Hall's Theorem for Enumerable Families of Finite Sets*

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Abstract. This work discusses the mechanisation in Isabelle/HOL of a general version of Hall's Theorem. It states that an enumerable family of finite sets has a system of distinct representatives (SDR) if it satisfies the "marriage condition". The marriage condition states that every finite subfamily of the possible infinite family of sets contains at least as many distinct members as the number of sets in the subfamily. The proof applies a formalisation of the Compactness Theorem for propositional logic. It checks the marriage condition for finite subfamilies of sets using Jiang and Nipkow's formalisation of the finite version of Hall's Theorem.

1 Hall's Theorem

Let $\mathcal{A}$ be a finite family of arbitrary subsets of a set S such that sets in the family may repeat. Hall's theorem (also known as the "marriage theorem") was proved initially by Philip Hall in 1935 [10]. It establishes a necessary and sufficient condition to select a distinct element for each set in the collection. This theorem is equivalent to other significant results applied in the study of combinatory and graph theory problems (cf. [2], [3], [18]): Menger's theorem (1929), König's minimax theorem (1931), König–Egerváry theorem (1931), Birkhoff-von Neumann's theorem (1946), Dilworth's theorem (1950), Max Flow-Min Cut theorem (Ford-Fulkerson algorithm) (1956), and also to probability theory results as Strassen's theorem (1965). For instance, the König–Egerváry theorem states that the number of lines (rows or columns) that cover all ones in a binary matrix is precisely the cardinality of a set of ones in different lines of the matrix. Taking the sets of ones in the matrix lines as the family of finite sets and selecting the ones that do not share lines as the system of distinct representatives, the equivalence between both problems is evident.

Hall's theorem is established using the notion of a system of distinct representatives (SDR) for a family of sets.

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Definition 1 (SDR). *Let S be an arbitrary set and $\{S_i\}_{i\in I}$ a collection of not necessarily distinct subsets of S with indices in the set I.*
A sequence $(x_i)_{i\in I}$ is a system of distinct representatives *for $\{S_i\}_{i\in I}$ if:*

1. *for all $i \in I$, $x_i \in S_i$, and;*
2. *for all $i, j \in I$, $x_i \neq x_j$, whenever $i \neq j$.*

Alternatively, one can define SDR as follows.
A function $f : I \to \bigcup_{i\in I} S_i$ is a SDR for $\{S_i\}_{i\in I}$ if:

1. for all $i \in I$, $f(i) \in S_i$, and;
2. f is an injective function.

Theorem 1 (Hall's Theorem | finite case). *Consider an arbitrary set S and a positive integer n. A finite collection $\{S_1, S_2, \ldots, S_n\}$ of finite subsets of S has a SDR if and only if the so called* marriage condition *(M) below is satisfied.*

For every $1 \leq k \leq n$ and an arbitrary set of k distinct indices $1 \leq i_1, \ldots, i_k \leq n$, one has that $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$. *(M)*

Hall's Theorem also holds for an infinite enumerable collection $\{S_i\}_{i\in I}$ of finite subsets of S (Theorem 2). Indeed, other versions of such a theorem are considered and proved in [17].

Theorem 2 (Hall's Theorem | enumerable case). *Let S be an arbitrary set and I an enumerable set of indices of finite subsets of S. The family $\{S_i\}_{i\in I}$ has a SDR if and only if the condition (M^*) below holds.*

For every finite subset of indices $J \subseteq I$, one has that $|\bigcup_{j\in J} S_j| \geq |J|$. *($M^*$)*

Jiang and Nipkow formalized the finite case of Hall's theorem in Isabelle/HOL ([13], [12]). The distinguishing feature of their formalisation was the use of functional indexations of collections of subsets of S instead of a representation of such collections as sequences. Indeed, using such indexation structure, they formalized this theorem applying both the Halmos and Vaughan's and the Rado's approaches (see [11], and [17], respectively). The former proof is nicely presented by Aigner and Ziegler using sequences in [1].

This work discusses a formalisation in Isabelle/HOL of the enumerable version of Hall's Theorem (Theorem 2). The demonstration consists in proving the sufficiency of the marriage condition for the existence of SDR: $M^* \Rightarrow$ SDR. The proof applies the Compactness Theorem for propositional logic, where the marriage condition for finite families is verified by using Jiang and Nipkow's formalisation. As in Jiang and Nipkow's approach, we use functional (infinite) indexations of families of sets. Such indexations representation allows us to apply their formalisation straightforwardly, allowing elegant and simple specifications.

The formalisation approach follows the logical constructive-model lines of reasoning of Cameron's informal proof in [3].

As far as we know, there is only another formalisation of the enumerable version of Hall's theorem in Lean that follows an approach different from the one used in this paper. Instead of applying the compactness theorem as we did, Gusakov, Mehta and Miller [9] formalised the theorem following a combinatorial approach that depends on a formalisation of König's lemma.

