Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$

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Abstract: In this paper, we find all Fibonacci and Lucas numbers written in the form $2^a + 3^b + 5^c$, in nonnegative integers a, b, c, with $\max\{a, b\} \le c$.

Key words: Fibonacci; Lucas; linear forms in logarithms; reduction method.

1. Introduction Let $(F_n)_{n\ge 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \ge 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [5] together with its very extensive annotated bibliography for additional references and history). We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n\ge 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

The problem of finding for Fibonacci and Lucas numbers of a particular form has a very rich history. Maybe the most outstanding result on this subject is due to Bugeaud, Mignotte and Siksek [1, Theorem 1] who showed that $0, 1, 8, 144$ and $1, 4$ are the only Fibonacci and Lucas numbers, respectively, of the form y^t, with $t > 1$ (perfect power). Other related papers searched for Fibonacci numbers of the forms $px^2 + 1, px^3 + 1$ [13], $k^2 + k + 2$ [7], $p^a \pm p^b + 1$ [8], $p^a \pm p^b$ [9], $y^t \pm 1$ [2] and $q^k y^t$ [3]. Also, in 1993, Pethő and Tichy proved that there are only finitely many Fibonacci numbers of the form $p^a + p^b + p^c$, with p prime. However, their proof uses the finiteness of solutions of S-unit equations, and as such is ineffective. Very recently, the authors [10] found all Fibonacci and Lucas numbers of the form $y^a + y^b + y^c$, with $2 \le y \le 9$.

In this paper, we are interested in Fibonacci and Lucas numbers which are sum of three perfect powers of some prescribed distinct bases. More precisely, our results are the following

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Theorem 1.1. *The only solutions of the Diophantine equation*

$$F_n = 2^a + 3^b + 5^c \tag{1}$$

in integers n, a, b, c, with $0 \le \max\{a, b\} \le c$ are

$$(n, a, b, c) \in \{(4, 0, 0, 0),\ (6, 1, 0, 1)\}.$$

Theorem 1.2. *The only solutions of the Diophantine equation*

$$L_n = 2^a + 3^b + 5^c \tag{2}$$

in integers n, a, b, c, with $0 \le \max\{a, b\} \le c$ are

$$(n, a, b, c) \in \{(2, 0, 0, 0),\ (4, 0, 0, 1),\ (7, 0, 1, 2)\}.$$

2. Auxiliary results First, we recall the well-known Binet's formulae for Fibonacci and Lucas sequences:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$. These formulas allow to deduce the bounds

$$\alpha^{n-2} \le F_n \le \alpha^{n-1} \text{ and } \alpha^{n-1} \le L_n \le 2\alpha^n,$$

which hold for all $n \ge 1$.

The next tools are related to the transcendental approach to solve Diophantine equations. First, we use a lower bound for a linear form logarithms *à la Baker* and such a bound was given by the following result of Matveev [11].

Lemma 1. *Let $\gamma_1, \dots, \gamma_s$ be real algebraic numbers and let $b_1, \dots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \dots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \ge \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\} \text{ for } j = 1, \dots, s.$$

Assume that

$$B \ge \max\{|b_1|, \dots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \ne 1$, then

$|\gamma_1^{b_1}\cdots\gamma_s^{b_s}-1|\geq \exp(-C(s,D)(1+\log B)A_1\cdots A_s),$

where $C(s,D):=1.4\cdot 30^{s+3}\cdot s^{4.5}\cdot D^2(1+\log D)$.

As usual, in the above statement, the *logarithmic height* of an s-degree algebraic number γ is defined as

$$h(\gamma)=\frac{1}{\ell}(\log|a|+\sum_{j=1}^{\ell}\log\max\{1,|\gamma^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1\leq j\leq \ell}$ are the conjugates of γ (over $\mathbb{Q}$).

