\langle\Gamma\vdash U_1\sqcap U_2\rangle}$. By IH, $\Gamma_n=V$ where $V\sqsubseteq U_1$ and $V\sqsubseteq U_2$. Then, by rule $\sqcap$, $V\sqsubseteq U_1\sqcap U_2$.
- Let $\dfrac{\underline{n}:\langle\Gamma\vdash U\rangle \quad \langle\Gamma\vdash U\rangle\sqsubseteq\langle\Gamma'\vdash U'\rangle}{\underline{n}:\langle\Gamma'\vdash U'\rangle}$. By IH, $\Gamma_n=V$ where $V\sqsubseteq U$. By lemma 9.5, $\Gamma'\sqsubseteq\Gamma$ and $U\sqsubseteq U'$. Thus, by lemma 9.2, $\Gamma'_n=V'\sqsubseteq V$. By rule tr, $V'\sqsubseteq U'$.

2. By induction on the derivation $\lambda.M:\langle\Gamma\vdash U\rangle$.

- If $\dfrac{}{\lambda.M:\langle env_\omega^{\lambda.M}\vdash\omega\rangle}$, nothing to prove.

- If $\dfrac{M:\langle U.\Gamma \vdash T\rangle}{\lambda.M:\langle \Gamma \vdash U \to T\rangle}$, nothing to prove.
- Let $\dfrac{\lambda.M:\langle \Gamma \vdash U_1\rangle \quad \lambda.M:\langle \Gamma \vdash U_2\rangle}{\lambda.M:\langle \Gamma \vdash U_1 \sqcap U_2\rangle}$. By IH, one has the following cases:
  - If $U_1 = U_2 = \omega$, then $U_1 \sqcap U_2 = \omega$.
  - If $U_1 = \omega$, $U_2 = \sqcap_{i=1}^{k}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M:\langle V_i.\Gamma \vdash T_i\rangle$, then, $U_1 \sqcap U_2 = U_2$
  - If $U_2 = \omega$, $U_1 = \sqcap_{i=1}^{k}(V'_i \to T'_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M:\langle V'_i.\Gamma \vdash T'_i\rangle$, then, $U_1 \sqcap U_2 = U_1$
  - If $U_1 = \sqcap_{i=1}^{k}(V_i \to T_i)$,$U_2 = \sqcap_{i=k+1}^{k+l}(V_i \to T_i)$, where $k, l \geq 1$ and $\forall 1 \leq i \leq k+l$, $M:\langle V_i.\Gamma \vdash T_i\rangle$, then $U_1 \sqcap U_2 = \sqcap_{i=1}^{k+l}(V_i \to T_i)$.
- Let $\dfrac{\lambda.M:\langle \Gamma \vdash U\rangle \quad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{\lambda.M:\langle \Gamma' \vdash U'\rangle}$. By lemma 9.5, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. By IH, one has the following:
  - If $U = \omega$, then, by lemma 7.3, $U' = \omega$.
  - Otherwise, $U = \sqcap_{i=1}^{k}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M:\langle V_i.\Gamma \vdash T_i\rangle$. By lemma 7.1, either $U' = \omega$, and then nothing to prove, or, by lemma 8.5, $U' = \sqcap_{i=1}^{p}(V'_i \to T'_i)$ where $p \geq 1$ and $\forall\, 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. By lemmas 9.1 and 9.5, $\langle V_{j_i}.\Gamma \vdash T_{j_i}\rangle \sqsubseteq \langle V'_i.\Gamma' \vdash T'_i\rangle$, for each $1 \leq i \leq p$, then, $M:\langle V'_i.\Gamma' \vdash T'_i\rangle$.

3. By lemma 1.2, $sup(\lambda.M) = 0$ and, by lemma 10.1, $|\Gamma| = nil$, thus, $\lambda.M : \langle nil \vdash U\rangle$. The proof is same as for 2, where $\to'_i$ is used on induction step, instead of $\to_i$.

□

The following lemma is an auxiliary lemma for substitution lemma 14, stating a property relating type judgements and the index update mechanism.

Lemma 13. *If $M:\langle \Gamma \vdash U\rangle$ and $0 \leq i < sup(M)$, then $M^{+i}:\langle \Gamma_{\leq i}.\omega.\Gamma_{>i} \vdash U\rangle$.*

Proof. By induction on the derivation $M:\langle \Gamma \vdash U\rangle$.

