A DIOPHANTINE EQUATION RELATED TO THE SUM OF POWERS OF TWO CONSECUTIVE GENERALIZED FIBONACCI NUMBERS

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ABSTRACT. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{m+2} = F_{m+1} + F_m$, for $m \geq 0$, where $F_0 = 0$ and $F_1 = 1$. There are several interesting identities involving this sequence such as $F_m^2 + F_{m+1}^2 = F_{2m+1}$, for all $m \geq 0$. In 2011, Luca and Oyono [8] proved that if $F_m^s + F_{m+1}^s$ is a Fibonacci number, with $m \geq 2$, then $s = 1$ or 2. One of the most known generalization of the Fibonacci sequence, is the k-generalized Fibonacci sequence $(F_n^{(k)})_n$ which is defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and such that each term afterwards is the sum of the k preceding terms. In this paper, we generalize Luca and Oyono's method to prove that the Diophantine equation

$$(F_m^{(k)})^s + (F_{m+1}^{(k)})^s = F_n^{(k)}$$

has no solution in positive integers n, m, k and s, if $3 \leq k \leq \min\{m, \log s\}$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.$$

The Fibonacci numbers are well-known for possessing wonderful and amazing properties (consult [7] together with their very extensive annotated bibliography for additional references and history).

Among the several pretty algebraic identities involving Fibonacci numbers, we are interested in the following one

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \text{ for all } n \geq 0. \tag{1.1}$$

In particular, this naive identity (which can be proved easily by induction) tell us that the sum of the square of two consecutive Fibonacci numbers is still a Fibonacci number. In order to check if the sum of higher powers of two consecutive Fibonacci numbers could also belong to this sequence, Marques and Togbé [9] showed that, for a fixed s, if $F_m^s + F_{m+1}^s$ is a Fibonacci number for infinitely many m, then $s = 1$ or 2. In 2011, Luca and Oyono [8] solved this problem completely, showing that the Diophantine equation

$$F_m^s + F_{m+1}^s = F_n \tag{1.2}$$

has no solutions (m, n, s) with $m \geq 2$ and $s \geq 3$.

Let $k \geq 2$ and denote $F^{(k)} := (F_n^{(k)})_{n\geq -(k-2)}$, the *$k$-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)}, \tag{1.3}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k-step Fibonacci sequence*, the *Fibonacci k-sequence*, or *k-bonacci* *sequence*. Clearly for $k = 2$, we obtain the well-known Fibonacci numbers $F_n^{(2)} = F_n$, for $k = 3$, the Tribonacci numbers $F_n^{(3)} = T_n$ and for $k = 4$, the Tetranacci numbers $F_n^{(3)} = Q_n$.

The aim of this paper is to study a generalization of the equation (1.2) in the k-generalized Fibonacci context. More precisely, we have the following result

Theorem 1.1. *The Diophantine equation*

$$(F_m^{(k)})^s + (F_{m+1}^{(k)})^s = F_n^{(k)} \tag{1.4}$$

has no solution in positive integers m, n, k *and* s, *with* $3 \leq k \leq \{m, \log s\}$.

We recall that for $-(k-2) \leq m \leq 1$ there are trivial solutions to the Eq. (1.4) for all $k \geq 2$.

Our method follows roughly the following steps: First, we use Matveev's result [11] on linear forms in logarithms to obtain an upper bound for s in terms of m. When m is small, say $m \leq 1394$, we use Dujella and Pethö's result [12] to decrease the range of possible values and then let the computer check the non existence of solutions in this case. To the case where $m \geq 1395$, we use again linear forms in logarithms to obtain an upper bound for s, now in terms of k, which, combined with the hypothesis $k < \log s$, gives us an absolute upper bound for s. In the final step, we use continued fractions to lower the bounds and then let the computer cover the range of possible values, showing that there are no solutions also in this case, which completes the proof.

2. Auxiliary results

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and 2 (as can be seen in [10]). Also, it was proved in [1, Lemma 1] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \text{ for all } n \geq 1, \tag{2.1}$$

where α is the dominant root of $\psi_k(x)$.

Recall that for $k = 2$, one has the useful Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $\alpha = (1 + \sqrt{5})/2 = -\beta^{-1}$. There are many closed formulas representing these k-generalized Fibonacci numbers, as can be seen in [3, 4, 5, 6]. However, we are interested in the simplified "Binet-like" formula due to G. Dresden [2, Theorem 1] :

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{2.2}$$

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. Also, the contribution of the roots inside the unit circle in formula (2.2) is almost trivial. More precisely, it was proved in [2] that

$$|E_n(k)| < \frac{1}{2}, \tag{2.3}$$

where $E_n(k) := F_n^{(k)} - g(\alpha, k)\alpha^{n-1}$ and $g(x, y) := (x - 1)/(2 + (y + 1)(x - 2))$.

