On the sum of powers of terms of a linear recurrence sequence

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Abstract

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. There are several interesting identities involving this sequence such as $F_n^2 + F_{n+1}^2 = F_{2n+1}$, for all $n \geq 0$. In a very recent paper, Marques and Togbé proved that if $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n, then $s = 1$ or 2. In this paper, we will prove, in particular, that if $(G_m)_m$ is a linear recurrence sequence (under weak assumptions) and $G_n^s + \cdots + G_{n+k}^s \in (G_m)_m$, for infinitely many positive integers n, then s is bounded by an effectively computable constant depending only on k and the parameters of G_m.

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1. Introduction

A sequence $(G_n)_{n\geq 0}$ is *a linear recurrence sequence* with coefficients c_0, c_1,...,c_{k-1}, with $c_0 \neq 0$, if

$$G_{n+k} = c_{k-1}G_{n+k-1} + \cdots + c_1 G_{n+1} + c_0 G_n, \tag{1}$$

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for all positive integer n. A recurrence sequence is therefore completely determined by the *initial values* $G_0, ..., G_{k-1}$, and by the coefficients $c_0, c_1, ..., c_{k-1}$. The integer k is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(G_n)_{n\geq 0}$ is given by

$$G(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0.$$

It is well-known that for all n

$$G_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n, \tag{2}$$

where r_j is a root of $G(x)$ and $g_j(x)$ is a polynomial over a certain number field, for $j = 1, ..., \ell$. In this paper, we consider only integer recurrence sequences, i.e., recurrence sequences whose coefficients and initial values are integers. Hence, $g_j(n)$ is an algebraic number, for all $j = 1, ..., \ell$, and $n \in \mathbb{Z}$.

A general Lucas sequence $(C_n)_{n\geq 0}$ given by $C_{n+2} = aC_{n+1} + bC_n$, for $n \geq 0$, where the values a, b, C_0 and C_1 are previously fixed, is an example of a linear recurrence of order 2 (also called *binary*). For instance, if $C_0 = 0$ and $C_1 = a = b = 1$, then $(C_n)_{n\geq 0} = (F_n)_{n\geq 0}$ is the well-known *Fibonacci sequence*:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

The Fibonacci numbers are known for their amazing properties (see [8, pp. 53-56] and [1]). For example, we have

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \text{ for all } n \geq 0. \tag{3}$$

Recently, Melham [5, 6, 7] have published a series of papers presenting identities that (according to him) can be regarded as higher order analogues of the above equality. Note that, in particular, the naive identity (3) (which can be easily proved by mathematical induction) tells us that the sum of the squares of two consecutive Fibonacci numbers is still a Fibonacci number. In a very recent paper, Marques and Togbé [3] searched for similar identities in higher powers, see [2] for a generalization. They proved that if $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n, then $s = 1$ or 2. Several related problems arise, such as:

- What happens if the Fibonacci sequence is replaced by another linear recurrence sequence (e.g., Lucas or Tribonacci sequences)?

- What is about the sum of many powers of Fibonacci numbers?

The aim of this paper is to work on these kind of problems. More precisely, our main result is the following.

Theorem 1. *Let $(G_n)_n$ be an integer linear recurrence sequence such that its characteristic polynomial has a simple positive root being the unique zero outside the unit circle. Let $s, k,$ and b be positive integer numbers and $\epsilon_j \in \{0,1\}$, with $1 \leq j \leq k-1$. Then, there exists an effectively computable constant C such that if*

$$G_n^s + \epsilon_1 G_{n+1}^s + \cdots + \epsilon_{k-1} G_{n+k-1}^s + G_{n+k}^s$$

belongs to the sequence $(b \cdot G_n)_n$, for infinitely many positive integers n, then $s < C$. The constant C depends only on k, b and the parameters of G_n.

Of course, there are also identities involving distinct powers of Fibonacci numbers such as

$$5F_{2n+2}^3 + 3F_{2n+1} + 3F_{2n} = F_{6(n+1)}, \text{ for all } n \geq 1.$$

As an application of our method, we will prove that there is not a similar identity under some weak hypotheses. More precisely, we have

Theorem 2. *Let $\ell, s_1, ..., s_\ell, a_1, ..., a_\ell$ be integers with $\ell > 1$ and $s_j \geq 1$. Suppose that there exists $1 \leq t \leq \ell$ such that $a_t \neq 0$ and $s_t > s_j$, for all $j \neq t$. If either s_t is even or a_t is not a positive power of 5, then the sum*

$$a_1 F_{n+1}^{s_1} + a_2 F_{n+2}^{s_2} + \cdots + a_\ell F_{n+\ell}^{s_\ell} \tag{4}$$

does not belong to the Fibonacci sequence for all sufficiently large n.