- Let $\dfrac{}{\underline{1}:\langle T.nil \vdash T\rangle}$. For $i = 0$, $\underline{1}^{+} = \underline{2}$ and, by rule varn, $\underline{2}:\langle \omega.T.nil \vdash T\rangle$.
- If $\dfrac{}{M:\langle env_{\omega}^{M} \vdash \omega\rangle}$, nothing to prove.
- Let $\dfrac{\underline{n}:\langle \Gamma \vdash U\rangle}{\underline{n+1}:\langle \omega.\Gamma \vdash U\rangle}$. If $i = 0$, then by rule varn $\underline{n+2} : \langle \omega.\omega.\Gamma \vdash U\rangle$. Otherwise, note that $\underline{n^{+i} + 1} = \underline{n+1}^{+(i+1)} = \underline{n+2}$. By IH one has $\underline{n}^{+i}:\langle \Gamma_{\leq i}.\omega.\Gamma_{>i} \vdash U\rangle$. By rule varn, $\underline{n+2}:\langle \omega.\Gamma_{\leq i}.\omega.\Gamma_{>i} \vdash U\rangle$.
- Let $\dfrac{M:\langle U.\Gamma \vdash T\rangle}{\lambda.M:\langle \Gamma \vdash U \to T\rangle}$. By lemma 1.2 one has $sup(M) > i+1$, hence, by IH, $M^{+(i+1)}:\langle U.\Gamma_{\leq i}.\omega.\Gamma_{>i} \vdash T\rangle$. Hence, by rule $\to_i$ and i-lift definition, $(\lambda.M)^{+i}:\langle \Gamma_{\leq i}.\omega.\Gamma_{>i} \vdash U \to T\rangle$.
- Let $\dfrac{M_1:\langle \Gamma \vdash U \to T\rangle \quad M_2:\langle \Delta \vdash U\rangle}{M_1\ M_2:\langle \Gamma \sqcap \Delta \vdash T\rangle}$. By lemma 1.1 one has $sup(M_1) > i$ or $sup(M_2) > i$. Suppose w.l.o.g. that $i < sup(M_1), sup(M_2)$. By IH,

$M_1^{+i}:\langle\Gamma_{\le i}.\omega.\Gamma_{>i}\vdash U\to T\rangle$ and $M_2^{+i}:\langle\Delta_{\le i}.\omega.\Delta_{>i}\vdash U\rangle$. Thus, by $\to_e$ and observing that $(\Gamma_{\le i}.\omega.\Gamma_{>i})\sqcap(\Delta_{\le i}.\omega.\Delta_{>i})=(\Gamma\sqcap\Delta)_{\le i}.\omega.(\Gamma\sqcap\Delta)_{>i}$, $(M_1\ M_2)^{+i}:\langle(\Gamma\sqcap\Delta)_{\le i}.\omega.(\Gamma\sqcap\Delta)_{>i}\vdash T\rangle$.

- Let $\dfrac{M:\langle\Gamma\vdash U_1\rangle\quad M:\langle\Gamma\vdash U_2\rangle}{M:\langle\Gamma\vdash U_1\sqcap U_2\rangle}$. By IH, $M^{+i}:\langle\Gamma_{\le i}.\omega.\Gamma_{>i}\vdash U_1\rangle$ and $M^{+i}:\langle\Gamma_{\le i}.\omega.\Gamma_{>i}\vdash U_2\rangle$. Thus, by rule $\sqcap_i$, $M^{+i}:\langle\Gamma_{\le i}.\omega.\Gamma_{>i}\vdash U_1\sqcap U_2\rangle$.
- Let $\dfrac{M:\langle\Gamma\vdash U\rangle\quad\langle\Gamma\vdash U\rangle\sqsubseteq\langle\Gamma'\vdash U'\rangle}{M:\langle\Gamma'\vdash U'\rangle}$. By IH, $M^{+i}:\langle\Gamma_{\le i}.\omega.\Gamma_{>i}\vdash U\rangle$ and, by lemma 9.5, $\Gamma'\sqsubseteq\Gamma$ and $U\sqsubseteq U'$. Hence, by lemma 9.2, $\Gamma'_{\le i}.\omega.\Gamma'_{>i}\sqsubseteq\Gamma_{\le i}.\omega.\Gamma_{>i}$. Thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $M^{+i}:\langle\Gamma'_{\le i}.\omega.\Gamma'_{>i}\vdash U'\rangle$.

□

Lemma 14 (Substitution). *Let $M:\langle\Gamma\vdash U\rangle$, for $sup(M)>0$, and $N:\langle\Delta\vdash\Gamma_i\rangle$:*

1. *If $\underline{i}\notin FI(M)$, then $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U\rangle$.*
2. *Otherwise, if $sup(N)\ge i-1$, then $\{\underline{i}/N\}M:\langle(\Gamma_{<i}.\Gamma_{>i})\sqcap\Delta\vdash U\rangle$.*

Proof. By induction on the derivation $M:\langle\Gamma\vdash U\rangle$.

