A New Fibonacci-Lucas Relation

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Very recently, B. Sury [1] proved an interesting and new Fibonacci-Lucas relation, namely,

$$2^{m+1}F_{m+1} = \sum_{i=0}^{m} 2^i L_i.$$

In his proof, Sury used a simple polynomial identity (see [2] for another proof). Naturally, a question arise: is there a similar formula for $3^{m+1}F_{m+1}$? In this note, we shall provide such formula. More precisely

Theorem. *For all $m \geq 1$, it holds that*

$$3^{m+1}F_{m+1} = \sum_{i=0}^{m} 3^i L_i + \sum_{i=0}^{m+1} 3^{i-1} F_i.$$

Proof. We can proceed by induction on m. Clearly, the basis case is straightforward, so we may suppose that the identity holds for $m \in \{k-1, k\}$. Now, we sum these relations and by adding $3^k L_k + 2 \cdot 3^{k+1} L_{k+1} + 3^k F_{k+1} + 2 \cdot 3^{k+1} F_{k+3}$ in each side, we arrive at

$$2\left(\sum_{i=0}^{k+1} 3^i L_i + \sum_{i=0}^{k+2} 3^{i-1} F_i\right) = 3^k(\underbrace{F_k + 4F_{k+1} + L_k + 6L_{k+1} + 6F_{k+2}}_{18F_{k+2}}),$$

where we used that $L_t = F_{t-1} + F_{t+1}$. ■

References

[1] B. Sury, A polynomial parent to a Fibonacci-Lucas relation. *The American Mathematical Monthly* **121**, No. 3, p. 236.

[2] H. Kwong, An alternate proof of Sury's Fibonacci-Lucas relation, *The American Mathematical Monthly* **121**, No. 6, p. 51.

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