We shall use a few times a result due to Matveev [11], which states the following

Theorem 2.1 (Matveev)**.** *Let* $\alpha_1, \ldots, \alpha_t$ *real algebraic numbers,* $b_1, \ldots,$ b_t *nonzero integers and* $\Lambda := \alpha_1^{b_1} \cdots \alpha_t^{b_t} - 1$*. Let* $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_t) : \mathbb{Q}]$ *and* $A_1, \ldots, A_t$ *positive real numbers satisfying*

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j = 1, \ldots, t.$$

Let $B \geq \max\{|b_1|, \ldots, |b_t|\}$*. Also define*

$$C_{t,D} := 1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D).$$

If $\Lambda \neq 0$*, then*

$$|\Lambda| > \exp(-C_{t,D}(1 + \log B)A_1 \cdots A_t) \ .$$

Where for an algebraic number η we write $h(\eta)$ for its logarithmic (or Weil's) height whose formula is

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\})\right) \ ,$$

with d being the degree of η over $\mathbb{Q}$, and

$$f(X) := a_0 \prod_{i=1}^{d}(X - \eta^{(i)}) \in \mathbb{Z}[X]$$

the minimal primitive polynomial over the integers having positive leading coefficient and $\eta = \eta^{(1)}$.

Another result which will play an important role in our proof is due to Dujella and Pethö [12]

Lemma 2.2. *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational* γ *such that* $q > 6M$*, and let* μ *be some real number. Let* $\epsilon := \parallel \mu q \parallel - M \parallel \gamma q \parallel$*. If* $\epsilon > 0$*, then there is no solution to the inequality*

$$0 < n\gamma - s + \mu < AB^{-n}$$

in positive integers n *and* s *with*

$$\frac{\log(Aq/\epsilon)}{\log B} \leq n \leq M \ .$$

The next theorem about continued fractions is due to Legendre, and will help us to finish the demonstration.

Theorem 2.3. *Let* ξ *be a real number. If a rational number* a/b *is such that*

$$\left|\xi - \frac{a}{b}\right| < \frac{1}{2b^2} \ ,$$

then it is a convergent of ξ*.*

Now, we are ready to deal with the proof of Theorem 1.1.

3. Proof of Theorem 1.1

First, observe that by using the estimates in (2.1), we obtain

$$\begin{aligned}\alpha^{n-1} > F_n^{(k)} = (F_m^{(k)})^s + (F_{m+1}^{(k)})^s &> \alpha^{(m-2)s} + \alpha^{(m-1)s} \\ &= \alpha^{(m-2)s}(1+\alpha^s) > \alpha^{(m-1)s},\end{aligned}$$

and

$$\begin{aligned}\alpha^{n-2} < F_n^{(k)} = (F_m^{(k)})^s + (F_{m+1}^{(k)})^s &< \alpha^{(m-1)s} + \alpha^{ms} \\ &= \alpha^{(m-1)s}(1+\alpha^s) \\ &< \alpha^{ms+1},\end{aligned}$$

where we used that $1+\alpha^s < \alpha^{s+1}$ for all $k \geq 3$, which is an immediate consequence of $\alpha^s(\alpha - 1) > (7/4)^3 \times (7/4-1) = 1029/256 > 1$. Now, the estimate $\alpha^{n-2} < F_n^{(k)} < \alpha^{n-1}$ together with the previous estimates yield $(m-1)s+1 < n < ms+3$. In conclusion, we have proved that if (m,n,k,s) is a solution of Eq. (1.4), then $n \in \{(m-1)s+2, (m-1)s+1, \ldots, ms+2\}$.

3.1. An inequality for s in terms of m.

Using formula (2.2), we rewrite (1.4) as

$$(F_{m+1}^{(k)})^s - g\alpha^{n-1} = (F_m^{(k)})^s - E_n(k) \; . \tag{3.1}$$

Since $|E_n(k)| < 1/2$, we have that $(F_{m+1}^{(k)})^s - g\alpha^{n-1} \in [(F_m^{(k)})^s - 1/2, (F_m^{(k)})^s + 1/2]$, which is positive. Now applying the absolute value and the triangle inequality in (3.1), we obtain

$$|g\alpha^{n-1} - (F_{m+1}^{(k)})^s| < \frac{1}{2} + (F_m^{(k)})^s < 2(F_m^{(k)})^s \; .$$

Dividing by $(F_{m+1}^{(k)})^s$ to get

$$\left| \frac{g\alpha^{n-1}}{(F_{m+1}^{(k)})^s} - 1 \right| < 2\left(\frac{F_m^{(k)}}{F_{m+1}^{(k)}}\right)^s \; . \tag{3.2}$$

In order to give an upper bound to the right side of (3.2), we use that

$$F_{m+1}^{(k)} = F_m^{(k)} + F_{m-1}^{(k)} + \cdots + F_{m-k+1}^{(k)} = 2F_m^{(k)} - F_{m-k}^{(k)} \; .$$

Thus, for $k \geq 3$

$$\frac{F_{m+1}^{(k)}}{F_m^{(k)}} = 2 - \frac{F_{m-k}^{(k)}}{F_m^{(k)}} \geq 2 - \frac{1}{k} > 1.65 \; . \tag{3.3}$$

Now, using estimate (3.3) in (3.2), we get our first key inequality:

$$\left| \frac{g\alpha^{n-1}}{(F_{m+1}^{(k)})^s} - 1 \right| < \frac{2}{1.65^s} \; . \tag{3.4}$$