1. Observe that $i<|\Gamma|=sup(M)$:
   - If $\dfrac{}{\underline{1}:\langle T.nil\vdash T\rangle}$, nothing to prove.
   - Let $\dfrac{}{M:\langle env_\omega^M\vdash\omega\rangle}$. By lemma 4.1, $sup(\{\underline{i}/N\}M)=sup(M)-1$. Thus, $env_\omega^{\{\underline{i}/N\}M}=(env_\omega^M)_{<i}.(env_\omega^M)_{>i}$ and the result holds trivially by rule ω.
   - Let $\dfrac{\underline{n}:\langle\Gamma\vdash U\rangle}{\underline{n+1}:\langle\omega.\Gamma\vdash U\rangle}$. By lemma 10.1, $|\omega.\Gamma|=n+1$, hence, $i<(n+1)$ and $\{\underline{i}/N\}\,\underline{n+1}=\underline{n}$. Note that $(\omega.\Gamma)_i=\Gamma_{(i-1)}$, thus, by IH one has $\{\underline{i-1}/N\}\,\underline{n}:\langle\Gamma_{<(i-1)}.\Gamma_{>(i-1)}\vdash U\rangle$. Since $(i-1)<n$, $\{\underline{i-1}/N\}\,\underline{n}=\underline{n-1}$, hence, by rule varn, $\underline{n}:\langle\omega.\Gamma_{<(i-1)}.\Gamma_{>(i-1)}\vdash U\rangle$.
   - Let $\dfrac{M:\langle U.\Gamma\vdash T\rangle}{\lambda.M:\langle\Gamma\vdash U\to T\rangle}$. If $sup(N)=0$, then, by lemma 2.1, $N^+\equiv N$, otherwise, by lemma 13, $N^+:\langle\omega.\Delta\vdash\Gamma_i\rangle$. By IH, $\{\underline{i+1}/N^+\}M:\langle U.\Gamma_{<i}.\Gamma_{>i}\vdash T\rangle$, thus, by $\to_i$, $\lambda.\{\underline{i+1}/N^+\}M:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U\to T\rangle$.
   - Let $\dfrac{M_1:\langle\Gamma\vdash U\to T\rangle\quad M_2:\langle\Gamma'\vdash U\rangle}{M_1\ M_2:\langle\Gamma\sqcap\Gamma'\vdash T\rangle}$. Suppose, w.l.o.g., $i<sup(M_1)$ and $i<sup(M_2)$, thus, $(\Gamma\sqcap\Gamma')_i=\Gamma_i\sqcap\Gamma'_i$. By rules $\sqcap_e$, $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$ one has $N:\langle\Delta\vdash\Gamma_i\rangle$ and $N:\langle\Delta\vdash\Gamma'_i\rangle$. Hence, by IH, $\{\underline{i}/N\}M_1:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U\to T\rangle$ and $\{\underline{i}/N\}M_2:\langle\Gamma'_{<i}.\Gamma'_{>i}\vdash U\rangle$. Thus, by rule $\to_e$, $(\{\underline{i}/N\}M_1\ \{\underline{i}/N\}M_2):\langle(\Gamma_{<i}\sqcap\Gamma'_{<i}).(\Gamma_{>i}\sqcap\Gamma'_{>i})\vdash T\rangle$.
   - Let $\dfrac{M:\langle\Gamma\vdash U_1\rangle\quad M:\langle\Gamma\vdash U_2\rangle}{M:\langle\Gamma\vdash U_1\sqcap U_2\rangle}$. By IH, $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U_1\rangle$ and $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U_2\rangle$. Thus, by rule $\sqcap_i$, one has that $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U_1\sqcap U_2\rangle$.
   - Let $\dfrac{M:\langle\Gamma\vdash U\rangle\quad\langle\Gamma\vdash U\rangle\sqsubseteq\langle\Gamma'\vdash U'\rangle}{M:\langle\Gamma'\vdash U'\rangle}$. By lemma 9.5, $\Gamma'\sqsubseteq\Gamma$ and $U\sqsubseteq U'$, hence, by lemma 9.2, $\Gamma'_i\sqsubseteq\Gamma_i$ and $\Gamma'_{<i}.\Gamma'_{>i}\sqsubseteq\Gamma_{<i}.\Gamma_{>i}$. Thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $N:\langle\Delta\vdash\Gamma_i\rangle$, and, by IH, $\{\underline{i}/N\}M:\langle\Gamma_{<i}.\Gamma_{>i}\vdash U\rangle$. By rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $\{\underline{i}/N\}M:\langle\Gamma'_{<i}.\Gamma'_{>i}\vdash U'\rangle$.