We organize this paper as follows. In Section 2, we will recall some useful properties such as a result of Matveev on linear forms in three logarithms, that we will use to prove Theorem 1. The third section is devoted to the proof of Theorem 1. In the last section, we combine our method with the useful fact that every non-zero integer power of $\alpha = (1+\sqrt{5})/2$ is irrational in order to prove Theorem 2.

2. Auxiliary results

In this section, we recall some results that will be very useful for the proof of the above theorems. Let $G(x)$ be the characteristic polynomial of a linear recurrence G_n. One can factor $G(x)$ over the set of complex numbers as

$$G(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_\ell)^{m_\ell},$$

where $r_1, ..., r_\ell$ are distinct non-zero complex numbers (called the *roots* of the recurrence) and $m_1, ..., m_\ell$ are positive integers. A root r_j of the recurrence is called a *dominant root* if $|r_j| > |r_i|$, for all $j \neq i \in \{1, ..., \ell\}$. The corresponding polynomial $g_j(n)$ is named the *dominant polynomial* of the recurrence. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined non-zero polynomials $g_1, ..., g_\ell \in \mathbb{Q}(\{r_j\}_{j=1}^{\ell})[x]$, with $\deg g_j \leq m_j - 1$, for $j = 1, ..., \ell$, such that

$$G_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n, \text{ for all } n. \tag{5}$$

For more details, see [9, Theorem C.1].

In the case of the Fibonacci sequence, the above formula is known as *Binet's formula*:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1 + \sqrt{5})/2$ (the golden number) and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$. Equation (5) and some tricks will allow us to obtain linear forms in three logarithms and then determine lower bounds *à la Baker* for these linear forms. From the main result of Matveev [4], we deduce the following lemma.

Lemma 1. *Let $\alpha_1, \alpha_2, \alpha_3$ be real algebraic numbers and let b_1, b_2, b_3 be non-zero integer rational numbers. Define*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ and let A_1, A_2, A_3 be positive real numbers which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j = 1, 2, 3.$$

Assume that $B' \geq \max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\}$. Define also

$$C_1 = 6750000 \cdot e^4(20.2 + \log(3^{5.5}D^2 \log(eD))).$$

If $\Lambda \neq 0$, then

$$-\log|\Lambda| \leq C_1 D^2 A_1 A_2 A_3 \log(1.5eDB' \log(eD)).$$

As usual, in the previous statement, the *logarithmic height* of an n-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{n}(\log|a| + \sum_{j=1}^{n} \log\max\{1, |\alpha^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1\leq j\leq n}$ are the conjugates of α.

The next lemma plays an important role in the proof of Theorem 1, since it will allow us to prove that a certain linear form is a real number.

Lemma 2. *Let $(G_n)_n$ be a linear recurrence having infinitely many positive terms and such that its characteristic polynomial has a simple positive dominant root. Then the dominant polynomial of $(G_n)_n$ is a positive constant.*

Proof We know that

$$G_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n,$$

where each r_j is a root of characteristic polynomial of G_n, with multiplicity m_j, and each $g_j(n)$ is a non-zero polynomial with degree $\leq m_j - 1$. Suppose that r_1 is the dominant root, since it is simple, we have immediately $m_1 = 1$ and then the degree of dominant polynomial is at most $m_1 - 1 = 0$, so it is a constant, say g_1. Now, dividing G_n by r_1^n, we get

$$\frac{G_n}{r_1^n} = g_1 + \sum_{j=2}^{\ell} \frac{g_j(n)}{\kappa_j^n}, \tag{6}$$

where $\kappa_j = r_1/r_j$. Since $|\kappa_j| > 1$, we have

$$\lim_{n\to\infty} \frac{g_j(n)}{\kappa_j^n} = 0, \text{ for all } 2 \leq j \leq \ell,$$

and so

$$0 \leq \limsup_{n\to\infty} \frac{G_n}{r_1^n} = g_1.$$

Therefore, $g_1 > 0$ as $g_1 \neq 0$.