The paper is organised as follows. Section 2 presents Cameron's informal proof followed in our formalisation approach. Section 3 briefly describes the formalisation of the compactness theorem (Subsection 3.1) for propositional logic, and the formalisation of the enumerable infinite version of Hall's theorem (Subsection 3.2). Section 4 discusses related work. Finally, Section 5 concludes and proposes future work. The formalisation is available through hyperlinks (↗) in the body of the paper.

2 Cameron's Informal proof

The formalisation approach follows the lines of reasoning of Cameron's informal proof given in [3], page 318.

Assume that the marriage condition (M^*) holds.

Consider the propositional language with constant symbols given by the set below.

$$\mathcal{P} = \{C_{n,x} \mid n \in \ I, x \in S_n\}$$

For each $n \in I$, the constant $C_{n,x}$ is interpreted as "select the element x from the set S_n."

The following three sets of propositional formulas describe the existence of a SDR for $\{S_n\}_{n \in I}$.

1. Select at least an element from each S_n:

   $$\mathcal{F} = \{\vee_{x \in S_n} C_{n,x} \mid n \in I\},$$

   The disjunction $\vee_{x \in S_n} C_{n,x}$ of atomic formulas is well-defined, since each constant corresponds to an element of the set S_n, that by hypothesis if finite.
2. Select at most an element from each S_n:

   $$\mathcal{G} = \{\neg(C_{n,x} \wedge C_{n,y}) \mid x, y \in S_n, x \neq y, n \in I\}.$$

3. Do not select more than once the same element from $\bigcup_{n \in I} S_n$:

   $$\mathcal{H} = \{\neg(C_{n,x} \wedge C_{m,x}) \mid x \in S_n \cap S_m, n \neq m, n, m \in I\}.$$

Let $\mathcal{T} = \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$. We apply the Compactness Theorem to prove that $\mathcal{T}$ is satisfiable.

Let $\mathcal{T}_0$ be any finite subset of formulas in $\mathcal{T}$ and let $J = \{i_1, \ldots, i_m\}$ be the corresponding finite subset of indices in I that are "referred" in $\mathcal{T}_0$, i.e., the set of all indices i such that $C_{i,x}$ for some $x \in S_i$, occurs in some formula of $\mathcal{T}_0$.

Let us consider the family of sets $\{S_{i_1}, \ldots, S_{i_m}\}$. Then, $\mathcal{T}_0$ is contained in the set $\mathcal{T}_1 = \mathcal{F}_0 \cup \mathcal{G}_0 \cup \mathcal{H}_0$, where

1. $\mathcal{F}_0 = \{\vee_{x \in S_n} C_{n,x} \mid n \in J\}$,
2. $\mathcal{G}_0 = \{\neg(C_{n,x} \wedge C_{n,y}) \mid x, y \in S_n, x \neq y, n \in J\}$,
3. $\mathcal{H}_0 = \{\neg(C_{n,x} \wedge C_{m,x}) \mid x \in S_n \cap S_m, n \neq m, n, m \in J\}$.

By hypothesis, $\{S_{i_1}, \ldots, S_{i_m}\}$ satisfies the condition (M^*) and, in particular, the condition (M). Therefore, by the finite version of Hall's Theorem there exists a function $f : J \to \bigcup_{i \in J} S_i$ such that f is a SDR for $\{S_{i_1}, \ldots, S_{i_m}\}$.

Consequently, a model for $\mathcal{T}_1$ is given by the interpretation $v : \mathcal{P} \to \{\mathsf{V}, \mathsf{F}\}$ defined by,

$$v(C_{n,x}) = \begin{cases} \mathsf{V}, & \text{if } n \in J \text{ and } f(n) = x, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Therefore, one has that $v(F) = \mathsf{V}$ for all formulas $F \in \mathcal{T}_1$ since f is a SDR for $\{S_{i_1}, \ldots, S_{i_m}\}$.

Thus, $\mathcal{T}_1$ is satisfiable and so is $\mathcal{T}_0$. In this manner, $\mathcal{T}$ is finitely satisfiable and consequently, by the Compactness Theorem, it is satisfiable.

Let $\mathcal{I} : \mathcal{P} \to \{\mathsf{V}, \mathsf{F}\}$ be a model of $\mathcal{T}$. We define the function $f : I \to \bigcup_{n \in I} S_n$ as

$$f(m) = x \text{ if and only if } \mathcal{I}(C_{m,x}) = \mathsf{V}.$$

Then, f is a SDR for $\{S_n\}_{n \in I}$:

Since $\mathcal{F}$ and $\mathcal{G}$ are satisfiable, for each $m \in I$ there is exactly an element in S_m (because f is a function). Also, since $\mathcal{H}$ is satisfiable, one has that f is an injective function.