In a first application of Matveev's result, take $t := 3$, $\gamma_1 := F_{m+1}^{(k)}$, $\gamma_2 := \alpha$, $\gamma_3 := g$, e $b_1 := -s$, $b_2 := n-1$, $b_3 := 1$. Now, consider

$$\Lambda_1 := g\alpha^{n-1}(F_{m+1}^{(k)})^{-s} - 1 \; ,$$

which is positive (an immediate consequence of what we observed right after (3.1)). The algebraic number field containing γ_1, γ_2 and γ_3 is $\mathbb{K} := \mathbb{Q}(\alpha)$, whose degree is $D := [\mathbb{K} : \mathbb{Q}] = k$. For the value of B, note that

$$\max\{|b_1|, |b_2|, |b_3|\} = \max\{s, n-1, 1\} = \max\{s, n-1\} ,$$

but since $s + 2 \leq (m-1)s + 2 \leq n$, then $s < s+1 \leq n-1$, and we can take $B := n-1$. Now we need to estimate the logarithmic heights $h(\gamma_1)$, $h(\gamma_2)$ and $h(\gamma_3)$.

Since $h(\gamma_1) = \log F_{m+1}^{(k)} < m \log 2$, and $\max\{Dh(\gamma_1), |\log \gamma_1|, 0.16\} < km \log 2$, we can take $A_1 := km \log 2$. Similarly, since $h(\gamma_2) = h(\alpha) = \log \alpha / k < \log 2/k$, where we used the fact that α is the only root of $\psi_k(x)$ outside the unit circle, and $\max\{Dh(\gamma_2), |\log \gamma_2|, 0.16\} < \log 2$, we can take $A_2 := \log 2$.

For $h(\gamma_3)$, we have

$$h(\gamma_3) = h(g) = \frac{1}{d}\left(\log |a_0| + \sum_{i=1}^{k} \log(\max\{|g_i|, 1\})\right) .$$

First, note that applying the conjugation (over $\mathbb{K}$) on $g = g(\alpha, k)$, we obtain $g_i = g(\alpha_i, k)$ for all $1 \leq i \leq k$, and since $[\mathbb{Q}(\alpha) : \mathbb{Q}(g)] = 1$, we have $d = k$. Put

$$G(x) = \prod_{i=1}^{k}\left(x - \frac{\alpha_i - 1}{2 + (\alpha_i - 2)(k+1)}\right) \in \mathbb{Q}[x] .$$

The leading coefficient a_0 of the minimal polynomial of g over the integers divides $\prod_{i=1}^{k}(2 + (\alpha_i - 2)(k+1))$. But,

$$\left|\prod_{i=1}^{k}(2 + (\alpha_i - 2)(k+1))\right| = (k+1)^k \left|\prod_{i=1}^{k}\left(2 - \frac{2}{k+1} - \alpha_i\right)\right| = (k+1)^k \left|\psi_k\left(2 - \frac{2}{k+1}\right)\right| ,$$

Since

$$|\psi_k(y)| < \max\{y^k, y^{k-1} + \cdots + y + 1\} < 2^k ,$$

for all $0 < y < 2$, we get the following upper bound for a_0:

$$a_0 \leq (k+1)^k \left|\psi_k\left(2 - \frac{2}{k+1}\right)\right| < 2^k (k+1)^k .$$

For all $k \geq 3$, we have $2(k+1) < k^3$ and then $\log |a_0| < k(\log(k+1) + \log 2) < 3k \log k$. Now, we need to estimate $|g_i|$. If $2 \leq i \leq k$, we have $|\alpha_i| < 1$, which gives us

$$|2 + (k+1)(\alpha_i - 2)| \geq (k+1)|2 - \alpha_i| - 2 > k - 1 \geq 2 .$$

Hence,

$$\left|\frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)}\right| \leq \frac{|\alpha_i| + 1}{2} < 1 \;\Rightarrow\; |g_i| < 1 .$$

In the case of the dominant root, α, is easy to see that $|g| < 1$, as follows

$$g = \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} < \frac{1}{2 - \dfrac{k+1}{2^{k-1}}} \leq 1 ,$$

where we used in the above inequality that $k+1 \leq 2^{k-1}$ for $k \geq 3$. Therefore, $\max\{|g_i|, 1\} = 1$, and we finally obtain

$$h(\gamma_3) = \frac{1}{k}\left(\log|a_0| + \sum_{i=1}^{k}\log\left(\max\{|g_i|, 1\}\right)\right) < \frac{1}{k}(3k\log k) < 3\log k \ .$$

So we can take $A_3 := 3k\log k$. Now, applying Theorem (2.1) to get a lower bound for $|\Lambda_1|$,

$$\begin{aligned} |\Lambda_1| &> \exp(-1.4 \times 30^{3+3} \times 3^{4.5} \times k^2 \times (1+\log k)(1+\log(n-1)) \\ &\quad \times (3k\log k)(\log 2)(km\log 2)) \\ \Rightarrow |\Lambda_1| &> \exp(-2.064 \times 10^{11} \times k^4 m\log k(2\log k)(1+\log(n-1))) \ . \end{aligned}$$

Using the previous estimates for n in terms of s and m, is easy to see that $n \geq 8$, where the inequality $1 + \log(n+1) < 2\log(n-1)$ holds, and $n - 1 \leq ms + 1$. Then,

$$|\Lambda_1| > \exp(-8.256 \times 10^{11} \times mk^4(\log k)^2 \log(ms+1)) \ .$$