2. - If $\dfrac{}{\underline{1}:\langle T.nil \vdash T\rangle}$, nothing to prove.
   - Let $\dfrac{}{M:\langle env_\omega^M \vdash \omega\rangle}$. One has the following cases:
     - If $FI(M) = \{\,\underline{i}\,\}$, then $|env_\omega^M| = i$, thus, $(env_\omega^M)_{<i}.(env_\omega^M)_{>i} = env_\omega^{M'}$, where M' is any term such that $sup(M') = i-1$. Hence, $env_\omega^{M'} \sqcap \Delta = \Delta$. By lemmas 4.3 and 10.1, $sup(\{\,\underline{i}/N\}M) = sup(N) = |\Delta|$, hence, by lemma 10.2, $\{\,\underline{i}/N\}M:\langle \Delta \vdash \omega\rangle$.
     - Otherwise, by lemma 4.3 and 10.1, $sup(\{\,\underline{i}/N\}M)$ is given by $max(sup(N), sup(M){-}1) = max(|\Delta|, |env_\omega^M|{-}1)$, which is equivalent to $|\Delta \sqcap ((env_\omega^M)_{<i}.(env_\omega^M)_{>i})|$. Thus, by lemma 10.2, $\{\,\underline{i}/N\}M:\langle \Delta \sqcap ((env_\omega^M)_{<i}.(env_\omega^M)_{>i}) \vdash \omega\rangle$.
   - Let $\dfrac{\underline{n}:\langle \Gamma \vdash U\rangle}{\underline{n{+}1}:\langle \omega.\Gamma \vdash U\rangle}$. For $i = n{+}1$, $\{\,\underline{n{+}1}/N\}\,\underline{n{+}1} = N$ and, by lemma 10.1, $|\Gamma| = n$. By lemma 12, $\Gamma_n = V$, where $V \sqsubseteq U$. Thus, by rule $\sqcap_e$ and lemma 9.2, $(\omega.\Gamma_{<n}.nil) \sqcap \Delta \sqsubseteq \Delta$ and, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $N:\langle(\omega.\Gamma_{<n}.nil) \sqcap \Delta \vdash U\rangle$.
   - Let $\dfrac{M:\langle U.\Gamma \vdash T\rangle}{\lambda.M:\langle \Gamma \vdash U \to T\rangle}$. Note that $(U.\Gamma)_{(i+1)} = \Gamma_i$. If $sup(N) = 0$, then, by lemma 2.1, $N^+ \equiv N$, otherwise, by lemma 13, $N^+:\langle \omega.\Delta \vdash \Gamma_i\rangle$. By IH, $\{\,\underline{i{+}1}/N^+\}M : \langle (U.\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta' \vdash T\rangle$, where Δ' is either nil or $\omega.\Delta$. If $\Delta' \equiv \omega.\Delta$, then $(U.\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta' = U.\big((\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta\big)$. Thus, by rule $\to_i$, $\lambda.\{\,\underline{i{+}1}/N^+\}M : \langle (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \vdash U \to T\rangle$. The case where $\Delta' \equiv nil$ is trivial.
   - Let $\dfrac{M_1:\langle \Gamma \vdash U \to T\rangle \quad M_2:\langle \Gamma' \vdash U\rangle}{M_1\ M_2:\langle \Gamma \sqcap \Gamma' \vdash T\rangle}$. If $\underline{i} \in FI(M_1)$ and $\underline{i} \in FI(M_2)$, then, $(\Gamma \sqcap \Gamma')_i = \Gamma_i \sqcap \Gamma'_i$, and, by rules $\sqcap_e$, $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $N:\langle \Delta \vdash \Gamma_i\rangle$ and $N:\langle \Delta \vdash \Gamma'_i\rangle$. By IH, $\{\,\underline{i}/N\}M_1 : \langle (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \vdash U \to T\rangle$ and $\{\,\underline{i}/N\}M_2 : \langle (\Gamma'_{<i}.\Gamma'_{>i}) \sqcap \Delta \vdash U\rangle$. Note that $(\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \sqcap (\Gamma'_{<i}.\Gamma'_{>i}) \sqcap \Delta = \big((\Gamma \sqcap \Gamma')_{<i}.(\Gamma \sqcap \Gamma')_{>i}\big) \sqcap \Delta$. Thus, by rule $\to_e$, $\{\,\underline{i}/N\}(M_1\ M_2):\langle \big((\Gamma \sqcap \Gamma')_{<i}.(\Gamma \sqcap \Gamma')_{>i}\big) \sqcap \Delta \vdash T\rangle$. The cases $\underline{i} \notin FI(M_1)$ and $\underline{i} \notin FI(M_2)$ are similar, using 1 on the induction step whenever necessary.
   - Let $\dfrac{M:\langle \Gamma \vdash U_1\rangle \quad M:\langle \Gamma \vdash U_2\rangle}{M:\langle \Gamma \vdash U_1 \sqcap U_2\rangle}$. By IH one has that $\{\,\underline{i}/N\}M : \langle (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \vdash U_1\rangle$ and $\{\,\underline{i}/N\}M:\langle (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \vdash U_2\rangle$. Thus, by rule $\sqcap_i$, $\{\,\underline{i}/N\}M:\langle (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \vdash U_1 \sqcap U_2\rangle$.
   - Let $\dfrac{M:\langle \Gamma \vdash U\rangle \quad \langle \Gamma \vdash U\rangle \sqsubseteq \langle \Gamma' \vdash U'\rangle}{M:\langle \Gamma' \vdash U'\rangle}$. By lemma 9.5, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$, hence, by lemma 9.2, $\Gamma'_i \sqsubseteq \Gamma_i$ and $\Gamma'_{<i}.\Gamma'_{>i} \sqsubseteq \Gamma_{<i}.\Gamma_{>i}$. Thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $N:\langle \Delta \vdash \Gamma_i\rangle$ and, by IH, one has $\{\,\underline{i}/N\}M : \langle (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta \vdash U\rangle$. By lemma 9.6, $(\Gamma'_{<i}.\Gamma'_{>i}) \sqcap \Delta \sqsubseteq (\Gamma_{<i}.\Gamma_{>i}) \sqcap \Delta$, thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $\{\,\underline{i}/N\}M:\langle (\Gamma'_{<i}.\Gamma'_{>i}) \sqcap \Delta \vdash U'\rangle$.