3 Formalisation

In this section, we discuss the formalisation of Hall's Theorem. This paper does not focus on the formalisation of the Compactness Theorem, but it is briefly explained for completeness.

The formalisation of the enumerable version of Hall's Theorem consists of less than 6.000 words in ca. 900 lines of code. It includes seven definitions and 46 lemmas and theorems.

Pertinently, we include links (↗) to the specific parts of the formalisation under analysis.

3.1 Notes on the formalisation of the Propositional Compactness Theorem

For completeness, this subsection sketches the formalisation of the Propositional Compactness Theorem, which is used here but is not part of this work. The formalisation was first given in [21] and follows closely Fitting's presentation in [8].

We present the most important definitions and proofs used in the formalisation.

The language of propositional formulas is specified through the following datatype.

Datatype *'b formula* =
  $\bot$
| $\top$
| *atom 'b*
| *negation 'b formula* ($\neg.$(-) [*110*] *110*)
| *conjuntion 'b formula 'b formula* (**infixl** $\wedge.$ *109*)
| *disjunction 'b formula 'b formula* (**infixl** $\vee.$ *108*)
| *implication 'b formula 'b formula* (**infixl** $\rightarrow.$ *100*)

To evaluate the *truth-value* of propositional formulas over an interpretation we specify the operator *t-v-evaluation*.

Primrec *t-v-evaluation* :: (*'b* $\Rightarrow$ *truth-value*) $\Rightarrow$ *'b formula* $\Rightarrow$ *truth-value*
where
  t-v-evaluation I $\bot$ = *Ffalse*
| *t-v-evaluation I* $\top$ = *Ttrue*
| *t-v-evaluation I* (*Atom P*) = *I P*
| *t-v-evaluation I* ($\neg.$ *F*) = (*v-negation* (*t-v-evaluation I F*))
| *t-v-evaluation I* (*F* $\wedge.$ *G*) = (*v-conjunction* (*t-v-evaluation I F*) (*t-v-evaluation I G*))
| *t-v-evaluation I* (*F* $\vee.$ *G*) = (*v-disjunction* (*t-v-evaluation I F*) (*t-v-evaluation I G*))
| *t-v-evaluation I* (*F* $\rightarrow.$ *G*) = (*v-implication* (*t-v-evaluation I F*) (*t-v-evaluation I G*))

The operator *t-v-evaluation* uses the definitions below.

Definition *v-negation* :: *truth-value* $\Rightarrow$ *truth-value* **where**
v-negation x $\equiv$ (*if x* = *Ttrue then Ffalse else Ttrue*)

Definition *v-conjunction* :: *truth-value* $\Rightarrow$ *truth-value* $\Rightarrow$ *truth-value* **where**
v-conjunction x y $\equiv$ (*if x* = *Ffalse then Ffalse else y*)

Definition *v-disjunction* :: *truth-value* $\Rightarrow$ *truth-value* $\Rightarrow$ *truth-value* **where**
v-disjunction x y $\equiv$ (*if x* = *Ttrue then Ttrue else y*)

Definition *v-implication* :: *truth-value* $\Rightarrow$ *truth-value* $\Rightarrow$ *truth-value* **where**
v-implication x y $\equiv$ (*if x* = *Ffalse then Ttrue else y*)

The notion of satisfiability is specified through the existence of *model*s.

Definition *model* :: $('b \Rightarrow$ *truth-value*$) \Rightarrow$ *'b formula set* $\Rightarrow$ *bool* (- *model* - [80,80] 80)
where *I model S* $\equiv (\forall F \in S.$ *t-v-evaluation I F* $=$ *Ttrue*$)$

Definition *satisfiable* :: *'b formula set* $\Rightarrow$ *bool* **where** *satisfiable S* $\equiv (\exists v.$ *v model S*$)$

The notion of compactness is specified using the Isabelle specification for finite sets and a specification for countable sets.

The next lemma, from Isabelle, formalised the fact that a *finite* set A is finite if and only if there exists a surjective function f from I_n onto A, where $I_n = \{m \in \mathbb{N} \mid m < n\}$, for some $n \in \mathbb{N}$.

Lemma *finite A* $\longleftrightarrow (\exists n\, f.\ A = f\ `\ \{i::nat.\ i < n\})$

We specify countable sets using the notion of *enumeration*, i.e., the existence of a surjective function with domain $\mathbb{N}$, given below.

Definition *enumeration* :: $(nat \Rightarrow 'b) \Rightarrow bool$ **where** *enumeration f* $= (\forall y. \exists n.\ y = (f\ n))$

König's lemma is used in classic textbooks to prove the Compactness Theorem. In the formalisation, we follow Fitting's textbook approach in [8] that instead applies the propositional model existence theorem.