After finding an upper bound on n which is general too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethő [4]. For a real number x, we use $\| x \| = \min\{|x-n| : n\in\mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q>6M$ and let $\epsilon = \|\mu q\| - M\|\gamma q\|$, where μ is a real number. If $\epsilon>0$, then there is no solution to the inequality*

$$0<m\gamma-n+\mu<AB^{-m}$$

in positive integers m,n with

$$\frac{\log(Aq/\epsilon)}{\log B}\leq m<M.$$

See Lemma 5, a.) in [4]. Now, we are ready to deal with the proofs of our results.

3. Proof of the Theorem 1.1 By combining Binet formula together with (2), we get

$$\frac{\alpha^n}{\sqrt{5}}-5^c=2^a+3^b+\frac{\beta^n}{\sqrt{5}}>0, \tag{3}$$

because $|\beta|<1$ while $2^a\geq 1$. Thus

$$\frac{\alpha^n 5^{-c}}{\sqrt{5}}-1=\frac{2^a}{5^c}+\frac{3^b}{5^c}+\frac{\beta^n}{5^c\sqrt{5}}$$

yields

$$\left|\frac{\alpha^n 5^{-c}}{\sqrt{5}}-1\right|<\frac{3}{5^{0.3c}},$$

where we used that $2<\sqrt{5}, 3<5^{0.7}$ and $c\geq\max\{a,b\}$. Therefore,

$$|e^{\Lambda_F}-1|<\frac{3}{5^{0.3c}}, \tag{4}$$

where $\Lambda_F=n\log\alpha-c\log 5+\log(1/\sqrt{5})$. By (3), $\Lambda_F>0$ and in particular $e^{\Lambda_F}\neq 1$. In order to apply Lemma 1, we take $s:=3$,

$$\gamma_1:=\alpha,\ \gamma_2:=5,\ \gamma_3:=1/\sqrt{5}$$

and

$$b_1:=n,\ b_2:=-c,\ b_3:=1.$$

For this choice we have $D=2$, $h(\gamma_1)=(\log\alpha)/2<0.25$, $h(\gamma_2)=\log 5<1.61$ and $h(\gamma_3)=\log\sqrt{5}<0.81$. In conclusion, $A_1:=0.5$, $A_2:=3.22$ and $A_3:=1.62$ are suitable choices. We also obtain the estimate

$$\alpha^{n-2}<F_n=2^a+3^b+5^c<2\cdot 5^c.$$

which yields $n<3.4c+3.5$ (here we used that $2^a+3^b\leq 2^c+3^c<5^c$). Thus we can choose $B:=3.4c+3.5>\max\{n,c\}$. By Lemma 1,

$$|e^{\Lambda_F}-1|>\exp(-3.5\cdot 10^9(1+\log(3.4c+3.5))). \tag{5}$$

We now combine (4) and (5) to get

$$c<7.3\cdot 10^9(1+\log(3.4c+3.5))$$

and so $c<3\cdot 10^{11}$ and $n<1.1\cdot 10^{12}$.

Also, $0<\Lambda_F<e^{\Lambda_F}-1<3/5^{0.3c}$ and this can be written as

$$0<n\log\alpha-c\log 5+\log(1/\sqrt{5})<3\cdot(1.6)^{-c}.$$

Since $c>(n-3.5)/3.4>0.3n-1.1$, we obtain (dividing by $\log 5$)

$$0<n\gamma-c+\mu<3\cdot(1.1)^{-n}, \tag{6}$$

with $\gamma:=\log\alpha/\log 5$ and $\mu:=\log(1/\sqrt{5})/\log 5=-1/2$.