□

Now, we are ready to deal with the proofs of our results.

3. The proof of Theorem 1

First, note that in the statement of Theorem 1, we have

$$G_n^s + \epsilon_1 G_{n+1}^s + \cdots + \epsilon_{k-1} G_{n+k-1}^s + G_{n+k}^s = bG_{t_n}, \text{ for all } n \in \mathcal{N}, \tag{7}$$

where $\{t_n : n \geq 0\}$ and $\mathcal{N}$ are infinite subsets of positive integers. Suppose that there is a positive integer n_0, such that $G_n \leq 0$, for all $n > n_0$. If s is even, then the left-hand side of (7) is a positive sequence, but the right-hand side is non-positive when $t_n > n_0$, which is a contradiction. In the case of an odd s, we define $H_n = -G_n$, for all n. Thus (7) becomes

$$H_n^s + \epsilon_1 H_{n+1}^s + \cdots + \epsilon_{k-1} H_{n+k-1}^s + H_{n+k}^s = bH_{t_n}, \text{ for all } n \in \mathcal{N},$$

which is an equivalent identity, but with $H_n > 0$, for all $n > n_0$. Summarizing, we may assume, without loss of generality, that $(G_n)_n$ has infinitely many positive terms.

According to equation (2), we have

$$G_n = g_1(n) r_1^n + \cdots + g_\ell(n) r_\ell^n, \text{ for } n \geq 1.$$

Assume that r_1 is the simple dominant root. So $r_1 > 0$ and by Lemma 2, $g_1(n) = g_1 > 0$. Thus, for any $0 \leq t \leq k$, the multinomial theorem yields

$$G_{n+t}^s = \sum_{\overline{\alpha} \in \mathcal{I}_s} \frac{s!}{\alpha_1! \cdots \alpha_\ell!} \prod_{j=1}^{\ell} g_j(n+t)^{\alpha_j} r_1^{\alpha_1(n+t)} \cdots r_\ell^{\alpha_\ell(n+t)},$$

where $\mathcal{I}_s = \{\overline{\alpha} = (\alpha_1, ..., \alpha_\ell) \in \mathbb{Z}^\ell : \alpha_i \geq 0 \text{ and } \sum_{i=1}^{\ell} \alpha_i = s\}$. A straightforward computation shows that

$$\frac{G_{n+t}^s}{r_1^{sn}} = g_1^s r_1^{ts} + \sum_{\overline{\alpha} \in \mathcal{I}_s \setminus \{s\mathfrak{e}_1\}} \frac{s!}{\alpha_1! \cdots \alpha_\ell!} \prod_{j=1}^{\ell} g_j(n+t)^{\alpha_j} \frac{r_1^{\alpha_1 t} r_2^{\alpha_2(n+t)} \cdots r_\ell^{\alpha_\ell(n+t)}}{r_1^{(s-\alpha_1)n}},$$

where $\mathfrak{e}_1 = (1, 0, ..., 0)$. Since $s - \alpha_1 = \alpha_2 + \cdots + \alpha_\ell > 0$ and $g_1(n+t) = g_1$ (by Lemma 2), we can write

$$\frac{G_{n+t}^s}{r_1^{sn}} = g_1^s r_1^{ts} + \sum_{\overline{\alpha} \in \mathcal{I}_s \setminus \{s\mathfrak{e}_1\}} \frac{s!}{\alpha_1! \cdots \alpha_\ell!} \prod_{j=2}^{\ell} \left(\frac{g_j(n+t)}{r_1^n} \right)^{\alpha_j} g_1^{\alpha_1} r_1^{\alpha_1 t} r_2^{\alpha_2(n+t)} \cdots r_\ell^{\alpha_\ell(n+t)}.$$