Theorem 3 (Propositional model existence (Th. 3.6.2 in [8])). *If $\mathcal{C}$ is a propositional consistency property, and $S \in \mathcal{C}$, then S is satisfiable.*

Theorem 4 (Propositional Compactness (Th. 3.6.3 in [8])). *Let S be a set of propositional formulas. If every finite subset of S is satisfiable, so is S.*

Both these theorems require the definition of propositional consistency. Let $\mathcal{C}$ be a collection of sets of propositional formulas. We call $\mathcal{C}$ a propositional consistency property if it meets the conditions for each $S \in \mathcal{C}$, given in the definition *consistenceP*, as specified below. In this definition *FormulaAlpha* and *FormulaBeta* correspond respectively to conjunctive and disjunctive propositional formulas as defined in [8].

Definition *consistenceP* :: *'b formula set set* $\Rightarrow$ *bool* **where**
consistenceP $\mathcal{C}$ =
$(\forall S.\ S \in \mathcal{C} \longrightarrow (\forall P.\ \neg\ (atom\ P \in S \wedge (\neg.atom\ P\ ) \in S)) \wedge$
$\bot\ \notin S \wedge (\neg.\top) \notin S \wedge$
$(\forall F.\ (\neg.\neg.F) \in S \longrightarrow S \cup \{F\} \in\ \mathcal{C}) \wedge$
$(\forall F.\ ((FormulaAlpha\ F) \wedge F{\in}S) \longrightarrow (\ S \cup \{Comp1\ F,\ Comp2\ F\}) \in \mathcal{C}) \wedge$
$(\forall F.\ ((FormulaBeta\ F) \wedge F{\in}S) \longrightarrow (\ S \cup \{Comp1\ F\} \in \mathcal{C}) \vee$
$(\ S \cup \{Comp2\ F\} \in \mathcal{C})))$

The formalisations of the model existence and the compactness theorems are given below.

Theorem *TheoremExistenceModels*:
assumes *h1*: $\exists g.\ enumeration\ (g{::}\ nat \Rightarrow {}'b\ formula)$
and *h2*: $consistenceP\ \mathcal{C}$
and *h3*: $(S{::}\ {}'b\ formula\ set) \in \mathcal{C}$
shows *satisfiable S*

The following auxiliary lemma is required to apply *TheoremExistenceModels* to obtain the compactness theorem. This lemma states that the collection of sets of propositional formulas given by $\mathcal{C}$ below is a propositional consistency property.

$$\mathcal{C} = \{W \mid \forall A\,(A \subseteq W \wedge A\ finite \rightarrow A\ satisfiable)\}$$

Lemma *ConsistenceCompactness*:
shows $consistenceP\{W{::}{}'b\ formula\ set.\ \forall A.\ (A\subseteq\ W \wedge finite\ A) \longrightarrow\ satisfiable\ A\}$

Finally, the compactness theorem is specified as below.

Theorem *Compactness-Theorem*:
assumes $\exists g.\ enumeration\ (g{::}\ nat \Rightarrow {}'b\ formula)$
and $\forall A.\ (A \subseteq (S{::}\ {}'b\ formula\ set) \wedge finite\ A) \longrightarrow satisfiable\ A$
shows *satisfiable S*

3.2 Formalisation of Hall's Theorem | Enumerable version

As in [13], we represent the collection of enumerable sets $\{S_n\}_{n\in I}$ in Isabelle/HOL as a function $S\ ::\ a \Rightarrow\ b\ set$ together with a set of indices $I\ ::\ a\ set$, where a and b are variable sets, and such that for all $i \in I$, the set $(S\ i)$ is finite. Unlike Jian and Nipkow's formalisation, for the enumerable version of the Hall's theorem, a and b are constrained to be arbitrary enumerable types.
A SDR for S and I is any function $R\ ::\ a \Rightarrow\ b$, which satisfies the predicate below.

Definition *system-representatives* $::\ ({}'a \Rightarrow {}'b\ set) \Rightarrow {}'a\ set \Rightarrow ({}'a \Rightarrow {}'b) \Rightarrow$ *bool* **where**
$system\text{-}representatives\ S\ I\ R\ \equiv (\forall\, i{\in}I.\ (R\ i) \in (S\ i)) \wedge (inj\text{-}on\ R\ I)$

Above, *(inj-on R I)* means that the function R is injective on I.
The marriage condition for S and I is formalized by the proposition,

$$\forall J \subseteq I.\, finite\, J \longrightarrow card\, J \leq card\, \left(\bigcup (S\ `\ J)\right)$$

where $S\ `\ J = \{S\,j \,|\, j \in J\}$.
Using the previous notions, Hall's Theorem is specified as:

Theorem *Hall*:
fixes $S :: 'a \Rightarrow 'b\ set$ **and** $I :: 'a\ set$
assumes $\exists\, g.\ enumeration\ (g{::}\ nat \Rightarrow 'a)$ **and** $\exists\, h.\ enumeration\ (h{::}\ nat \Rightarrow 'b)$

and *Finite*: $\forall\, i{\in}I.\ finite\ (S\ i)$
and *Marriage*: $\forall\, J{\subseteq}I.\ finite\ J \longrightarrow\ card\ J \leq card\ (\bigcup\ (S\ `\ J))$
shows $\exists\, R.\ system\text{-}representatives\ S\ I\ R$

The following four definitions in Isabelle correspond to the formalisation of the sets $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\mathcal{T}$ used in the informal proof. The definition of $\mathcal{F}$ uses *disjunction-atomic* to build the disjunction associated with each finite set in the collection.

Primrec *disjunction-atomic* :: $'b\ list \Rightarrow 'a \Rightarrow ('a \times 'b) formula$ **where**
$disjunction\text{-}atomic\ []\ i = \bot$
$|\ disjunction\text{-}atomic\ (x\#D)\ i = (atom\ (i,\ x))\ \lor.\ (disjunction\text{-}atomic\ D\ i)$

Definition $\mathcal{F}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow (('a \times 'b) formula)\ set$ **where**
$\mathcal{F}\ S\ I\ \equiv (\bigcup i{\in}I.\ \{\ disjunction\text{-}atomic\ (set\text{-}to\text{-}list\ (S\ i))\ i\ \})$

Definition $\mathcal{G}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \times 'b) formula\ set$ **where**
$\mathcal{G}\ S\ I \equiv \{\neg.(atom\ (i,x)\ \land.\ atom(i,y))$
$|x\ y\ i\ .\ x{\in}(S\ i) \land y{\in}(S\ i) \land\ x{\neq}y \land i{\in}I\}$

Definition $\mathcal{H}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \times 'b) formula\ set$ **where**
$\mathcal{H}\ S\ I \equiv \{\neg.(atom\ (i,x)\ \land.\ atom(j,x))$
$|\ x\ i\ j.\ x \in (S\ i) \cap (S\ j) \land (i{\in}I \land j{\in}I \land i{\neq}j)\}$

Definition $\mathcal{T}$ :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \times 'b) formula\ set$ **where**
$\mathcal{T}\ S\ I\ \equiv (\mathcal{F}\ S\ I) \cup (\mathcal{G}\ S\ I) \cup (\mathcal{H}\ S\ I)$

The above definitions illustrate the benefit of using sets of indices in our specification. The set of indices occurring in a set of formulas (*indices-set-formulas*) is the union of set of indices occurring in each formula (*indices-formula*) that are defined recursively.

If the associated collection of finite subsets with indices in *To* ($\mathcal{T}_0$, in the informal proofs), (*indices-set-formulas To*) satisfies the marriage condition, then there is a SDR. The proof uses Jiang and Nipkow's finite version of Hall's Theorem given in [13]. Indeed, the proof can apply either Halmos and Vaughan's or Rado's formalisations in [13] without any modification. This is possible since our specification of predicates, as SDR, are independent of any definition in Jiang and Nipkow's formalisation.

Lemma *system-distinct-representatives-finite*:
assumes
$\forall i{\in}I.\ (S\ i){\neq}\{\}$ **and** $\forall i{\in}I.$ *finite* $(S\ i)$ **and** $To \subseteq (\mathcal{T}\ S\ I)$ **and** *finite To*
and $\forall J{\subseteq}(\textit{indices-set-formulas}\ To).\ \textit{card}\ J \le \textit{card}\ (\bigcup\ (S\ `\ J))$
shows $\exists R.$ *system-representatives* S (*indices-set-formulas To*) R

The following lemma states that if there exists a SDR R for a collection of finite sets given by A and $\mathcal{I}$, then any subset of formulas $X \subseteq (\mathcal{T}\ A\ \mathcal{I})$ is satisfiable. A model for X is given by the next interpretation of formulas.

Fun *Hall-interpretation* :: $('a \Rightarrow 'b\ set) \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow (('a \times 'b) \Rightarrow \textit{truth-value})$ **where**
Hall-interpretation $A\ I\ R = (\lambda(i,x).(\textit{if}\ \ i \in I \wedge x \in (A\ i) \wedge (R\ i) = x\ \ \textit{then}\ \textit{Ttrue}\ \textit{else}\ \textit{Ffalse}))$

Lemma *SDR-satisfiable*:
assumes $\forall i{\in}I.\ (A\ i) \neq \{\}$ **and** $\forall i{\in}I.$ *finite* $(A\ i)$ **and** $X \subseteq (\mathcal{T}\ A\ I)$
and *system-representatives* $A\ I\ R$
shows *satisfiable* X

Lemma *SDR-satisfiable* above is the kernel of the formalisation. It proves that the set of formulas $(\mathcal{T}\ A\ I)$ built from A and I is satisfiable building and evaluating the model given by the function *Hall-interpretation*.