□

As a consequence of lemma 10 and the possibility of some free indices be eliminated during a β-reduction, we need the following definition.

Definition 9. *Let M be a term and $sup(M) = m$. For a context Γ, let $\Gamma\lfloor_M$ be the restriction of Γ to $FI(M)$, given by $\Gamma_{\leq m}.nil$.*

The definition above will allow us to type the resulting term from a β-reduction in a shorter context, related to the original one. First, we prove some properties about the restriction on contexts.

Lemma 15. *1. If $sup(N) \leq sup(M)$, then $env_\omega^M\lfloor_N = env_\omega^N$.*

2. If $|\Gamma| \leq sup(M)$, then $(\Gamma \sqcap \Delta)\lfloor_M = \Gamma \sqcap \Delta\lfloor_M$.

3. If $sup(N) > 0$, then $(U.\Gamma)\lfloor_N = U.\Gamma\lfloor_{(\lambda.N)}$.

Proof. 1. Straightforward from definition 9 and the definition of env_ω^M.

2. Let $sup(M)=m$. Thus, $(\Gamma \sqcap \Delta)\lfloor_M = (\Gamma \sqcap \Delta)_{\leq m}.nil = (\Gamma_{\leq m} \sqcap \Delta_{\leq m}).nil = (\Gamma_{\leq m}.nil) \sqcap (\Delta_{\leq m}.nil) = \Gamma \sqcap (\Delta_{\leq m}.nil) = \Gamma \sqcap \Delta\lfloor_M$.

3. If $sup(N) > 0$, by lemma 1.2, $sup(\lambda.N) = sup(N) - 1$. Thus, $(U.\Gamma)\lfloor_N = (U.\Gamma)_{\leq sup(N)}.nil = U.\Gamma_{\leq (sup(N)-1)}.nil = U.\Gamma\lfloor_{(\lambda.N)}$.

$\square$

Finally, we have theorem 2 stating the proof for β-redices and then theorem 3 for any β-contraction.

Theorem 2. *If $(\lambda.M\ N) : \langle \Gamma \vdash U \rangle$ then $\{\underline{1}/N\}M : \langle \Gamma\lfloor_{\{\underline{1}/N\}M} \vdash U \rangle$*

Proof. By induction on the derivation $(\lambda.M\ N) : \langle \Gamma \vdash U \rangle$.