Comparing the above inequality with (3.4), we get

$$\frac{2}{1.65^s} > \exp(-8.256 \times 10^{11} \times mk^4(\log k)^2 \log(ms+1)) \ .$$

Now, taking logarithms in the previous inequality, we have

$$\log 2 - s\log 1.65 > -8.256 \times 10^{11} \times mk^4(\log k)^2 \log(ms+1) \ ,$$

which leads to

$$s < 16.7 \times 10^{11} \times mk^4(\log k)^2 \log(ms+1) \ .$$

Since $m, s \geq 3$, then $\log m, \log s \geq 1$ and so $\log(ms+1) < \log(ms)^2 < 4\log m\log s$. Thus, we obtain

$$\frac{s}{\log s} < 66.8 \times 10^{11} \times m^5(\log m)^3 \ , \tag{3.5}$$

where we also used the hypothesis $k \leq m$.

Now, we are going to use the following argument showed by Luca and Oyono [8], for $x > e$,

$$\frac{x}{\log x} < A \ \Rightarrow \ x < 2A\log A \ , \tag{3.6}$$

whenever $A \geq 3$. Thus, taking $A := 66.8 \times 10^{11} \times m^5(\log m)^3$, inequality (3.5) yields

$$\begin{aligned} s &< 2 \times (66.8 \times 10^{11} \times m^5(\log m)^3)\log(66.8 \times 10^{11} \times m^5(\log m)^3) \\ &< 133.6 \times 10^{11} \times m^5(\log m)^3(29.54 + 5\log m + 3\log\log m) \\ &< 133.6 \times 10^{11} \times m^5(\log m)^3(34.9\log m) \ , \end{aligned}$$

In the last chain of inequalities, we have used that $\log\log m < \log m$ and $29.54 + 8\log m < 34.9\log m$ holds for all $m \geq 3$. Hence, we have the following result.

Lemma 3.1. *If (m, n, k, s) is a nontrivial solution in positive integers of Eq. (1.4) with $3 \leq k \leq \min\{m, \log s\}$, then*

$$s < 4.7 \times 10^{14} \times m^5(\log m)^4 \ . \tag{3.7}$$

3.2. **The case of small** m**.** Following our plan, we next consider the cases when $m \in [3, 1394]$, and after finding an upper bound for n, the next step is to reduce it and then let the computer handle with the possible solutions. To do that, first observe that in this case

$$s < 4.7 \times 10^{14} \times (1394)^5 (\log 244)^4 \ \Rightarrow \ s < 6.75 \times 10^{33} \ .$$

Thus, we obtain the following upper bounds for k and n:

$$\begin{aligned} n \leq ms + 2 \quad &\Rightarrow \quad n < 1394 \times 6.75 \times 10^{33} + 2 \ \Rightarrow \ n < 9.41 \times 10^{36} \ , \\ k \leq \log s \quad &\Rightarrow \quad k \leq \log(6.75 \times 10^{33}) \ \Rightarrow \ k \leq 77 \ . \end{aligned}$$

Also note that $n < ms + 2$, gives us $s > (n-2)/1394$. Now, in order to use the reduction method due to Dujella and Pethö [12], take

$$\Gamma_1 := (n-1)\log\alpha - \log\left(\frac{1}{g}\right) - s \log F_{m+1}^{(k)} \ . \tag{3.8}$$

Then $\Lambda_1 = e^{\Gamma_1} - 1 > 0$, since $\Lambda_1 > 0$, and from (3.4) we have

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \Lambda_1 < \frac{2}{1.65^s} \ .$$

Dividing both sides of the previous inequality by $\log F_{m+1}^{(k)}$, and using that $s > (n-2)/1394$, we obtain

$$0 < n\left(\frac{\log\alpha}{\log F_{m+1}^{(k)}}\right) - s - \left(\frac{\log(\alpha/g)}{\log F_{m+1}^{(k)}}\right) < 2.01 \times (1.65)^{-\frac{n}{1394}} \ .$$

With,

$$\gamma_{m,k} := \frac{\log\alpha}{\log F_{m+1}^{(k)}} \ , \ \mu_{m,k} := -\frac{\log(\alpha/g)}{\log F_{m+1}^{(k)}} \ , \ A := 2.01 \text{ and } B := (1.65)^{\frac{1}{1394}} \ ,$$

the previous inequality yields

$$0 < n\gamma_{m,k} - s - \mu_{m,k} < AB^{-n} \ . \tag{3.9}$$

Let us show that $\gamma_{m,k}$ is an irrational number. Indeed, if $\gamma_{m,k} \in \mathbb{Q}$, we have $\alpha^q = \left(F_{m+1}^{(k)}\right)^p$ for some $p, q \in \mathbb{Q}$, with $q > 0$. Conjugating this relation (over $\mathbb{K}$), taking the product and then the absolute value, we get

$$\left|\prod_{i=1}^{k} \alpha_i\right|^q = \left(F_{m+1}^{(k)}\right)^{kp} \neq 1 \ .$$

