A DIOPHANTINE EQUATION RELATED TO THE SUM OF SQUARES OF CONSECUTIVE k-GENERALIZED FIBONACCI NUMBERS

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ABSTRACT. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. There are several interesting identities involving this sequence such as $F_n^2 + F_{n+1}^2 = F_{2n+1}$, for all $n \geq 0$. One of the most known generalization of the Fibonacci sequence, is the k-generalized Fibonacci sequence $(F_n^{(k)})_n$ which is defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and such that each term afterwards is the sum of the k preceding terms. In this paper, we prove that contrarily to the Fibonacci case, the Diophantine equation

$$(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(k)}$$

has no any solution in positive integers n, m and k, with $n > 1$ and $k \geq 3$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.$$

The Fibonacci numbers are well-known for possessing wonderful and amazing properties (consult [7] together with their very extensive annotated bibliography for additional references and history).

Among the several pretty algebraic identities involving Fibonacci numbers, we are interested in the following one

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \text{ for all } n \geq 0. \tag{1.1}$$

In particular, this naive identity (which can be proved easily by induction) tell us that the sum of the square of two consecutive Fibonacci numbers is still a Fibonacci number.

Let $k \geq 2$ and denote $F^{(k)} := (F_n^{(k)})_{n\geq -(k-2)}$, the *$k$-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)}, \tag{1.2}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k-step Fibonacci sequence*, the *Fibonacci k-sequence*, or *k-bonacci sequence*. Clearly for $k = 2$, we obtain the well-known Fibonacci numbers $F_n^{(2)} = F_n$, for $k = 3$, the Tribonacci numbers $F_n^{(3)} = T_n$ and for $k = 4$, the Tetranacci numbers $F_n^{(3)} = Q_n$. Recently, Melham [8, 9, 10] has published a series of papers presenting identities that (acording to him) can be regarded as higher order analogues of the identity (1.1). For instance, he proved the following identities for Tribonacci and Tetranacci numbers:

$$T_{n+3}^2 + T_{n+2}^2 + T_{n+1}^2 - T_n^2 = 2T_{2n} + 32T_{2n+1} + 3T_{2n+2}$$

and

$$Q_{n+6}^2 + Q_{n+5}^2 + 2Q_{n+4}^2 + 2Q_{n+3}^2 - 2Q_{n+2}^2 + Q_{n+1}^2 - Q_n^2$$
$$= 46Q_{2n} + 70Q_{2n+1} + 82Q_{2n+2} + 88Q_{2n+3}.$$

The aim of this paper is to study a similar equation than (1.1) in the k-generalized Fibonacci context. More precisely, we have the following

Theorem 1.1. *The Diophantine equation*

$$(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(k)} \tag{1.3}$$

has no any solution in positive integers n, m and k, with $n > 1$ and $k \geq 3$.

We recall that for $n = 1$, the triple $(n, m, k) = (1, 3, k)$ is a solution of Eq. (1.3) for all $k \geq 2$. When $n = 2$, one has $(F_2^{(k)})^2 + (F_3^{(k)})^2 = 1 + 2^2 = 5$, but on the other hand $F_m^{(k)}$ belongs to the increasing sequence $4, 7, 8, 13, 15, \ldots$, for $k \geq 3$ and $m \geq 4$. Thus, there is no solution of Eq. (1.3) for $n = 2$. In conclusion, we may suppose that $n \geq 3$.

2. Auxiliary results

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and 2 (as can be seen in [11]). Also, it was proved in [1, Lemma 1] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \text{ for all } n \geq 1, \tag{2.1}$$

where α is the dominant root of $\psi_k(x)$.

Recall that for $k = 2$, one has the useful Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $\alpha = (1 + \sqrt{5})/2 = -\beta^{-1}$. There are many closed formulas representing these k-generalized Fibonacci numbers, as can be seen in [3, 4, 5, 6]. However, we are interested in the simplified "Binet-like" formula due to G. Dresden [2, Theorem 1] :

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{2.2}$$

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. Also, the contribution of the roots inside the unit circle in formula (2.2) is almost trivial. More precisely, it was proved in [2] that

$$|E_n(k)| < \frac{1}{2}, \tag{2.3}$$

where $E_n(k) := F_n^{(k)} - g(\alpha, k)\alpha^{n-1}$ and $g(x, y) := (x - 1)/(2 + (y + 1)(x - 2))$.

