A FAMILY OF MINIMAL IMMERSED TORI IN $\mathbb{S}^3$ WITH DENSE PRINCIPAL LINES

RONALDO GARCIA AND DIEGO MARQUES

ABSTRACT. In this paper is given an example of a discrete family of minimal tori $\mathbb{T}^2$ immersed in $\mathbb{S}^3$ such that all their principal lines are dense. A relation between dynamics and transcendental number theory is established.

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1. Introduction

Let $\alpha : \mathbb{M} \to \mathbb{S}^3$ be an immersion of class $C^r, r \geq 3$, of a smooth, compact and oriented two-dimensional manifold $\mathbb{M}$ into the three dimensional sphere $\mathbb{S}^3$ endowed with the canonical inner product $\langle .,. \rangle$ of $\mathbb{R}^4$.

The *Fundamental Forms* of α at a point p of $\mathbb{M}$ are the symmetric bilinear forms on $\mathbb{T}_p\mathbb{M}$ defined as follows, see [16]:

$$I_\alpha(p; v, w) = \langle D\alpha(p; v), D\alpha(p; w)\rangle,$$
$$II_\alpha(p; v, w) = \langle -DN_\alpha(p; v), D\alpha(p; w)\rangle.$$

Here, N_α is the positive unit normal of the immersion α and $\langle N_\alpha, \alpha\rangle = 0$.

In a local chart (u, v) the two fundamental forms are denoted by $I_\alpha = E_\alpha du^2 + 2F_\alpha dudv + G_\alpha dv^2$ and $II_\alpha = e_\alpha du^2 + 2f_\alpha dudv + g_\alpha dv^2$.

We recall that in $\mathbb{S}^3$, with the second fundamental form relative to the normal vector $N = \alpha \wedge \alpha_u \wedge \alpha_v$, it follows that:

$$e = \frac{\det[\alpha, \alpha_u, \alpha_v, \alpha_{uu}]}{\sqrt{E_\alpha G_\alpha - F_\alpha^2}},\ f = \frac{\det[\alpha, \alpha_u, \alpha_v, \alpha_{uv}]}{\sqrt{E_\alpha G_\alpha - F_\alpha^2}},\ g = \frac{\det[\alpha, \alpha_u, \alpha_v, \alpha_{vv}]}{\sqrt{E_\alpha G_\alpha - F_\alpha^2}}.$$

The eigenvalues $k_1 \leq k_2$ of $II_\alpha - kI_\alpha = 0$ are called *principal curvatures* and the corresponding eigenspaces are called *principal directions.*

The umbilic set of α is defined by $\mathcal{U}_\alpha = \{p \in \mathbb{M} :\ k_1(p) = k_2(p)\}$.

These two line fields, called the *principal line fields* of α are of class C^{r-2} on $\mathbb{M} \setminus \mathcal{U}_\alpha$; they are distinctly defined.

The principal directions of α are defined by the implicit differential equation

$$(F_\alpha g_\alpha - G_\alpha f_\alpha)dv^2 + (E_\alpha g_\alpha - G_\alpha e_\alpha)dudv + (E_\alpha f_\alpha - F_\alpha e_\alpha)du^2 = 0 \quad (1)$$

When the surface $\mathbb{M}^2$ is oriented the principal lines can be assembled in two one-dimensional orthogonal foliations which will be denoted by $\mathcal{F}_1(\alpha)$

and $\mathcal{F}_2(\alpha)$. The umbilic set $\mathcal{U}_\alpha$ is the singular set of both foliations. The triple $\mathcal{P}_\alpha = \{\mathcal{F}_1(\alpha), \mathcal{F}_2(\alpha), \mathcal{U}_\alpha\}$ is called the *principal configuration* of the immersion α, [6, 7].

A principal line γ is called *recurrent* if $\gamma \subseteq L(\gamma)$, where $L(\gamma)$ is the limit set of γ, and it is called *dense* if $L(\gamma) = \mathbb{M}$.