On the other hand, we already know that this module is equal to one, since $\alpha, \alpha_2, \cdots, \alpha_k$ are the roots of $\psi_k(x)$, which contradicts the relation above. Thus, $\gamma_{m,k} \notin \mathbb{Q}$. Take $M := 9.41 \times 10^{36}$. Let $q_{t,m,k}$ be the denominator of the t-th convergent to $\gamma_{m,k}$. To do the following calculation, we have used the *Mathematica 9* software on a OSX 10.8.4, 1.8 GHz Intel Core i5 with 4GB of RAM. Calculating the smallest value of $q_{700,m,k}$, for $4 \leq m \leq 1394$ and $3 \leq k \leq \min\{m, 77\}$, we have that $q_{700,m,k} > 2.1 \times 10^{425} > 6M$, and for the same range, $\epsilon_{700,m,k} > 1.8 \cdot 10^{-189}$, which means that $\epsilon_{700,m,k}$ is always positive (this is not true for $\epsilon_{600,m,k}$). Hence, by Lemma 2.2, there are no integer solutions for (3.9) when

$$\left\lfloor \max_{\substack{3 \leq k \leq 77 \\ 3 \leq m \leq 1394}} \frac{\log(Aq_{700,m,k}/\epsilon_{700,m,k})}{\log B} \right\rfloor \leq n \leq 9.41 \times 10^{36}$$

$$\Rightarrow \left\lfloor \frac{\log(2.01 \cdot 2.1 \times 10^{425}/1.8 \cdot 10^{-189})}{\log((1.65)^{\frac{1}{1394}})} \right\rfloor \leq n \leq 9.41 \times 10^{36}$$
$$\Rightarrow 1515054 \leq n \leq 9.41 \times 10^{36} .$$

Therefore, we have $n \leq 1515053$ and consequently $s \leq 757526$, since $s \leq (n-2)/(m-1)$. Also, using that $k \leq \log s$, we get $k \leq 13$. A computer search with *Mathematica* revealed no solutions to the equation (1.1) in the range $3 \leq m \leq 1394$, $3 \leq k \leq 13$, $21 \leq s \leq 757526$ and $(m-1)s+2 \leq n \leq ms+2$. This finishes the case $m \in [3, 1394]$.

3.3. **Finding absolute upper bounds.** From now on, we assume that $m \geq 1395$. Set

$$\mathcal{X}_m := \frac{|E_m(k)|s}{g\alpha^{m-1}} .$$

Lemma 3.1 gives us

$$\mathcal{X}_m < \frac{|E_m(k)|4.7 \times 10^{11} \times m^5(\log m)^4}{g\alpha^{m-1}} < \frac{1}{\alpha^{(m-1)/2}} , \tag{3.10}$$

where we used that

$$4.7 \times 10^{11} \times m^5(\log m)^4 < \left(\frac{7}{4}\right)^{\frac{m-1}{2}} < \alpha^{\frac{m-1}{2}} ,$$

holds for $m \geq 1395$. In particular, $\mathcal{X}_m < \alpha^{-697} < (7/4)^{-697} < 2.3 \times 10^{-30}$. Similarly,

$$\mathcal{X}_{m+1} = \frac{|E_{m+1}(k)|s}{g\alpha^{m+1}} < \frac{(1/2)4.7 \times 10^{11} \times m^5(\log m)^4}{(1/2)\alpha^{m+1}} < \frac{1}{\alpha^{(m-1)/2}} .$$

We now write

$$(F_m^{(k)})^s = g^s\alpha^{(m-1)s}\left(1 + \frac{E_m(k)}{g\alpha^{m-1}}\right)^s \quad \text{and} \quad (F_{m+1}^{(k)})^s = g^s\alpha^{ms}\left(1 + \frac{E_{m+1}(k)}{g\alpha^m}\right)^s . \tag{3.11}$$

If $E_m(k) > 0$, then

$$1 < \left(1 + \frac{E_m(k)}{g\alpha^{m-1}}\right)^s = \left(1 + \frac{|E_m(k)|}{g\alpha^{m-1}}\right)^s < e^{\left(s|E_m(k)|/g\alpha^{m-1}\right)} < 1 + 2\mathcal{X}_m ,$$

because $e^x < 1 + 2x$, for $0 < x < 1.25$, while if $E_m(k) < 0$, then

$$\begin{aligned} 1 > \left(1 + \frac{E_m(k)}{g\alpha^{m-1}}\right)^s = \left(1 - \frac{|E_m(k)|}{g\alpha^{m-1}}\right)^s &= \exp\left(s\log\left(1 - \frac{|E_m(k|}{g\alpha^{m-1}}\right)\right) \\ &> 1 - 2\mathcal{X}_m , \end{aligned}$$

now because $\log(1-x) > -2x$, for $0 < x < 0.79$. The same inequalities are true if we replace m by $m+1$. Combining these two facts with (3.11), we can see how $(F_m^{(k)})^s$ is *well approximated* by $g^s\alpha^{(m-1)s}$, as follows

$$\begin{aligned} (F_m^{(k)})^s = \quad & g^s\alpha^{(m-1)s}\left(1 + \tfrac{E_m(k)}{g\alpha^{m-1}}\right)^s && < g^s\alpha^{(m-1)s}(1 + 2\mathcal{X}_m) \\ \Rightarrow \quad & (F_m^{(k)})^s - g^s\alpha^{(m-1)s} && < 2\mathcal{X}_m g^s\alpha^{(m-1)s} , \end{aligned} \tag{3.12}$$