- Let $\dfrac{}{(\lambda.M\ N) : \langle env_\omega^{(\lambda.M\ N)} \vdash \omega \rangle}$. By lemma 5, one has $sup(\{\underline{1}/N\}M) \leq sup(\lambda.M\ N)$, hence, by lemma 15.1, $env_\omega^{\lambda.M\ N}\lfloor_{\{\underline{1}/N\}M} = env_\omega^{\{\underline{1}/N\}M}$. By rule ω the result is obtained, trivially.

- Let $\dfrac{\lambda.M : \langle \Delta \vdash U \to T \rangle \quad N : \langle \Delta' \vdash U \rangle}{(\lambda.M\ N) : \langle \Delta \sqcap \Delta' \vdash T \rangle}$. One has the following cases.

  If $sup(M)=0$, then, by lemma 12.3, $\Delta = nil$ and $M : \langle nil \vdash T \rangle$. By lemma 3.3, $\{\underline{1}/N\}M \equiv M$, thus, $\Delta \sqcap \Delta' = \Delta'$ and $\Delta'\lfloor_{\{\underline{1}/N\}M} = \Delta'\lfloor_M = nil$.

  If $sup(M)>0$, then, by lemma 12.2, $M : \langle U.\Delta \vdash T \rangle$:

  - If $\underline{1} \notin FI(M)$, then, by lemma 14.1, $\{\underline{1}/N\}M : \langle \Delta \vdash T \rangle$. By lemma 15.2, $(\Delta \sqcap \Delta')\lfloor_{\{\underline{1}/N\}M} = \Delta \sqcap (\Delta'\lfloor_{\{\underline{1}/N\}M})$, hence, by rule $\sqcap_e$ and lemma 9.2, $(\Delta \sqcap \Delta')\lfloor_{\{\underline{1}/N\}M} \sqsubseteq \Delta$. Thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $\{\underline{1}/N\}M : \langle (\Delta \sqcap \Delta')\lfloor_{\{\underline{1}/N\}M} \vdash T \rangle$.
  - Otherwise, by lemma 14.2, $\{\underline{1}/N\}M : \langle \Delta \sqcap \Delta' \vdash T \rangle$. By lemma 10.1, $|\Delta \sqcap \Delta'| = sup(\{\underline{1}/N\}M)$, thus, $(\Delta \sqcap \Delta')\lfloor_{\{\underline{1}/N\}M} = \Delta \sqcap \Delta'$.

- Let $\dfrac{(\lambda.M\ N) : \langle \Gamma \vdash U_1 \rangle \quad (\lambda.M\ N) : \langle \Gamma \vdash U_2 \rangle}{(\lambda.M\ N) : \langle \Gamma \vdash U_1 \sqcap U_2 \rangle}$. By IH one has $\{\underline{1}/N\}M : \langle \Gamma\lfloor_{\{\underline{1}/N\}M} \vdash U_1 \rangle$ and $\{\underline{1}/N\}M : \langle \Gamma\lfloor_{\{\underline{1}/N\}M} \vdash U_2 \rangle$. Thus, by rule $\sqcap_i$, $\{\underline{1}/N\}M : \langle \Gamma\lfloor_{\{\underline{1}/N\}M} \vdash U_1 \sqcap U_2 \rangle$.

- Let $\dfrac{(\lambda.M\ N) : \langle \Gamma \vdash U \rangle \quad \langle \Gamma \vdash U \rangle \sqsubseteq \langle \Gamma' \vdash U' \rangle}{(\lambda.M\ N) : \langle \Gamma' \vdash U' \rangle}$. By IH, one has $\{\underline{1}/N\}M : \langle \Gamma\lfloor_{\{\underline{1}/N\}M} \vdash U \rangle$. By lemma 9.5, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$, hence, by lemma 9.2, $\Gamma'\lfloor_{\{\underline{1}/N\}M} \sqsubseteq \Gamma\lfloor_{\{\underline{1}/N\}M}$. Thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $\{\underline{i}/N\}M : \langle \Gamma'\lfloor_{\{\underline{1}/N\}M} \vdash U' \rangle$.

$\square$

Theorem 3 (SR for β-contraction). *If $M:\langle\Gamma\vdash U\rangle$ and $M\longrightarrow_\beta N$, then $N:\langle\Gamma\lfloor_N\vdash U\rangle$.*

Proof. Induction on the derivation $M:\langle\Gamma\vdash U\rangle$

- Let $\dfrac{}{M:\langle env^M_\omega\vdash\omega\rangle}$. One has that $FI(N)\subseteq FI(M)$, hence, $sup(N)\leq sup(M)$. By lemma 15.1, $env^M_\omega\lfloor_N = env^N_\omega$, thus, by rule ω, $N:\langle env^N_\omega\vdash\omega\rangle$.