The qualitative behavior of principal lines on surfaces was initiated by G. Monge [13] who introduced this concept and described the global behavior of these curves on the triaxial ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

The global behavior of principal lines is known only in very special class of surfaces, including quadrics and cyclides of Dupin, which are part of a triple orthogonal system of surfaces. In these cases the principal lines are closed, or a connection of umbilic separatrices. See Fig. 1.

FIGURE 1. Principal lines of the ellipsoid and of a Dupin cyclide.

The first examples of nontrivial recurrent principal lines was given by Gutierrez and Sotomayor, see [6, 7, 8] and [4].

In a recent work of the first author with Sotomayor [5], using methods of perturbation theory, is presented examples of embedded tori (deformation of the Clifford torus) with both principal foliations having dense leaves.

For a survey of recent works about qualitative theory of principal lines see [3].

The study of foliations with dense leaves goes back to Poincaré, Birkhoff, Denjoy, Peixoto, among others, [14].

In this paper is presented an explicit example of a family of minimal immersed tori having all principal lines dense. The result is obtained showing that both Poincaré return maps have transcendental rotation number and therefore a strong relation between number theory and dynamical aspects of principal foliations is obtained.

2. Family of Immersed Minimal Tori

Consider the family of immersions $\alpha_{m,n} = \alpha$ defined by:

$$\alpha_{m,n}(u, v) = (\cos mv \sin u, \sin mv \sin u, \cos nv \cos u, \sin nv \cos u). \quad (2)$$

Let $N_\alpha = (\alpha \wedge \alpha_u \wedge \alpha_v)/|\alpha \wedge \alpha_u \wedge \alpha_v|$. Then

$$N_\alpha = \frac{(-n \cos u \sin mv, n \cos u \cos mv, m \sin u \sin nv, -m \sin u \cos nv)}{\sqrt{m^2 + (n^2 - m^2)\cos^2 u}}.$$

The coefficients of the first and second fundamental forms of α are given by:

$$
\begin{aligned}
E_\alpha(u,v) =& \langle \alpha_u, \alpha_u\rangle = 1\\
F_\alpha(u,v) =& \langle \alpha_u, \alpha_v\rangle = 0\\
G_\alpha(u,v) =& \langle \alpha_v, \alpha_v\rangle = m^2 + (n^2 - m^2)\cos^2 u\\
f_\alpha(u,v) =& \langle \alpha_{uv}, N_\alpha\rangle = \frac{mn}{\sqrt{m^2 + (n^2 - m^2)\cos^2 u}}\\
e_\alpha(u,v) =& \langle \alpha_{uu}, N_\alpha\rangle = 0\\
g_\alpha(u,v) =& \langle \alpha_{vv}, N_\alpha\rangle = 0.
\end{aligned}
\tag{3}
$$

Therefore the mean curvature $\mathcal{H}_\alpha = (E_\alpha g_\alpha + e_\alpha G_\alpha - 2f_\alpha F_\alpha)/(E_\alpha G_\alpha - F_\alpha^2) = 0$ and the extrinsic Gaussian curvature is $\mathcal{K}_{\alpha,ext} = (e_\alpha g_\alpha - f_\alpha^2)/(E_\alpha G_\alpha - F_\alpha^2) < 0$.

The coordinate curves are closed and they are the asymptotic lines of $\alpha_{m,n}$.

