On the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$

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Abstract. In this paper, we find all solutions of the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$ in positive integers $x, y \geq 1$, $\alpha, \beta, \gamma, n \geq 3$ with $\gcd(x, y) = 1$.

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1. Introduction

The interest for the diophantine equation

$$x^2 + C = y^n, \quad x \geq 1, \quad y \geq 1, \quad n \geq 3 \tag{1.1}$$

started with a paper due to Lebesgue [18] and dating back to 1850, where he proved that the above equation has no solutions for $C = 1$. More recently, other values of C were considered. Tengely [24] solved the equation with $C = b^2$ and $3 \leq b \leq 501$. The case where $C = p^k$, a power of a prime number, was studied in [7, 16, 17] for $p = 2$, in [5, 6, 19] for $p = 3$, in [1, 2] for $p = 5$, and in [22] for $p = 7$. For arbitrary primes, some advances can be found in [4]. In [9], the cases with $1 \leq C \leq 100$ were completely solved. The solutions for the cases $C = 2^a \cdot 3^b$, $C = 2^a \cdot 5^b$ and $C = 5^a \cdot 13^b$, when x and y are coprime, can be found in [3, 20, 21], respectively. Recent progress on the subject were made in the cases $C = 5^a \cdot 11^b$, $C = 2^a \cdot 11^b$, $C = 2^a \cdot 3^b \cdot 11^c$, $C = 2^a \cdot 5^b \cdot 13^c$ and can be found in [10, 11, 12, 14].

In this paper, we are interested in solving the Diophantine equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = y^n, \ \gcd(x, y) = 1, \ x, y \geq 1, \ \alpha, \beta, \gamma \geq 0, \ n \geq 3. \tag{1.2}$$

Our result is the following.

Theorem 1.1. *The equation* (1.2) *has no solution except for:*

$n = 3$ *the solutions given in Table* 1*;*
$n = 4$ *the solutions given in Table* 2*;*
$n = 5$ $(x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)$;

$n = 6 \quad (x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0);$
$n = 8 \quad (x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0).$

One can deduce from the above result the following corollary.

Corollary 1.2. *The equation*

$$x^2 + 5^k 17^l = y^n, \quad x \geq 1, \quad y \geq 1, \quad \gcd(x, y) = 1, \quad n \geq 3, \quad k \geq 0, \ l \geq 0 \quad (1.3)$$

has only the solution

$$(x, y, k, l, n) = (94, 21, 2, 1, 3), \ (2034, 161, 3, 2, 3), \ (8, 3, 0, 1, 4).$$

Therefore, our work extends that of Pink and Rábai [23].

α_1	β_1	γ_1	z	α	β	γ	x	y
1	0	0	1	1	0	0	5	3
1	0	0	$2 \cdot 5$	7	6	0	383	129
2	0	0	1	2	0	0	11	5
4	0	1	5	4	6	1	5369	321
3	0	2	5	3	6	2	167589	3041
1	1	1	2^2	13	1	1	93	89
1	1	1	5	1	7	1	1531	171
1	1	1	1	1	1	1	453	59
3	1	1	1	3	1	1	7	9
1	1	2	1	1	1	2	63	19
2	1	2	1	2	1	2	59	21
1	1	3	2	7	1	3	5471	321
1	1	3	5	1	7	3	17052501	66251
3	2	0	1	3	2	0	23	9
3	2	0	2	9	2	0	17771	681
5	2	0	1	5	2	0	261	41
0	2	1	1	0	2	1	94	21
0	2	1	2	6	2	1	55157	1449
3	3	1	2	9	3	1	10763	489
3	3	1	2^2	15	3	1	4617433	27729
0	3	2	1	0	3	2	2034	161
3	3	5	2^5	33	3	5	2037783243169	160733121
1	4	0	1	1	4	0	9	11
4	4	1	$2 \cdot 5$	10	10	1	3274947	22169
5	4	2	$2 \cdot 5$	11	10	2	699659581	788121
1	5	0	17	1	5	6	916769	9971
1	5	1	17	1	5	7	846227	14859
1	5	1	2	7	5	1	17579	681

TABLE 1. Solutions for $n = 3$.

α_1	β_1	γ_1	z	α	β	γ	x	y
1	0	0	2	5	0	0	7	3
0	1	0	2	4	1	0	1	3
0	0	1	2^2	8	0	1	1087	33
0	0	1	1	0	0	1	8	3
0	0	1	2^2	8	0	1	47	9
1	0	1	2	5	0	1	9	5
3	0	1	2	7	0	1	15	7
3	0	1	2^2	11	0	1	495	23
2	1	0	2	6	1	0	79	9
2	2	1	2	6	2	1	409	21
3	2	2	2	7	2	2	511	33
1	0	3	2^2	9	0	3	4785	71

TABLE 2. Solutions for $n = 4$.

2. The case $n = 3$

Lemma 2.1. *When $n = 3$, all the solutions to equation* (1.2) *are given in Table* 1.