Here are the values of $g(\alpha, k)$ with six decimal digits, respectively for $k = 3, 4, \ldots, 11$:

$$\{0.618419, 0.566342, 0.537926, 0.521772, 0.512454, 0.507071, 0.503980, 0.502220, 0.501227\}$$

We remark that, for $k \geq 12$, the value of $g(\alpha, k)$ is not greater than 0.502, as can be seen below

$$g(\alpha, k) = \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} < \frac{2 - 1}{2 - (k+1)/2^{k-1}} < 0.502 \quad \text{for } k \geq 12, \tag{2.4}$$

and that certainly, $g := g(\alpha, k) < 4/3$ in all cases. Moreover, we claim that $g > 1/\alpha$. In fact, we have

$$g\alpha - 1 = \frac{\alpha(\alpha-1) - (2+(k+1)(\alpha-2))}{(2+(k+1)(\alpha-2))} = \frac{\alpha^2 + k\overbrace{(2-\alpha)}^{>0} - 2\alpha}{2+(k+1)(\alpha-2)} \geq \frac{\alpha^2 - 5\alpha + 6}{2+(k+1)(\alpha-2)} > 0, \tag{2.5}$$

where (2.5) follows from the fact that $k \geq 3$, the inequality $x^2 - 5x + 6 > 0$ for all $1 < x < 2$ and that $2 + (k+1)(\alpha - 2) > 2 - (k+1)/2(k-1) \geq 1$, for all $k \geq 3$. On the other hand, (2.5) gives

$$\begin{aligned} \frac{\alpha(\alpha-1) - (2+(k+1)(\alpha-2))}{2+(k+1)(\alpha-2)} > 0 \quad &\Rightarrow \quad \frac{\alpha(\alpha-1)}{2+(k+1)(\alpha-2)} - 1 > 0 \\ &\Rightarrow \quad \frac{\alpha-1}{2+(k+1)(\alpha-2)} > \frac{1}{\alpha} \\ &\therefore \quad g > \frac{1}{\alpha} \end{aligned} \tag{2.6}$$

Now, we are ready to deal with the proof of Theorem 1.1.

3. Proof of Theorem 1.1

First, observe that by using the estimates in (2.1), we obtain

$$(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 > \alpha^{2n-4} + \alpha^{2n-2} = \alpha^{2n-4}(1+\alpha^2) > \alpha^{2n-2}$$

and

$$(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 < \alpha^{2n-2} + \alpha^{2n} = \alpha^{2n-2}(1+\alpha^2) < \alpha^{2n+1},$$

where we used that $1 + \alpha^2 < \alpha^3$ for $k \geq 3$. Now, the estimate $\alpha^{m-2} < F_m^{(k)} < \alpha^{m-1}$ together with the previous estimates yield $2n - 1 < m < 2n + 3$. In conclusion, we have proved that if (m, n, k) is a solution of Eq. (1.3), then $m \in \{2n, 2n+1, 2n+2\}$.

First, if $n \leq 20$ and $3 \leq k \leq 42$, then a finite computation shows no solutions for (1.3), whereas if $n \leq 20$ and $k > 42$ then all three $F_n^{(k)}$, $F_{n+1}^{(k)}$ and $F_m^{(k)}$, are distinct powers of 2:

$$F_n^{(k)} = 2^{n-2}, \quad F_{n+1}^{(k)} = 2^{n-1} \quad \text{and} \quad F_m^{(k)} \in \{2^{2n-2}, 2^{2n-1}, 2^{2n}\}.$$