Proposition 1. *Let $\alpha_{m,n} : [0,2\pi]\times[0,2\pi] \to \mathbb{S}^3$ be defined by equation 2. Then $\alpha_{m,n}$ is an immersed torus of $\mathbb{S}^3$, free of umbilic points. Both principal foliations $\mathcal{F}_i(\alpha_{m,n})$, (i = 1, 2) are regular and the Poincaré return maps defined $\pi^i : \{u = 0\} \to \{u = 2\pi\}$ (i = 1, 2) have the same rotation number*

$$\rho_{m,n} = \frac{2\mathbf{K}(\frac{\sqrt{m^2-n^2}}{m})}{\pi m}.$$

Here $\mathbf{K}$ is the elliptic integral defined by $\mathbf{K}(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$

Proof. The principal lines are defined by the following differential equation.

$$(F_\alpha g_\alpha - f_\alpha G_\alpha)dv^2 + (E_\alpha g_\alpha - G_\alpha e_\alpha)dudv + (E_\alpha f_\alpha - F_\alpha e_\alpha)du^2 = 0.$$

By equation (3) it follows that this equation can be solved as:

$$\frac{dv}{du} = \pm\sqrt{\frac{E_\alpha}{G_\alpha}}, \quad v(0, v_0) = v_0.$$

Therefore,

$$v(2\pi) - v(0) = \pm\int_0^{2\pi} \frac{1}{\sqrt{m^2 + (n^2 - m^2)\cos^2 u}} du = \pm\frac{4\mathbf{K}(\frac{\sqrt{m^2-n^2}}{m})}{m}.$$

So the rotation number of both Poincaré maps defined by

$$\pi^i : \{u = 0\} \to \{u = 2\pi\},\ \pi^i(v_0) = v_0 \pm v(2\pi, v_0) = v_0 \pm \frac{4\mathbf{K}(\frac{\sqrt{m^2-n^2}}{m})}{m}.$$

is given by

$$\rho_{m,n} = \frac{2\mathbf{K}(\frac{\sqrt{m^2-n^2}}{m})}{\pi m}.$$

□

Remark 1. *For all positive integers m and n, we have*

$$\frac{\mathbf{K}(\frac{\sqrt{n^2-m^2}}{n})}{n} = \frac{\mathbf{K}(\frac{\sqrt{m^2-n^2}}{m})}{m},$$

and so $\rho_{m,n} = \rho_{n,m}$.

Remark 2. *For all positive integers m and n, we have*

$$\rho_{m,n} \cdot MG(m,n) = 1,$$

where $MG(m,n)$ is the arithmetic-geometric Gauss mean. See [2].

Remark 3. *Consider the homogeneous polynomial of degree 3, $f(x,y,z,w) = -2xyz + w(x^2 - y^2)$. For $n = 2$ and $m = 1$ the trace of the immersion $\alpha_{1,2}$ defined by equation* (2) *is contained in the algebraic set $f^{-1}(0)$. See also* [11] *and* [15].

FIGURE 2. Stereographic projection in $\mathbb{R}^3$ of the immersed torus $\alpha_{1,2}$

3. On the arithmetic nature of $\rho_{m,n}$

The aim of this section is to study the arithmetic nature (i.e., to decide when a given complex number is a rational number, an irrational algebraic number or else a transcendental number) of the numbers $\rho_{m,n}$ for positive integers m and n. Since $\rho_{m,m} = \frac{1}{m}$ is rational, we may consider $m \neq n$.

Here, we shall prove the following result

Theorem 1. For any distinct positive integers m and n, the number $\rho_{m,n}$ is transcendental.

First, in order to prove the previous result, some tools will be essential ingredients.

Let $F(a,b,c;z)$ be the *Gauss hypergeometric function*, given by

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(x)_n = x(x+1)\cdots(x+n-1)$, if $n > 0$ and $(x)_0 = 1$.