$$\begin{aligned} (F_m^{(k)})^s = \quad & g^s\alpha^{(m-1)s}\left(1 + \tfrac{E_m(k)}{g\alpha^{m-1}}\right)^s && > g^s\alpha^{(m-1)s}(1 - 2\mathcal{X}_m) \\ \Rightarrow \quad & (F_m^{(k)})^s - g^s\alpha^{(m-1)s} && > -2\mathcal{X}_m g^s\alpha^{(m-1)s} , \end{aligned} \tag{3.13}$$

thus

$$|(F_m^{(k)})^s - g^s\alpha^{(m-1)s}| < 2\mathcal{X}_m g^s\alpha^{(m-1)s} \quad \text{and} \quad |(F_{m+1}^{(k)})^s - g^s\alpha^{ms}| < 2\mathcal{X}_{m+1} g^s\alpha^{ms} \ . \tag{3.14}$$

We now go back to (1.4) and rewrite it as

$$\begin{aligned} g\alpha^{n-1} + E_n(k) = F_n^{(k)} = (F_m^{(k)})^s + (F_{m+1}^{(k)})^s &= g^s\alpha^{(m-1)s} + g^s\alpha^{ms} \\ &+ \left((F_m^{(k)})^s - g^s\alpha^{(m-1)s}\right) \\ &+ \left((F_{m+1}^{(k)})^s - g^s\alpha^{ms}\right) \ , \end{aligned}$$

which gives us

$$\begin{aligned} |g\alpha^{n-1} - g^s\alpha^{(m-1)s}(1+\alpha^s)| &\leq |((F_m^{(k)})^s - g^s\alpha^{(m-1)s})| \\ &\quad + |((F_{m+1}^{(k)})^s - g^s\alpha^{ms})| + |E_n(k)| \\ &< 2\mathcal{X}_m g^s\alpha^{(m-1)s} + 2\mathcal{X}_{m+1} g^s\alpha^{ms} + \frac{1}{2}. \end{aligned}$$

Dividing both sides by $g^s\alpha^{ms}$,

$$|g^{1-s}\alpha^{n-(ms+1)} - (1+\alpha^{-s})| < \frac{1}{2g^s\alpha^{ms}} + 0.38\mathcal{X}_m + 2\mathcal{X}_{m+1} \tag{3.15}$$

where (3.15) holds since $\alpha^s > (7/4)^3 > 5.35$, and then $2/\alpha^s < 2/5.35 < 0.38$. Now, we need a lower bound to $2g^s\alpha^{ms}$ in terms of $\alpha^{\frac{m-1}{2}}$:

$$\begin{aligned} 2g^s\alpha^{ms-\frac{m-1}{2}} &> 2\left(\frac{1}{2}\right)^s \times \left(\frac{7}{4}\right)^{2s} \times \left(\frac{7}{4}\right)^{(m-2)s-\frac{m-1}{2}} \\ &> 2\left(\frac{49}{32}\right)^3 \times \left(\frac{7}{4}\right)^{607} > 3\times 10^{147} > 10^3 \ , \end{aligned}$$

therefore $(2g^s\alpha^{ms})^{-1} < 0.001/\alpha^{(m-1)/2}$. Using it in (3.15) jointly with (3.10), we obtain

$$|g^{1-s}\alpha^{n-(ms+1)} - (1+\alpha^{-s})| < \frac{0.001}{\alpha^{\frac{m-1}{2}}} + \frac{0.38}{\alpha^{\frac{m-1}{2}}} + \frac{2}{\alpha^{\frac{m}{2}}} < \frac{2.39}{\alpha^{\frac{m-1}{2}}} \ , \tag{3.16}$$

Hence, we conclude that

$$|g^{1-s}\alpha^{n-(ms+1)} - 1| < \frac{1}{\alpha^s} + \frac{2.39}{\alpha^{\frac{m-1}{2}}} < \frac{3.39}{\alpha^l} \ , \tag{3.17}$$

where we put $l := \min\{s, \frac{m-1}{2}\}$. Having in mind to use Matveev's result once more, we now set

$$\Lambda_2 := g^{1-s}\alpha^{n-(ms+1)} - 1 \ , \tag{3.18}$$

but before this, we must show that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then conjugating over $\mathbb{Q}(\alpha)$ and taking the product of all conjugates, we have

$$\left(\prod_{i=1}^{k}|g_i|\right)^{s-1} = \left|\prod_{i=1}^{k}(g_i)^{s-1}\right| = \left|\prod_{i=1}^{k}\alpha_i\right|^{n-(ms+1)} = 1 \ ,$$

but on the other hand, we saw that $|g_i| < 1$ for all $1 \leq i \leq k$ which contradicts the above identity. Thus, $\Lambda_2 \neq 0$.