For $n = 6$, we have $(x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0)$.

Proof. Equation (1.2) can be rewritten as

$$\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3, \tag{2.1}$$

where A is sixth-power free and defined implicitly by $2^\alpha 5^\beta 17^\gamma = Az^6$. One can see that $A = 2^{\alpha_1} 5^{\beta_1} 17^{\gamma_1}$ with α_1, β_1, γ_1, $\in \{0, 1, 2, 3, 4, 5\}$. We thus get

$$V^2 = U^3 - 2^{\alpha_1} 5^{\beta_1} 17^{\beta_1}, \tag{2.2}$$

with $U = y/z^2$, $V = x/z^3$ and α_1, β_1, γ_1 $\in \{0, 1, 2, 3, 4, 5\}$. We need to determine all the $\{2, 5, 17\}$-integral points on the above 216 elliptic curves. Recall that if $\mathcal{S}$ is a finite set of prime numbers, then an $\mathcal{S}$-integer is rational number a/b with coprime integers a and b, where the prime factors of b are in $\mathcal{S}$. We use MAGMA [13] to determine all the $\{2, 5, 17\}$-integer points on the above elliptic curves. Here are a few remarks about the computations:

1. We eliminate the solutions with $UV = 0$ because they yield to $xy = 0$.
2. We consider only solutions such that the numerators of U and V are coprime.
3. If U and V are integers then $z = 1$. So $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\gamma_1 = \gamma$.
4. If U and V are rational numbers which are not integers, then z is determined by the denominators of U and V. The numerators of these rational numbers give x and y. Then α, β, γ are computed knowing that $2^\alpha 5^\beta 17^\gamma = Az^6$.

Therefore, we first determine $(U, V, \alpha_1, \beta_1, \gamma_1)$ and then we use the relations

$$U = \frac{y}{z^2}, \;\; V = \frac{x}{z^3}, \;\; 2^\alpha 5^\beta 17^\gamma = Az^6,$$

to find the solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 1.

For $n = 6$, equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = y^6 \tag{2.3}$$

becomes equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = \left(y^2\right)^3. \tag{2.4}$$

We look in the list of solutions of equation Table 1 and observe that the only solutions in Table 1 whose y is a perfect square. Therefore, the only solutions to equation (1.2) for $n = 6$ are the two solutions listed in Theorem 1.1. This completes the proof of Lemma 2.1. □

3. The case $n = 4$

Here, we have the following result.

Lemma 3.1. *If $n = 4$, then the only solutions to equation* (1.2) *are given in Table* 2.

If $n = 8$, then the only solution to equation (1.2) *is* $(x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0)$.

Proof. Equation (1.2) can be written as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4, \tag{3.1}$$

where A is fourth-power free and defined implicitly by $2^\alpha\, 5^\beta\, 17^\gamma = Az^4$. One can see that $A = 2^{\alpha_1}\, 5^{\beta_1}\, 17^{\gamma_1}$ with $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3\}$. Hence, the problem consists of determining the $\{2, 5, 17\}$-integer points on the totality of the 64 elliptic curves

$$V^2 = U^4 - 2^{\alpha_1}\, 5^{b_1}\, 17^{\gamma_1}, \tag{3.2}$$

with $U = y/z,\ V = x/z^2$ and $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3\}$. Here, we use again `MAGMA` [13] to determine the $\{2, 5, 17\}$-integer points on the above elliptic curves. As in Section 2, we first find $(U, V, \alpha_1, \beta_1, \gamma_1)$, and then using the coprimality conditions on x and y and the definition of U and V, we determine all the corresponding solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 2.

Looking in the list of solutions of equation Table 2, we observe the 8 solutions in Table 2 whose values for y are perfect squares. Thus, the only solutions to equation (1.2) with $n = 8$ are those listed in Theorem 1.1. This concludes the proof of Lemma 3.1. □

4. The case $n \geq 5$

The aim of this section is to determine all solutions of equation (1.2), for $n \geq 5$ and to prove its unsolubility for $n = 7$ and $n \geq 9$. The cases when n is of the form $2^a 3^b$ were treated in previous sections. So, apart from these cases, in order to prove that (1.2) has no solution for $n \geq 7$, it suffices to consider n prime. In fact, if $(x, y, \alpha, \beta, \gamma, n)$ is a solution for (1.2) and $n = pk$, where $p \geq 7$ is prime and $k > 1$, then $(x, y^k, \alpha, \beta, \gamma, p)$ is also a solution. So, from now on, n will denote a prime number.