Lemma 1. *For all $z \in \mathbb{C}$, with $|z| < 1$, we have*

$$\mathbf{K}(z) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; z^2\right). \tag{4}$$

Proof. A well-known integral representation of $F(a,b,c;z)$, see [9] page 1040, is given by

$$F(a,b,c;w)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-wt)^{-a}dt,$$

where $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$ is the *Gamma function.* Taking $a=b=\frac{1}{2}, c=1$ and $w=z^2$, we obtain

$$F\left(\frac{1}{2},\frac{1}{2},1;z^2\right)=\frac{\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1\frac{dt}{\sqrt{t(1-t)(1-z^2t)}}$$

Since $\Gamma(1)=1,\ \Gamma(\frac{1}{2})=\sqrt{\pi}$ and by applying the trigonometric change of variables $t=\sin^2 x$, we get

$$F\left(\frac{1}{2},\frac{1}{2},1;z^2\right)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{(1-z^2\sin^2 x)}}=\frac{2}{\pi}\mathbf{K}(z),$$

as desired. □

Note that we can deduce from Lemma 1 that the rotation number can be written as

$$\rho_{m,n}=\frac{1}{n}F\left(\frac{1}{2},\frac{1}{2},1;\frac{n^2-m^2}{n^2}\right). \tag{5}$$

Some of these values, can be seen in the following table

n	$\rho_{1,n}$	numeric	$\rho_{1,n}$
2	$2\mathbf{K}(-\sqrt{3})/\pi$	0.68644...	$\frac{1}{2}F(\frac{1}{2},\frac{1}{2},1;\frac{3}{4})$
3	$2\mathbf{K}(-2\sqrt{2})/\pi$	0.53659...	$\frac{1}{2}F(\frac{1}{2},\frac{1}{2},1;\frac{1}{4})$
4	$\mathbf{K}(-\sqrt{15})/2\pi$	0.44582...	$\frac{1}{4}F(\frac{1}{2},\frac{1}{2},1;\frac{15}{16})$
5	$2\mathbf{K}(-2\sqrt{6})/\pi$	0.38402...	$\frac{1}{3}F(\frac{1}{2},\frac{1}{2},1;\frac{4}{9})$

The next lemma, crucial to Theorem 1, is already known. Indeed, the transcendence of hypergeometric functions at algebraic values have been studied for several authors (see [1] and its extensive bibliography). However, we shall prove it for the sake of completeness.

Lemma 2. *If* $\alpha\in\overline{\mathbb{Q}}\backslash\{0,1\}$, *with* $|\alpha|<1$ *and* $|\arg\alpha|<\pi$, *then* $F\left(\frac{1}{2},\frac{1}{2},1;\alpha\right)$ *is transcendental. Here, as usual,* $\overline{\mathbb{Q}}$ *is the algebraic closure of* $\mathbb{Q}$ *in* $\mathbb{C}$.

Before the proof, we shall give a brief overview of this standard approach.

Let ω_1 and ω_2 be generators of a non-degenerated lattice in $\mathbb{C}$. The *Weierstrass elliptic function*, with *periods* ω_1 and ω_2, is defined as

$$\wp(z)=\frac{1}{z^2}+\sum_{(s,\ell)\neq(0,0)}\left\{\frac{1}{(z-s\omega_1-\ell\omega_2)^2}-\frac{1}{(s\omega_1+\ell\omega_2)^2}\right\}.$$

It is well-known that this function satisfies the differential equation

$$y'^2=4y^3-g_2y-g_3,$$

where

$$g_2 = 60 \sum_{(s,\ell)\neq(0,0)} (s\omega_1 + \ell\omega_2)^{-4} \text{ and } g_3 = 140 \sum_{(s,\ell)\neq(0,0)} (s\omega_1 + \ell\omega_2)^{-6}$$

are called *invariants* of $\wp(z)$. Such a function has the corresponding elliptic curve $y^2 = 4x^3 - g_2x - g_3$ and vice-versa. The *Weierstrass* ζ*-function*, associated to $\wp(z)$, is defined by the formula $\zeta'(z) = -\wp(z)$ and $\eta_j = \zeta(\omega_j)$, $j = 1, 2$, are called *quasi-periods* of $\wp(z)$. Now, we can state our key lemma

Lemma 3. *Let ω be a nonzero period of $\wp(z)$, then ω/π is transcendental.*

Proof. Let ω be a nonzero period of $\wp(z)$ and let η be its corresponding quasi-period. We use Corollary 1.19 of [10, p. 302] to conclude that ω/η and ω/π are algebraically independent. In particular, they are both transcendental. □