As $|r_j| < 1$ and $\lim_{n\to\infty} g_j(n+t)/r_1^n = 0$, each term in the previous summation tends to zero as $n \to \infty$, because $\alpha_j > 0$, for some $2 \le j \le \ell$. So,

$$\lim_{n\to\infty} \frac{G_{n+t}^s}{r_1^{sn}} = g_1^s r_1^{ts}. \tag{8}$$

On the other hand, let $\mathcal{N}$ be an infinite set of positive integers and let $(t_n)_n \subseteq \mathbb{N}$ be a sequence such that $\sum_{j=0}^k \epsilon_j G_{n+j}^s = bG_{t_n}$, for all $n \in \mathcal{N}$, where we set $\epsilon_0 = \epsilon_k = 1$. By the formula in (6), we have that $G_m = O(r_1^m)$, for all m (where as usual, O denotes the 'big-oh' Landal symbol). Thus the equality $\sum_{j=0}^k \epsilon_j G_{n+j}^s = bG_{t_n}$ yields $O(r_1^{ns}) = O(r_1^{t_n})$ and so $t_n = O(n)$. Also, Equation (8) implies that

$$g_1^s(1 + \epsilon_1 r_1^s + \cdots + \epsilon_{k-1} r_1^{(k-1)s} + r_1^{ks}) = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{bG_{t_n}}{r_1^{ns}}. \tag{9}$$

However,

$$\frac{G_{t_n}}{r_1^{ns}} = g_1 r_1^{t_n - ns} + g_2(t_n)\frac{r_2^{t_n}}{r_1^{ns}} + \cdots + g_\ell(t_n)\frac{r_\ell^{t_n}}{r_1^{ns}}.$$

Now, we use that any exponential function of n dominates any polynomial function of n, together with the fact that $t_n = O(n)$ and $|r_j| < 1$, for $j = 2, ..., \ell$, to conclude that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{g_j(t_n)}{r_1^{ns}} \cdot r_j^{t_n} = 0, \text{ for } j = 2, ..., \ell.$$

So equation (9) becomes

$$g_1^{s-1}(1 + \epsilon_1 r_1^s + \cdots + \epsilon_{k-1} r_1^{(k-1)s} + r_1^{ks}) = b \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} r_1^{t_n - ns}.$$

Since $t_n - ns$ is an integer and $r_1 > 1$, then $t_n - ns$ must be constant with respect to n, say t, for all $n \in \mathcal{N}$ sufficiently large. Hence equation (9) yields

$$g_1^{s-1}(1 + \epsilon_1 r_1^s + \cdots + \epsilon_{k-1} r_1^{(k-1)s} + r_1^{ks}) = br_1^t. \tag{10}$$

Observe that equation (10) can be rewritten into the form

$$e^\Lambda - 1 = r_1^{-ks} + \epsilon_1 r_1^{-(k-1)s} + \cdots + \epsilon_{k-1} r_1^{-s} > 0,$$

where

$$\Lambda = (s-1)\log(1/g_1) - (ks - t)\log r_1 + \log b.$$

As g_1 and r_1 are positive, then $\Lambda \in \mathbb{R}$ and the above inequality implies that $\Lambda > 0$. Thus

$$\Lambda < e^{\Lambda} - 1 = r_1^{-ks} + \epsilon_1 r_1^{-(k-1)s} + \cdots + \epsilon_{k-1} r_1^{-s} < k r_1^{-s}.$$

Therefore, we get

$$\log \Lambda < \log k - s \log r_1. \tag{11}$$

In order to apply Lemma 1, we take

$$\alpha_1 = 1/g_1, \quad \alpha_2 = r_1, \quad \alpha_3 = b, \quad b_1 = s-1, \quad b_2 = -(ks-t), \quad b_3 = 1.$$

However, b_1, b_2 and b_3 must be non-zero. Since, we may suppose $s > 1$, the only point that requires extra care is the verification of $b_2 \neq 0$. In fact, if $b_2 = 0$, i.e., $ks = t$, then equation (10) gives

$$g_1^{s-1} r_1^{ks} < g_1^{s-1}(1 + \epsilon_1 r_1^s + \cdots + \epsilon_{k-1} r_1^{(k-1)s} + r_1^{ks}) = b r_1^t.$$

So, we get $b > g_1^{s-1}$ and Theorem 1 is proved with the choice of $C = \log b / \log g_1 + 1$. Therefore, we assume that $ks \neq t$.