- Let $\dfrac{M':\langle V.\Gamma\vdash T\rangle}{\lambda.M':\langle\Gamma\vdash V\to T\rangle}$. By IH, $N':\langle(V.\Gamma)\lfloor_{N'}\vdash T\rangle$, where $M'\longrightarrow_\beta N'$.

  If $sup(N')=0$, then $N':\langle nil\vdash T\rangle$. By $\to'_i$, $\lambda.N':\langle nil\vdash\omega\to T\rangle$, hence, by rules $\to$, $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $\lambda.N':\langle nil\vdash V\to T\rangle$.

  If $sup(N')>0$, then, by lemma 15.3, $(V.\Gamma)\lfloor_{N'} = V.\Gamma\lfloor_{\lambda.N'}$. Thus, by rule $\to_i$, $\lambda.N':\langle\Gamma\lfloor_{\lambda.N'}\vdash V\to T\rangle$.

- Let $\dfrac{M':\langle nil\vdash T\rangle}{\lambda.M':\langle nil\vdash\omega\to T\rangle}$. Thus, $M'\longrightarrow_\beta N'$ and, by theorem 1, $sup(N')\leq sup(M')=0$. By IH, $N':\langle nil\vdash T\rangle$, hence, by rule $\to'_i$, $\lambda.N':\langle nil\vdash\omega\to T\rangle$.

- Let $\dfrac{M_1:\langle\Delta\vdash U\to T\rangle\quad M_2:\langle\Delta'\vdash U\rangle}{M_1\ M_2:\langle\Delta\sqcap\Delta'\vdash T\rangle}$. Suppose that $N\equiv(N_1\ M_2)$, where $M_1\longrightarrow_\beta N_1$, hence, by IH, $N_1:\langle\Delta\lfloor_{N_1}\vdash U\to T\rangle$. By rule $\to_e$, $(N_1\ M_2):\langle\Delta\lfloor_{N_1}\sqcap\Delta'\vdash T\rangle$.

  - If $sup(N_1)\geq sup(M_2)$, then $sup(N)=sup(N_1)$ and, by lemma 15.2, $(\Delta\sqcap\Delta')\lfloor_{N_1} = \Delta\lfloor_{N_1}\sqcap\Delta'$.
  - If $sup(M_2)>sup(N_1)$, then $sup(N)=sup(M_2)$ and, by lemma 15.2, $(\Delta\sqcap\Delta')\lfloor_{M_2} = \Delta\lfloor_{M_2}\sqcap\Delta'$. By rule $\sqcap_e$ and lemma 9.2, one has that $(\Delta\lfloor_{M_2})_{>sup(N_1)}\sqcap\Delta'_{>sup(N_1)}\sqsubseteq\Delta'_{>sup(N_1)}$, thus, by lemma 9.2, $(\Delta\sqcap\Delta')\lfloor_{N_1}.((\Delta\lfloor_{M_2})_{>sup(N_1)}\sqcap\Delta'_{>sup(N_1)})\sqsubseteq(\Delta\sqcap\Delta')\lfloor_{N_1}.\Delta'_{>sup(N_1)}$. Observe, by lemma 6.4 and definition 9, that $(\Delta\sqcap\Delta')\lfloor_{N_1}.\Delta'_{>sup(N_1)} = \Delta\lfloor_{N_1}\sqcap\Delta'$ and that $(\Delta\sqcap\Delta')\lfloor_{N_1}.((\Delta\lfloor_{M_2})_{>sup(N_1)}\sqcap\Delta'_{>sup(N_1)}) = \Delta\lfloor_{M_2}\sqcap\Delta'$. Thus, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $N:\langle\Delta\lfloor_{M_2}\sqcap\Delta'\vdash T\rangle$.

- Let $\dfrac{M:\langle\Gamma\vdash U_1\rangle\quad M:\langle\Gamma\vdash U_2\rangle}{M:\langle\Gamma\vdash U_1\sqcap U_2\rangle}$. By IH, one has $N:\langle\Gamma\lfloor_N\vdash U_1\rangle$ and $N:\langle\Gamma\lfloor_N\vdash U_2\rangle$, thus, by rule $\sqcap_i$, $N:\langle\Gamma\lfloor_N\vdash U_1\sqcap U_2\rangle$.