Previous results allow us to prove the following lemma. It states that any finite subset of formulas $To \subseteq (\mathcal{T}\ S\ I)$, such that the collection of finite sets of formulas with indices used by the formulas in *To* hold the marriage condition, is satisfiable.

Lemma *finite-is-satisfiable*:
assumes
$\forall i{\in}I.\ (S\ i){\neq}\{\}$ **and** $\forall i{\in}I.$ *finite* $(S\ i)$ **and** $To \subseteq (\mathcal{T}\ S\ I)$ **and** *finite To*
and $\forall J{\subseteq}(\textit{indices-set-formulas}\ To).\ \textit{card}\ J \le \textit{card}\ (\bigcup\ (S\ `\ J))$
shows *satisfiable To*

The lemma *finite-is-satisfiable* and the Compactness Theorem are then used to prove that the set of formulas $(\mathcal{T}\ S\ I)$ is satisfiable.

Lemma *all-formulas-satisfiable*:

fixes S :: $'a \Rightarrow\ 'b$ *set* **and** I :: $'a$ *set*
assumes $\exists\, g.$ *enumeration* $(g$:: *nat* $\Rightarrow' a)$ **and** $\exists\, h.$ *enumeration* $(h$:: *nat* $\Rightarrow' b)$

and $\forall\, i{\in}I.$ *finite* $(S\ i)$
and $\forall\, J{\subseteq}I.$ *finite* $J \longrightarrow$ *card* $J \leq$ *card* $(\bigcup\ (S\ `\ J))$
shows *satisfiable* $(\mathcal{T}\ S\ I)$

The lemma below, *satisfiable-representant*, states that if $(\mathcal{T}\,S\,I)$ is satisfiable then the corresponding (enumerable) collection of finite sets $\{S_i\}_{i\in I}$, given by S and I, has a SDR. For its proof we use the function *SDR* and lemma *function-SDR*.

Fun *SDR* :: $(('a \times\ 'b) \Rightarrow$ *truth-value*$) \Rightarrow ('a \Rightarrow\ 'b$ *set*$) \Rightarrow\ 'a$ *set* $\Rightarrow ('a \Rightarrow' b$
)
where
SDR M S I = $(\lambda i.$ (*THE* $x.$ (*value M* (*atom* $(i{,}x)) =$ *Ttrue*$) \wedge x{\in}(S\ i)))$

Soundness of the function *SDR* is proved by the lemma *function-SDR* below.

Lemma *function-SDR* :
assumes $i \in I$ **and** M *model* $(\mathcal{F}\ S\ I)$ **and** M *model* $(\mathcal{G}\ S\ I)$ **and** *finite*$(S\ i)$
shows $\exists!x.$ (*value M* (*atom* $(i{,}x)) =$ *Ttrue*$) \wedge\ x \in (S\ i) \wedge$ *SDR M S I i* $= x$

Lemma *satisfiable-representant* :
assumes *satisfiable* $(\mathcal{T}\ S\ I)$ **and** $\forall\, i{\in}I.$ *finite* $(S\ i)$
shows $\exists\, R.$ *system-representatives S I R*
Finally, we obtain the formalisation of Hall's Theorem.

Theorem *Hall* :
fixes S :: $'a \Rightarrow\ 'b$ *set* **and** I :: $'a$ *set*
assumes $\exists\, g.$ *enumeration* $(g$:: *nat* $\Rightarrow' a)$ **and** $\exists\, h.$ *enumeration* $(h$:: *nat* $\Rightarrow' b)$

and *Finite*: $\forall\, i{\in}I.$ *finite* $(S\ i)$
and *Marriage*: $\forall\, J{\subseteq}I.$ *finite* $J \longrightarrow$ *card* $J \leq$ *card* $(\bigcup\ (S\ `\ J))$
shows $\exists\, R.$ *system-representatives S I R*
proof−
have *satisfiable* $(\mathcal{T}\ S\ I)$ **using** *assms all-formulas-satisfiable*[*of I*] **by** *auto*
thus *?thesis* **using** *Finite Marriage satisfiable-representant*[*of S I*] **by** *auto*
qed

4 Related Work

There are two preliminary Isabelle formalisations over which this one is developed. The first is the formalisation of the finite case of Hall's Theorem by Jiang and Nipkow [16] and the second is the formal verification of the Compactness Theorem given by Serrano in [21]. Also, there exists another formalisation in Isabelle of the Compactness Theorem for propositional logic developed by Michaelis and Nipkow [14]. Nevertheless, we prefer to use one above mentioned developed by Serrano.