So, we take $t = 2$, $\lambda_1 := g$, $\lambda_2 := \alpha$ and $c_1 := 1-s$, $c_2 := n-(ms+1)$. Again, $\mathbb{K} := \mathbb{Q}(\alpha)$ and $D := [\mathbb{K} : \mathbb{Q}] = k$. Now, to choose the value of $B \geq \max\{|c_1|, |c_2|\} = \max\{s-1, |n-(ms+1)|\}$, observe that

$$\begin{aligned} n \geq (m-1)s + 2 \quad &\Rightarrow \quad n - (ms+1) \geq -(s-1) \\ n \leq ms + 2 \quad &\Rightarrow \quad n - (ms+1) \leq s - 1 \ . \end{aligned}$$

Hence $|n - (ms+1)| < s - 1$, and we can take $B := s - 1$. As in the previous application of Theorem 2.1, we can take $A_1 := 3k \log k$ and $A_2 := \log 2$. We thus get that

$$\begin{aligned} |\Lambda_2| \;&>\; \exp(-1.4 \times 30^5 \times 2^{4.5} \times k^2(1 + \log k)(1 + \log(s-1)) \\ &\qquad \times (3k \log k)(\log 2)) \\ &>\; \exp(-7.74 \times 10^8 \times k^3 \log k(1 + \log(s-1))) \ . \end{aligned}$$

Combining the last inequality with (3.17) we obtain

$$\begin{aligned} \frac{3.39}{\alpha^l} \;&>\; \exp(-7.74 \times 10^8 \times k^3 \log k(1 + \log(s-1))) \\ \Rightarrow \ l \;&<\; \frac{\log 3.39}{\log \alpha} + \frac{7.74}{\log \alpha} \times 10^8 \times k^3 \log k(1 + \log(s-1)) \\ \Rightarrow \ l \;&<\; 1.4 \times 10^9 \times k^3 \log k(1 + \log(s-1)) \ . \end{aligned} \tag{3.19}$$

If $l = s$, then (3.19) becomes

$$s < 1.4 \times 10^9 \times k^3 \log k(1 + \log(s-1)) \ ,$$

and using that $k < \log s$,

$$s < 1.4 \times 10^9 \times (\log s)^3 \log \log s(1 + \log(s-1)) \ ,$$

which is valid only for $s \leq 9.55 \times 10^{15}$.

If $l = (m-1)/2$, we use Lemma 3.1 and (3.19) to get

$$\begin{aligned} \frac{m-1}{2} \;&<\; 1.4 \times 10^9 \times k^3 \log k(1 + \log(4.7 \times 10^{14} \times m^5 (\log m)^4)) \\ &<\; 1.4 \times 10^9 \times k^3 \log k(34.79 + 5 \log m + 4 \log \log m) \ , \end{aligned}$$

thus,

$$\frac{m}{\log m} < 3.7 \times 10^{10} \times k^3 \log k \ , \tag{3.20}$$

where we used that, for $m \leq 1395$, the inequality $\log \log m < 0.31 \log m$ holds, giving $34.79 + 5 \log m + 4 \log \log m < 34.79 + 6.24 \log m < 13 \log m$, and that $m - 1 > m/1.004$. Now, using again (3.6), we gain an upper bound for m in terms of k:

$$\begin{aligned} m \;&<\; 2(3.7 \times 10^{10} \times k^3 \log k) \log(3.7 \times 10^{10} \times k^3 \log k) \\ &< 1.93 \times 10^{12} k^3 (\log k)^2 \ . \end{aligned} \tag{3.21}$$

Again, by Lemma 3.1, now combined with (3.21), we have an upper bound for s in terms of k, which will give us, as in the previous case, an absolute upper bound for s:

$$\begin{aligned} s \;&<\; 4.7 \times 10^{11} \times (1.93 \times 10^{12} k^3 (\log k)^2)^5 \times (\log(1.93 \times 10^{12} k^3 (\log k)^2))^4 \\ &<\; 1.26 \times 10^{73} \times k^5 (\log k)^{10} (28.29 + 3 \log k + 2 \log \log k)^4 \\ \Rightarrow \ s \;&<\; 5.33 \times 10^{78} \times k^5 (\log k)^{14} \ , \end{aligned}$$

and since $k \leq \log s$, we have

$$s < 5.33 \times 10^{78} \times (\log s)^5 (\log\log s)^{14} \ ,$$

which is true only for $s < 7.31 \times 10^{100}$, so $k < \log(7.31 \times 10^{100}) < 232$.

In both cases, we have $s < 7.31 \times 10^{100}$ and $k < 232$, which are still very high to let the computer do the calculation. In order to reduce these bounds, we use a criterion due to Legendre (Theorem 2.3) about the convergents in a continued fraction. To use it, we go back to (3.17) to get, using that $s \geq 20$ and $m \geq 1394$, the following upper bound:

$$|\Lambda_2| < \frac{1}{\alpha^s} + \frac{2.39}{\alpha^{(m-1)/2}} < \frac{1}{\alpha^{20}} + \frac{2.39}{\alpha^{697}} < 1.38 \times 10^{-5}. \tag{3.22}$$

Set,

$$\Gamma_2 := (s-1)\log(g^{-1}) - (ms + 1 - n)\log\alpha \ .$$

Note that $\Lambda_2 = e^{\Gamma_2} - 1$, then by the previous inequality, we have

$$1.38 \times 10^{-5} > |\Lambda_2| = |e^{\Gamma_2} - 1| \geq |e^{\Gamma_2}| - 1 \ \Rightarrow \ e^{|\Gamma_2|} < 1.38 \times 10^{-5} + 1. \tag{3.23}$$