Proof. (Lemma 2) Let γ be a closed path on the Riemann surface on the elliptic curve

$$y^2 = 4x^3 - g_2x - g_3,$$

with $g_2, g_3 \in \overline{\mathbb{Q}}$. Then

$$\omega = \int_\gamma \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} = \frac{1}{2}\int_\gamma \frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}$$

is a period of the corresponding elliptic function $\wp(z)$, where e_1, e_2, e_3 are the roots of $4x^3 - g_2x - g_3$. Then, for any $\alpha \in \overline{\mathbb{Q}}\backslash\{0,1\}$, we choose the elliptic curve

$$C_\alpha : \; y^2 = 4x(1-x)(\alpha^{-1} - x).$$

Denoting $\wp_\alpha(z)$ by the corresponding elliptic function to C_α, one deduces that

$$\omega_\alpha = \frac{\sqrt{\alpha}}{2}\int_\gamma \frac{dx}{\sqrt{x(1-x)(1-\alpha x)}}$$

is a nonzero period of $\wp_\alpha(z)$, here γ is any closed path on the Riemann surface of C_α. Thus, by Lemma 3, the number $\frac{\omega_\alpha}{\pi}$ is transcendental. Hence, since $\frac{2}{\sqrt{\alpha}}$ is a nonzero algebraic number, we get that

$$\frac{2\omega_\alpha}{\sqrt{\alpha}\pi} = \frac{1}{\pi}\int_\gamma \frac{dx}{\sqrt{x(1-x)(1-\alpha x)}} \tag{6}$$

is also transcendental.

On the other hand, it is known (see [12, 4.2.2]) the existence of a path γ' on the Riemann surface of C_α, such that

$$\frac{1}{\pi}\int_{\gamma'} \frac{dx}{\sqrt{x(1-x)(1-\alpha x)}} = F\left(\frac{1}{2}, \frac{1}{2}, 1; \alpha\right),$$

where $|\arg\alpha| < \pi$. Since the number in Eq. (6) is transcendental, for the choice of $\gamma = \gamma'$, then $F\left(\frac{1}{2}, \frac{1}{2}, 1; \alpha\right)$ is transcendental. The proof is then complete. □

Finally, we are able to deal with the proof of theorem

Proof. (Theorem 1) Since $\rho_{m,n} = \rho_{n,m}$, by Remark 1, we may suppose $n > m$. Thus, if we set $z_{m,n} := \frac{\sqrt{n^2-m^2}}{n}$, one obtains that $z_{m,n} \in \overline{\mathbb{Q}}$, $\arg z_{m,n} = 0$ (since it is a positive real number) and $|z_{m,n}| < 1$. Therefore, Lemma 2 together with identity in (5) yield that

$$\rho_{m,n} = \frac{1}{n} F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{n^2-m^2}{n^2}\right)$$

is transcendental. □

4. Dense Principal Lines

In this section the main result of this paper is formulated.

Theorem 2. Consider the immersions $\alpha_{m,n}$ defined by equation (2) with $m \neq n$. The all leaves of the principal configuration $\mathcal{P}_{\alpha_{m,n}}$ are dense.

Proof. Follows from Proposition 1 and Theorem 1.

□

Remark 4. *A stereographic projection (conformal map) of the family $\alpha_{m,n}$ give examples of immersions in $\mathbb{R}^3$ having all principal lines dense.*

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Ronaldo Garcia
Instituto de Matemática e Estatística,
Universidade Federal de Goiás,
CEP 74001-970, Caixa Postal 131,
Goiânia, GO, BRAZIL
e-mail: ragarcia@mat.ufg.br

Diego Marques
Departamento de Matemática
Universidade de Brasília
CEP 70910-900, Brasília, DF, BRAZIL
e-mail: diego@mat.unb.br