Regarding the finite case of Hall's Theorem, the first formalisation of such theorem was developed by Romanowicz and Grabowski [19] in Mizar following Rado's analytical proof [17]. Also, there is a formalisation in Coq of the finite version of Hall's theorem that uses formalisations of combinatorial arguments as Dilworth's decomposition theorem and existence of bi-partitions in graphs [22]. Indeed, there are earlier combinatorial formalisations of Dilworth's theorem in Mizar as the one presented in [20]. This theorem states that in a finite partially ordered set, the size of minimal chains and maximal anti-chains are the same. Recently, Gusakov, Mehta and Miller [9] presented three different proofs of the finite version of Hall's theorem formalised in Lean in terms of indexed families of finite subsets, of existence of *matchings* (injections) that saturate binary relations over finite sets, and of matchings in bipartite graphs.

There are a myriad of formalisations related to Hall's theorem, which are based on combinatorial approaches and not on the compactness approach followed in this paper. Among them, we could mention recent works by Doczkal et al. in their graph theory Coq library (e.g., [5], [7], and [6]). Finally, Singh and Natarajan formalized in Coq other combinatorial resuls as the perfect graph theorem and a weak version of this theorem (e.g., [23], [24]).

As far as we know, the unique formalisation of the enumerable version of Hall's theorem is the one by Gusakov, Mehta and Miller cited in the introduction [9]. As above mentioned, the authors formalised three versions of the finite case of Hall's theorem in Lean. Also, they apply an *inverse limit* version of the König's lemma to conclude the enumerable case as specified in this paper. The inverse limit version of the König's lemma states that if $\{X_i\}, i \in \mathbb{N}$ is an indexed family of sets with functions $f_i : X_{i+1} \to X_i$, for each i, then if each X_i is a nonempty finite set, then there exists a family of elements $x \in \prod_i X_i$ such that $x_i = f_i(x_{i+1})$, for all i. The usual version of the König's lemma follows form this one, by choosing as set X_i the paths of length i from the root vertex v_0 in a tree. So, the function f_i maps paths in X_{i+1} into the paths without their last edge in X_i. The inverse limit consist of the infinite chain of functions $f_1, f_2, \ldots$. König's lemma is applied to prove the enumerable version of Hall's theorem by taking M_n as the set of all *matchings* on the first n indices of I (i.e., the set of all possible SDRs for the sets $S_1, \ldots, S_n$), and $f_n : M_{n+1} \to M_n$ as the restriction of a matching to a smaller index set. Since the marriage condition holds for the finite indexed families, each M_n is nonempty and by König's lemma an element of the inverse limit gives a matching on I. Differently for our formalisation, Gusakov, Mehta and Miller proof does not follow a constructive approach as the one given in our development in which a model is built to guarantee the hypotheses of the compactness theorem for propositional logic.

5 Conclusion

This paper presented a formalisation of Hall's theorem for infinite enumerable collections of finite sets in Isabelle. The proof uses a formalisation of the compactness theorem for propositional logic and, in addition, Jiang and Nipkow's formalisation of Hall's theorem for considering the case of finite collections of finite sets.

The distinctive characteristics of our formalisation are:

- it inherits the advantages of the representation of collections of sets through set-indexations from Jiang and Nipkow's formalisation of the finite version of Hall's theorem [13];
- it profits from the Isabelle/HOL deductive features to follow a line of reasoning that remains close to the analytical proofs, and;
- in contrast with combinatorial proofs, it follows the logical constructive-model approach that applies the compactness theorem.

Interesting applications include the formalisation of the extension of Hall's theorem to non-enumerable families of finite sets, and the formalisation of other related combinatorial theorems (applying Hall's theorem) as those mentioned in the introduction.

Other applications of the compactness theorem, which are not discussed in this paper, were formalised similarly and are also available in the distribution. For instance, the De Bruijn-Erdös's graph colouring theorem ([4]), and König's lemma (cf., exercise in Chapter I.6 in [15]). These formalisations follow the logical constructive-model approach described in this paper. Of course, a variety of consequences of the propositional compactness theorem would also be welcome as those presented in textbooks (e.g., [3], [8], and [15]).

References

1. M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, sixth edition, 2018. Chapter 30: Three famous theorems on finite sets, https://doi.org/10.1007/978-3-662-57265-8.
2. R. D. Borgersen. Equivalence of seven major theorems in combinatorics, 2004. Talk available at Department of Mathematics, University of Manitoba. https://home.cc.umanitoba.ca/~borgerse/Presentations/GS-05R-1.pdf.
3. P. J. Cameron. *Combinatorics: Topics, Techniques, Algorithms.* Cambridge University Press, 1994.
4. N. G. De Bruijn and P. Erdös. A Colour Problem for Infinite Graphs and a Problem in the Theory of Relations. *Indagationes Mathematicae (Proceedings)*, 54:371–373, 1951. https://doi.org/10.1016/S1385-7258(51)50053-7.