So the sum of the first two squares cannot be the third value, since no two distinct powers of 2 sum up to another power of 2. So, from now on $n > 20$ independently of k. Let us denote

$$F_n^{(k)} = g\alpha^{n-1} + E_n(k), \tag{3.1}$$

where $E_n(k)$ is defined as before. Let $m = 2n + i$, $i \in \{0, 1, 2\}$. Then the equation (1.3) can be written as

$$(g\alpha^{n-1} + E_n(k))^2 + (g\alpha^n + E_{n+1}(k))^2 = g\alpha^{2n+i-1} + E_{2n+i}(k).$$

Divide across by α^{2n-2} to get

$$(g + E_n(k)/\alpha^{n-1})^2 + (g\alpha + E_{n+1}(k)/\alpha^{n-1})^2 = g\alpha^{i+1} + E_{2n+i}(k)/\alpha^{2n-2}. \tag{3.2}$$

We write

$$(g + E_n(k)/\alpha^{n-1})^2 = g^2 + C_1,$$

where

$$\begin{aligned} |C_1| &= |2gE_{n+1}(k)/\alpha^{n-1} + (E_n(k)/\alpha^{n-1})^2| \\ &\leq 2 \times (4/3) \times (1/2) \times \alpha^{-(n-1)} + (1/4) \times \alpha^{-2(n-1)} \\ &< 2 \times 2 \times (4/3) \times (1/2) \times \alpha^{-(n-1)} \\ &< 3/\alpha^{n-1}. \end{aligned} \tag{3.3}$$

Similarly,

$$(g\alpha + E_{n+1}(k)/\alpha^{n-1})^2 = g^2\alpha^2 + C_2,$$

where

$$\begin{aligned} |C_2| &= |2g\alpha E_{n+1}(k)/\alpha^{n-1} + (E_{n+1}(k)/\alpha^{n-1})^2| \\ &< 2 \times (4/3) \times 2 \times (1/2) \times (2/\alpha^{n-1}) \\ &< 6/\alpha^{n-1}. \end{aligned} \tag{3.4}$$

Since $C_3 = E_{2n+i}/\alpha^{2n-2} < 1/\alpha^{n-1}$, we obtain from (3.2), (3.3) and (3.4) that

$$\begin{aligned} |g + g\alpha^2 - \alpha^{i+1}| &= \frac{1}{g}|C_3 - C_1 - C_2| \\ &< 2 \times \left(\frac{1}{\alpha^{n-1}} + \frac{3}{\alpha^{n-1}} + \frac{6}{\alpha^{n-1}}\right) = \frac{20}{\alpha^{n-1}} \end{aligned} \tag{3.5}$$

Computing the left hand side of (3.5) for $k = 3, 4, \ldots, 11$ and $i \in \{0, 1, 2\}$ with the values of g given previously, we obtain

$$0.505 < |g + g\alpha^2 - \alpha^{i+1}| < \frac{20}{\alpha^{n-1}} < \frac{20}{1.5^{n-1}},$$

which contradicts the fact that $n > 20$.

Now, suppose that $k \geq 12$. If $i = 0$, we see that in the left-hand side of inequality (3.5) we have, since $g\alpha > 1$, that

$$g + g\alpha^2 - \alpha = g + (g\alpha)\alpha - \alpha > g > 0.5,$$

leading to $1.5^{n-1} < 40$, which is false for $n > 20$. So, $i \in \{1, 2\}$.

If $i = 1$, then

$$\alpha > 2(1 - 2^{-12}) = \frac{4095}{2048} > 1.99,$$

and $g < 0.502$, so $\alpha^2 - g\alpha^2 - g > (1.99)^2 - 0.502 \times 2 - 0.502 > 2.45$, and we get

$$2.45 < \alpha^2 - c\alpha^2 - c \leq |g + g\alpha^2 - \alpha^2| < \frac{20}{\alpha^{n-1}} \Rightarrow \alpha^{n-1} < 8.17$$

which is false for $\alpha > 1.99$ and $n > 20$.

Similarly, if $i = 2$ we also have $g < 0.502$ and $\alpha > 1.99$, so $\alpha^3 - g\alpha^2 - g > (1.99)^3 - 0.502 \times 2 - 0.502 > 6.37$, which is even larger than the previous one, giving us again a contradiction.

This completes the proof. □

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