- Let $\dfrac{M:\langle\Gamma'\vdash U'\rangle\quad\langle\Gamma'\vdash U'\rangle\sqsubseteq\langle\Gamma\vdash U\rangle}{M:\langle\Gamma\vdash U\rangle}$. By IH, $N:\langle\Gamma'\lfloor_N\vdash U'\rangle$ and, by lemma 9.5, $\Gamma\sqsubseteq\Gamma'$ and $U'\sqsubseteq U$. Thus, by lemma 9.2, $\Gamma\lfloor_N\sqsubseteq\Gamma'\lfloor_N$ and, by rules $\sqsubseteq_{\langle\rangle}$ and $\sqsubseteq$, $N:\langle\Gamma\lfloor_N\vdash U\rangle$.

$\square$

5 Conclusions and Future Work

We introduced an intersection type system in de Bruijn notation and proved it to preserve subject reduction. One particular difference between the type system presented in definition 8 and the one in [KN07] is that the former allows some kind of weakenig, while the latter does not. This characteristic may be relevant while investigating the principal typing property [Wel02]. A type inference algorithm for it might need Expansions to be performed [CW04.2]. Apparently, the way to achieve it is adding expansion variables to the type system [CW04, CW04.2].

The investigation of type inference, principal types, principal typings and other relevant properties in this system of intersection types as well as its adaptation for explicit substitution calculi in de Bruijn notation is an interesting work to be done.

References

[ACCL91] M. Abadi, L. Cardelli, P.-L. Curien, and J.-J. Lévy. Explicit Substitutions. *J. of Func. Programming*, 1(4):375–416, 1991.

[ARK01] M. Ayala-Rincón and F. Kamareddine. Unification via the λs_e-Style of Explicit Substitution. *The Logical Journal of the Interest Group in Pure and Applied Logics*, 9(4):489–523, 2001.

[BCDC83] H. Barendregt, M. Coppo, and M. Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *J. Symbolic Logic*, 48:931–940, 1983.

[CDC78] M. Coppo and M. Dezani-Ciancaglini. A new type assignment for lambda-terms. *Archiv für Mathematische Logik und Grundlagenforschung*, 19:139–156, 1978.

[CDC80] M. Coppo and M. Dezani-Ciancaglini. An Extension of the Basic Functionality Theory for the λ-Calculus. *Notre Dame Journal of Formal Logic*, 21(4):685–693, 1980.

[CF58] H. B. Curry and R. Feys. *Combinatory Logic*, volume 1. North Holland, 1958.

[CW04] S. Carlier and J. B. Wells. Type Inference with Expansion Variables and Intersection Types in System E and an Exact Correspondence with β-reduction. In *PPDP '04: Proceedings of the 6^{th} ACM SIGPLAN international conference on Principles and practice of declarative programming*, pages 132–143. ACM, 2004.

[CW04.2] S. Carlier and J. B. Wells. Expansion: the Crucial Mechanism for Type Inference with Intersection Types: a Survey and Explanation. In *ITRS '04 workshop*, 2004.

[dB72] N.G. de Bruijn. Lambda-Calculus Notation with Nameless Dummies, a Tool for Automatic Formula Manipulation, with Application to the Church-Rosser Theorem. *Indag. Mat.*, 34(5):381–392, 1972.

[dB78] N.G. de Bruijn. A namefree lambda calculus with facilities for internal definition of expressions and segments. T.H.-Report 78-WSK-03, Technische Hogeschool Eindhoven, Nederland, 1978.

[Kam03] F. Kamareddine, editor. *Thirty Five Years of Automating Mathematics*. Kluwer, 2003.

[KN07] F. Kamareddine and K. Nour. A completeness result for a realisability semantics for an intersection type system. *Annals of Pure and Applied Logic*, 146:180–198, 2007.

[KR95] F. Kamareddine and A. Ríos. A λ-calculus à la de Bruijn with Explicit Substitutions. In *Proc. of PLILP'95*, volume 982 of *LNCS*, pages 45–62. Springer, 1995.

[Mil78] Robin Milner. A theory of type polymorphism in programming. *Journal of computer and System Science*, 17(3):348–375, 1978.

[NGdV94] R. P. Nederpelt, J. H. Geuvers, and R. C. de Vrijer. *Selected papers on Automath*. North-Holland, 1994.

[Pot80] G. Pottinger. A type assignment for the strongly normalizable λ-terms. In J.P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 561–578. Academic Press, 1980.

[Wel02] J.B. Wells. The essence of principal typings. In *Proc. 29th International Colloquium on Automata, Languages and Programming, ICALP 2002*, volume 2380 of *LNCS*, pages 913–925. Springer, 2002.