We claim that γ is irrational. In fact, if $\gamma=p/q$, then $\alpha^{2q}\in\mathbb{Q}$, which is an absurdity. Let q_n be the denominator of the n-th convergent of the continued fraction of γ. Taking $M:=1.1\cdot 10^{12}$, we have

$$q_{29}=971159673756047>6M$$

and then $\epsilon:=\|\mu q_{29}\|-M\|\gamma q_{29}\|=0.4999\ldots$. Note that the conditions to apply Lemma 2 are fulfilled for $A=3$ and $B=1.1$, and hence there is no solution to inequality (6) (and then no solution to the Diophantine equation (2)) for n in the range

$$\left[\left\lfloor\frac{\log(Aq_{29}/\epsilon)}{\log B}\right\rfloor+1,M\right)=[381,1.1\cdot 10^{12}).$$

Thus $n\leq 380$ and the estimate $5^c<F_n\leq F_{380}$ yields $c\leq 180$.

Note that $\nu_5(F_n-2^a-3^b)=c$. In order to get an upper bound for this 5-adic valuation, we need to exclude the trivial cases when $F_n - 2^a - 3^b = 0$ (e.g. $(n, a, b) = (5, 1, 1)$), because clearly they don't give any solution. Thus, `Mathematica` returns $\nu_5(F_n - 2^a - 3^b) \leq 10$, for $n \geq 380$, $0 \leq \max\{a, b\} \leq 180$. Therefore $c \leq 10$ and then $n \leq 37$.

Finally, we use `Mathematica` to find the solutions of Eq. (1) in the range $0 \leq \max\{a, b\} \leq c \leq 10$ and $n \leq 37$. Fastly, the program returns us

$$(n, a, b, c) \in \{(4, 0, 0, 0), (6, 1, 0, 1)\}.$$

This completes the proof. □

4. Proof of the Theorem 1.2 By combining Binet formula together with (2), we get

$$\alpha^n - 5^c = 2^a + 3^b - \beta^n > 0$$

and similarly as in the proof of previous theorem, we obtain

$$|e^{\Lambda_L} - 1| < \frac{3}{5^{0.3c}},$$

where $\Lambda_L := n \log \alpha - c \log 5$. The estimates $\Lambda_L > 0$ and $\Lambda_L < e^{\Lambda_L} - 1$ lead to

$$\log |\Lambda_L| < \log 3 - 0.48c. \tag{7}$$

Now, we will determine a lower bound for Λ_L. We remark that the bounds available for linear forms in two logarithms are substantially better than those available for linear forms in three logarithms. Here we choose to use a result due to Laurent [6, Corollary 2] with $m = 24$ and $C_2 = 18.8$. First let us introduce some notations. Let α_1, α_2 be real algebraic numbers, with $|\alpha_j| \geq 1$, b_1, b_2 be positive integer numbers and

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let A_j be real numbers such that

$$\log A_j \geq \max\{h(\alpha_j), |\log \alpha_j|/D, 1/D\}, j \in \{1, 2\},$$

where D is the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$ over $\mathbb{Q}$. Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

Laurent's result asserts that if α_1, α_2 are multiplicatively independent, then

$$\log |\Lambda| \geq -18.8 \cdot D^4 \left(\max\{\log b' + 0.38, m/D, 1\}\right)^2 \cdot \log A_1 \log A_2.$$

We then take

$$D = 2,\ b_1 = c,\ b_2 = n,\ \alpha_1 = 5,\ \alpha_2 = \alpha.$$

We choose $\log A_1 = 1.61$ and $\log A_2 = 0.25$. So we get

$$b' = \frac{c}{0.5} + \frac{n}{3.22} < 3.1c + 0.8,$$

where we used $n < 3.4c + 2.5$, which is obtained from $\alpha^{n-1} < L_n < 2 \cdot 5^c$.

As α and y are multiplicatively independent, by Corollary 2 of [6] we get

(8)
$$\log |\Lambda_L| \geq -121 \cdot \left(\max\{\log(3.1c + 0.8) + 0.38, 11\}\right)^2.$$

Now, we combine the estimates (7) and (8) to obtain

(9)
$$c < 252.1 \cdot \left(\max\{\log(3.1c + 0.8) + 0.38, 11\}\right)^2 + 2.3.$$

Therefore inequality (9) gives $c \leq 36382$ and so $n \leq 123704$.

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