Exercises on Real Number Proving

These exercises are intended to illustrate basic techniques to prove formulas involving real numbers. The exercises in this section refer to the theory `realproving.pvs`.

1. A googol is the number 10^{100} and a googolplex is the number 10^{googol}.

   Problem: Prove that a googol plex is strictly greater than the multiplication of two googols. Lemma `googolplex_gt_googol2` specifies this statement in PVS.

   Hint: The following lemmas are defined in the PVS prelude.

   ```
   expt_plus: LEMMA n0x^(i + j) = n0x^i * n0x^j

   both_sides_expt_gt1_gt: LEMMA gt1x ^ i > gt1x ^ j IFF i > j
   ```

   Use both lemmas to reduce the problem to proving that $10^{100} > 200$. At this point, prove that $100^{100} > 10^3$, using lemma `both_sides_expt_gt1_gt`, and that $10^3 > 200$, using `(eval-formula)`.

2. **Problem:** Prove that for $x, y \in \mathbb{R}$, $y(1-x)(1-x) \leq 0$ if $y \leq 0$.

   Hint: Use `grind-reals`, but before that you may have to use `name-replace`.

3. **Problem:** Prove that for all $x \in \mathbb{R}$, $x(1-x) \leq \frac{1}{4}$.

   Hint: Notice that $x\,(1-x) \leq \frac{1}{4}$ is a single variable polynomial.

4. **Problem:** Prove that for all $x \in \mathbb{R}$, $x^2 \leq 1$ if and only if $-1 \leq x \leq 1$.

   Hint: Notice that the above statement only involves polynomial expressions on one variable. Flatten and split the sequent before using a decision procedure.

5. The polynomial $r(x) = x - \frac{11184811}{33554432}x^3 - \frac{13421773}{67108864}x^5$ is a quick approximation of the arc tangent function, i.e., atan, for $|x| \leq \frac{1}{3}$.

   Problem: Using interval arithmetic, prove that $r(x)$ provides at least two decimals of precision of atan for $|x| \leq \frac{1}{3}$.

6. **Homework:** Using exact real arithmetic, prove that the 10-th decimal of π is 5.