Now, since

$$|\Gamma_2| \leq e^{|\Gamma_2|}|e^{\Gamma_2} - 1| < (1.38 \times 10^{-5} + 1)|\Lambda_2| < (1.38 \times 10^{-5} + 1)\left(\frac{1}{\alpha^s} + \frac{2.39}{\alpha^{(m-1)/2}}\right) \ ,$$

we have

$$|(s-1)\log(g^{-1}) - (ms+1-n)\log\alpha| < (1.38 \times 10^{-5} + 1)\left(\frac{1}{\alpha^s} + \frac{2.39}{\alpha^{(m-1)/2}}\right)$$

then dividing it by $(s-1)\log\alpha$,

$$\left|\frac{\log(g^{-1})}{\log\alpha} - \frac{ms+1-n}{s-1}\right| < \frac{(1.38 \times 10^{-5} + 1)}{(s-1)\log\alpha}\left(\frac{1}{\alpha^s} + \frac{2.39}{\alpha^{(m-1)/2}}\right) \ . \tag{3.24}$$

Assume next that $s \geq 292$. Then $\alpha^s > (7/4)^{292} > 2.58 \times 10^{68} s$ and, since $m \geq 1395$, we have $\alpha^{(m-1)/2} > (7/4)^{697} > 2.58 \times 10^{68} s$, which in (3.24) gives us

$$\left|\frac{\log(g^{-1})}{\log\alpha} - \frac{ms+1-n}{s-1}\right| < \frac{1}{4.2 \times 10^{67}(s-1)^2} \ . \tag{3.25}$$

By Theorem 2.3, inequality (3.25) implies that the rational number $(ms+1-n)/(s-1)$ is a convergent to $\beta_k = (\log(g^{-1}))/(\log\alpha)$. Let $[a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \ldots]$ be the continued fraction of β_k and $p_t^{(k)}/q_t^{(k)}$ its t-th convergent. Assume that $(ms+1-n)/(s-1) = p_{t_k}^{(k)}/q_{t_k}^{(k)}$ for some t_k. Then $s - 1 = d_k q_{t_k}^{(k)}$, for some positive integer d_k, so $s - 1 \geq q_{t_k}^{(k)}$. On the other hand, using again the *Mathematica* software, we get that

$$\min_{k \in \{3,232\}} q_{250}^{(k)} > 4.87 \times 10^{118} > 7.31 \times 10^{100} - 1 > s - 1,$$

therefore $1 \leq t_k \leq 250$, for all $3 \leq k \leq 232$. Also using *Mathematica*, we observe that $a_{t_k+1} \leq \max\{a_t^{(k)}\} < 4.15 \times 10^{67}$, for $k \in \{3, 232\}$ and $t \in \{1, 251\}$. From the properties of continued fractions, we have

$$\begin{aligned}\left|\beta_k - \frac{ms+1-n}{s-1}\right| = \left|\beta_k - \frac{p_{t_k}^{(k)}}{q_{t_k}^{(k)}}\right| &> \frac{1}{(a_{t_k+1}+2)(q_{t_k}^{(k)})^2}\\ &\geq \frac{d_k^2}{4.15\times 10^{67}(s-1)^2}\\ &\geq \frac{1}{4.15\times 10^{67}(s-1)^2}\ ,\end{aligned}$$

which contradicts (3.25). So, $s \leq 291$, and $k <\in \{3,4,5\}$.

3.4. **The final step.** Let's go back to (3.16). Divide it across by $(1+\alpha^{-s})$ to obtain,

$$|\alpha^{n-(ms+1)}g^{1-s}(1+\alpha^{-s})^{-1}-1| < \frac{2.39}{\alpha^{(m-1)/2}}\ . \tag{3.26}$$

Now, set $t := ms+1-n$. Using inequality (3.22), we get the following relations

$$\begin{aligned} g^{1-s}\alpha^{-t}-1 &< 1.38\times 10^{-5}\\ \Rightarrow\ t &> \frac{(s-1)\log g^{-1}}{\log\alpha} - \frac{\log(1+1.38\times 10^{-5})}{\log\alpha}\\ &> 0.68s-0.69\ ,\end{aligned}$$

and,

$$\begin{aligned} g^{1-s}\alpha^{-t}-1 &> -1.38\times 10^{-5}\\ \Rightarrow\ t &< \frac{(s-1)\log g^{-1}}{\log\alpha} - \frac{\log(1-1.38\times 10^{-5})}{\log\alpha}\\ &< 1.27s-1.26\ .\end{aligned}$$

Therefore, $t \in [\lfloor 0.68s+0.31\rfloor, \lfloor 1.27s-1.26\rfloor]$. A computational search for the range $20 \leq s \leq 291$, $3 \leq k \leq 5$ and t in the previous interval, returned

$$\min\left\{\left|\frac{\alpha^{-t}g^{1-s}}{(1+\alpha^{-s})}-1\right|\right\} > 0.0003\ \Rightarrow \frac{2.39}{\alpha^{(m-1)/2}} > 0.0003 \Rightarrow m \leq 33\ ,$$

which contradicts the fact that $m \geq 1394$. Hence, the theorem is proved.

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