5. C. Doczkal, G. Combette, and D. Pous. A Formal Proof of the Minor-Exclusion Property for Treewidth-Two Graphs. In *Proceedings 9th International Conference on Interactive Theorem Proving ITP*, volume 10895 of *Lecture Notes in Computer Science*, pages 178–195. Springer, 2018. https://doi.org/10.1007/978-3-319-94821-8_11.
6. C. Doczkal and D. Pous. Completeness of an axiomatization of graph isomorphism via graph rewriting in Coq. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs CPP*, pages 325–337. ACM, 2020. https://doi.org/10.1145/3372885.3373831.
7. C. Doczkal and D. Pous. Graph Theory in Coq: Minors, Treewidth, and Isomorphisms. *J. Autom. Reason.*, 64(5):795–825, 2020. https://doi.org/10.1007/s10817-020-09543-2.
8. M. Fitting. *First-Order Logic and Automated Theorem Proving.* Springer-Verlag, second edition, 1996. https://doi.org/10.1007/978-1-4612-2360-3.
9. A. Gusakov, B. Mehta, and K. A. Miller. Formalizing Hall's Marriage Theorem in Lean. 2021. https://doi.org/10.48550/arXiv.2101.00127.
10. P. Hall. On representatives of subsets. *London Mathematical Society*, 10:26–30, 1935. https://doi.org/10.1112/jlms/s1-10.37.26.
11. P. R. Halmos and H. E. Vaughan. The Marriage Problem. *American Journal of Mathematics*, 72(1):214–215, 1950. https://doi.org/10.2307/2372148.
12. D. Jiang and T. Nipkow. Hall's Marriage Theorem. *Archive of Formal Proofs*, Dec. 2010. https://www.isa-afp.org/browser_info/current/AFP/Marriage/document.pdf.
13. D. Jiang and T. Nipkow. Proof Pearl: The Marriage Theorem. In *Proceedings First International Conference on Certified Programs and Proofs CPP*, volume 7086 of *Lecture Notes in Computer Science*, pages 394–399, 2011. https://doi.org/10.1007/978-3-642-25379-9_28.
14. J. Michaelis and T. Nipkow. Formalized Proof Systems for Propositional Logic. In *Proceedings 23rd International Conference on Types for Proofs and Programs TYPES 2017*, volume 104 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:16, 2018. http://10.4230/LIPIcs.TYPES.2017.5.
15. A. Nerode and Shore. *Logic for Applications.* Graduate Text in Computer Science. Springer, second edition, 2012. https://doi.org/10.1007/978-1-4612-0649-1.
16. T. Nipkow. Linear quantifier elimination. *Journal of Automated Reasoning*, 45:189–212, 2010. https://doi.org/10.1007/s10817-010-9183-0.
17. R. Rado. Note on the transfinite case of Hall's theorem on representatives. *London Mathematical Society*, S1-42(1):321–324, 1967. https://doi.org/10.1112/jlms/s1-42.1.321.
18. P. F. Reichmeider. *The equivalence of some combinatorial matching theorems.* Polygonal Publishing House, 1985.
19. E. Romanowicz and A. Grabowski. The Hall Marriage Theorem. *Formalized Mathematics*, 12(3):315–320, 2004. https://fm.mizar.org/2004-12/pdf12-3/hallmar1.pdf.
20. P. Rudnicki. Dilworth's Decomposition Theorem for Posets. *Formalized Mathematics*, 17(4):223–232, 2009. https://doi.org/10.2478/v10037-009-0028-4.
21. F. F. Serrano. *Formalización en Isar de la Meta-Lógica de Primer Orden.* PhD thesis, Universidad de Sevilla, Spain, 2012. https://idus.us.es/handle/11441/57780.
22. A. K. Singh. Formalization of some central theorems in combinatorics of finite sets, 2017. Short presentation at LPAR-21. https://arxiv.org/abs/1703.10977.

23. A. K. Singh and R. Natarajan. Towards a Constructive Formalization of Perfect Graph Theorems. In *Proceedings 8th Indian Conference on Logic and Its Applications ICLA*, volume 11600 of *Lecture Notes in Computer Science*, pages 183–194. Springer, 2019. https://doi.org/10.1007/978-3-662-58771-3_17.
24. A. K. Singh and R. Natarajan. A constructive formalization of the weak perfect graph theorem. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs CPP*, pages 313–324. ACM, 2020. https://doi.org/10.1145/3372885.3373819.