Note that $D = [\mathbb{Q}(g_1, r_1) : \mathbb{Q}] \leq \ell!$, since $g_1 \in \mathbb{Q}(\{r_j\}_{1 \leq j \leq \ell})$. Now, set $h(\alpha_1) = h(1/g_1) = h_1$. Since r_1 is a root of $G(x)$, the minimal polynomial of r_1 is a divisor of $G(x)$, we have

$$h(\alpha_2) \leq (\log r_1)/\ell$$

and finally $h(\alpha_3) = \log b$. Then, we take

$$A_1 = \ell! h_1 + |\log g_1| + 0.16, \quad A_2 = (\ell-1)! \log r_1 + 0.16, \quad A_3 = \ell! \log b + 0.16.$$

Also, we get

$$\begin{aligned} B' &\geq \max\{1, \max\{|b_j| A_j / A_1; 1 \leq j \leq 3\}\} \\ &= \max\left\{ s-1, |ks-t| \cdot \left(\frac{(\ell-1)! \log r_1 + 0.16}{\ell! h_1 + |\log g_1| + 0.16} \right) \right\}, \end{aligned}$$

for s sufficiently large. Thus, we obtain

$$\begin{aligned} -\log|\Lambda| &< 3.7 \cdot 10^4 \cdot (20.2 + \log(3^{5.5} \ell!^2 \log(e\ell!))) \ell!^2 (\ell! h_1 + \log g_1 + 0.16) \times \\ &\quad ((\ell-1)! \log r_1 + 0.16)(\ell! \log b + 0.16) \cdot \log(1.5 e \ell! B' \log(e\ell!)). \end{aligned} \tag{12}$$

Combining estimates (11) and (12), we get a constant $C > 0$, which depends only on b, k and the parameters of G_n, such that $s < C$.

□

4. The proof of Theorem 2

For $1 \leq j \leq \ell$, the Binet's formula and the binomial formula imply

$$F_{n+j}^{s_j} = \left(\frac{1}{\sqrt{5}}\right)^{s_j} \cdot \sum_{k=0}^{s_j} \binom{s_j}{k} (-1)^{(n+j+1)k} \alpha^{(n+j)s_j - 2(n+j)k}.$$

Thus

$$\frac{F_{n+j}^{s_j}}{\alpha^{(n+t)s_t}} = \left(\frac{1}{\sqrt{5}}\right)^{s_j} \cdot \sum_{k=0}^{s_j} \binom{s_j}{k} (-1)^{(n+j+1)k} \alpha^{(s_j - s_t)n + js_j - ts_t - 2(n+j)k}.$$

Since $s_t > s_j$, for all $j \neq t$, we conclude that

$$\lim_{n\to\infty} \frac{F_{n+j}^{s_j}}{\alpha^{(n+t)s_t}} = \begin{cases} 0, & \text{if } j \neq t \\ (\sqrt{5})^{-s_t}, & \text{if } j = t \end{cases}$$

Suppose that Theorem 2 is false. Then, there exists a sequence $(t_n)_n \subseteq \mathbb{N}$ such that

$$a_1 F_{n+1}^{s_1} + a_2 F_{n+2}^{s_2} + \cdots + a_\ell F_{n+\ell}^{s_\ell} = F_{t_n}, \tag{13}$$

for infinitely many positive integers n. Thus,

$$\limsup_{n\to\infty} \frac{F_{t_n}}{\alpha^{(n+t)s_t}} = \limsup_{n\to\infty} \sum_{j=0}^{\ell} \frac{a_j F_{n+j}^{s_j}}{\alpha^{(n+t)s_t}} = a_t \left(\frac{1}{\sqrt{5}}\right)^{s_t}. \tag{14}$$

On the other hand, a straightforward computation gives

$$\lim_{n\to\infty} \frac{F_{t_n}}{\alpha^{(n+t)s_t}} = \frac{1}{\sqrt{5}} \cdot \lim_{n\to\infty} \alpha^{t_n - (n+t)s_t}. \tag{15}$$

Since $|\alpha| > 1$, combining (14) and (15), we get the identity

$$\frac{\alpha^\nu}{\sqrt{5}} = a_t \left(\frac{1}{\sqrt{5}}\right)^{s_t},$$

where $\nu = \lim_{n\to\infty}(t_n - (n+t)s_t)$. Thus $\alpha^{2\nu} \in \mathbb{Q}$ and then $\nu = 0$. Therefore the above identity becomes

$$(\sqrt{5})^{s_t - 1} = a_t, \tag{16}$$

which is impossible for an even s_t (indeed, its left-hand side is irrational). Therefore, s_t is odd and a_t is a non-negative power of 5. By hypothesis, one concludes that $s_t = 1$. However, with $k \geq 2$, this leads to an absurdity as

$$1 \leq \min\{s_1, s_2\} < s_t = 1$$

and completes the proof of Theorem 2.

$\square$

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