Lemma 4.1. *The Diophantine equation* (1.2) *has no solution with* $n \geq 5$ *prime except for*

$n = 5 \qquad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0);$

Proof. Let $(x, y, \alpha, \beta, \gamma, n)$ be a solution for (1.2). We claim that y is odd. In fact, if y is even and since $\gcd(x, y) = 1$, one has that x is odd, and then $-2^\alpha 5^\beta 17^\gamma \equiv x^2 - y^n \equiv 1 \pmod 4$, but this contradicts the fact that $-2^\alpha 5^\beta 17^\gamma \equiv 0, 2$ or $3 \pmod 4$. Now, write equation (1.2) as $x^2 + dz^2 = y^n$, where

$$d = 2^{\alpha - 2\lfloor \alpha/2 \rfloor} 5^{\beta - 2\lfloor \beta/2 \rfloor} 17^{\gamma - 2\lfloor \gamma/2 \rfloor},$$

and $z = 2^{\lfloor \alpha/2 \rfloor} 5^{\lfloor \beta/2 \rfloor} 17^{\lfloor \gamma/2 \rfloor}$. Since $x - 2\lfloor x/2 \rfloor \in \{0, 1\}$, we have

$$d \in \{1, 2, 5, 10, 17, 34, 85, 170\}.$$

We then factor the previous equation over $\mathbb{K} = \mathbb{Q}[i\sqrt{d}] = \mathbb{Q}[\sqrt{-d}]$ as

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n.$$

Now, we claim that the ideals $(x + i\sqrt{d}z)\mathcal{O}_\mathbb{K}$ and $(x - i\sqrt{d}z)\mathcal{O}_\mathbb{K}$ are coprime. If this is not the case, there must exist a prime ideal $\mathfrak{p}$ containing these ideals. Therefore, $x \pm i\sqrt{d}z$ and y^n (and so y) belong to $\mathfrak{p}$. Thus $2x \in \mathfrak{p}$ and hence either 2 or x belongs to $\mathfrak{p}$. Since $\gcd(2, y) = \gcd(x, y) = 1$, then 1 belongs to the ideals $\langle 2, y \rangle$ and $\langle x, y \rangle$, then $1 \in \mathfrak{p}$ leading to an absurdity of $\mathfrak{p} = \mathcal{O}_\mathbb{K}$. By the unique factorization of ideals, it follows that $(x + i\sqrt{d}z)\mathcal{O}_\mathbb{K} = \mathfrak{j}^n$, for some ideal $\mathfrak{j}$ of $\mathcal{O}_\mathbb{K}$. Using the Mathematica command

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NumberFieldClassNumber[Sqrt[-d]]
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we obtain that the class number of $\mathbb{K}$ is either $1, 2, 4$ or 12 and so coprime to n, then $\mathfrak{j}$ is a principal ideal yielding

$$x + i\sqrt{d}z = \varepsilon \eta^n, \tag{4.1}$$

for some $\eta \in \mathcal{O}_\mathbb{K}$ and ε a unit of $\mathbb{K}$. Since the group of units of $\mathbb{K}$ is a subset of $\{\pm 1, \pm i\}$ and n is odd, then ε is a n-th power. Thus, (4.1) can be reduced to $x + i\sqrt{d}z = \eta^n$. Since $\mathbb{K}$ is an imaginary quadratic field and $-d \not\equiv 1 \pmod 4$, then $\{1, i\sqrt{d}\}$ is an integral basis and we can write $\eta = u + i\sqrt{d}v$, for some integers u and v. We then get

$$\frac{\eta^n - \overline{\eta}^n}{\eta - \overline{\eta}} = \frac{2^{\lfloor \alpha/2 \rfloor} 5^{\lfloor \beta/2 \rfloor} 17^{\lfloor \gamma/2 \rfloor}}{v}, \tag{4.2}$$

where, as usual, $\overline{w}$ denotes the complex conjugate of w.

Let $(L_m)_{m\geq 0}$ be the Lucas sequence given by

$$L_m = \frac{\eta^m - \overline{\eta}^m}{\eta - \overline{\eta}}, \text{ for } m \geq 0.$$

We recall that the Primitive Divisor Theorem for Lucas sequences ensures for primes $n \geq 5$, that there exists a *primitive divisor* for L_n, except for the finitely many (*defective*) pairs $(\eta, \overline{\eta})$ given in Table 1 of [8] (a primitive divisor of L_n is a prime that divides L_n but does not divide $(\eta-\overline{\eta})^2 \prod_{j=1}^{n-1} L_j$). And a helpful property of a primitive divisor p is that $p \equiv \pm 1 \pmod{n}$.

For $n = 5$, we find in Table 1 in [8] that L_5 has a primitive divisor except for $(u, d, v) = (1, 10, 1)$ which leads to a number $\eta = 1 + i\sqrt{10} \in \mathbb{Q}[i\sqrt{10}]$ ($d = 10$ is one of the possible values of d described in the beginning of this proof), which gives the solution with $n = 5$.

Apart from this case, let p be a primitive divisor of L_n, $n \geq 7$. The identity (4.2) implies that $p \in \{2, 5, 17\}$ and so $p = 17$, since $p \not\equiv \pm 1 \pmod{n}$, for $p = 2, 5$. Hence, n is a prime dividing 17 ± 1 and so $n = 2$ or 3 which contradicts the fact that $n \geq 7$. This completes the proof of Theorem 1.1.

□

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