Algebra and Discrete Mathematics RESEARCH ARTICLE
Number ?. **(????).** pp. 1–**??**

Weighted zero-sum problems over C_3^r

Hemar Godinho, Abílio Lemos, Diego Marques

Communicated by Communicated person

January 11, 2015

Abstract. Let C_n be the cyclic group of order n and set $s_A(C_n^r)$ as the smallest integer ℓ such that every sequence $\mathcal{S}$ in C_n^r of length at least ℓ has an A-zero-sum subsequence of length equal to $\exp(C_n^r)$, for $A=\{-1,1\}$. In this paper, among other things, we give estimates for $s_A(C_3^r)$, and prove that $s_A(C_3^3)=9$, $s_A(C_3^4)=21$ and $41 \leq s_A(C_3^5) \leq 45$.

Introduction

Let G be a finite abelian group (written additively), and $\mathcal{S}$ be a finite sequence of elements of G and of length $\mathfrak{m}$. For simplicity we are going to write $\mathcal{S}$ in a *multiplicative* form

$$\mathcal{S} = \prod_{i=1}^{\ell} g_i^{v_i},$$

where v_i represents the number of times the element g_i appears in this sequence. Hence $\sum_{i=1}^{\ell} v_i = \mathfrak{m}$.

Let $A=\{-1,1\}$. We say that a subsequence $a_1 \cdots a_s$ of $\mathcal{S}$ is an *A-zero-sum subsequence*, if we can find $\epsilon_1,\ldots,\epsilon_s \in A$ such that

$$\epsilon_1 a_1 + \cdots + \epsilon_s a_s = 0 \quad \text{in} \quad G.$$

The first two authors were partially supported by a grant from CNPq-Brazil. The third author is partially supported by FAP-DF, FEMAT and CNPq-Brazil

2010 Mathematics Subject Classification: *20D60, 20K01.*

Key words and phrases: *Weighted zero-sum, abelian groups.*

Here we are particularly interested in studying the behavior of $s_A(G)$ defined as the smallest integer ℓ such that every sequence $\mathcal{S}$ of length greater than or equal to ℓ, satisfies the condition (s_A), which states that there must exist an A-zero-sum subsequence of $\mathcal{S}$ of length $\exp(G)$ (the exponent of G).

For this purpose, two other invariants will be defined to help us in this study. Thus, define $\eta_A(G)$ as the smallest integer ℓ such that every sequence $\mathcal{S}$ of length greater than or equal to ℓ, satisfies the condition (η_A), which says that there exists an A-zero-sum subsequence of $\mathcal{S}$ of length *at most* $\exp(G)$. Define also $g_A(G)$ as the smallest integer ℓ such that every sequence $\mathcal{S}$ of *distinct* elements and of length greater than or equal to ℓ, satisfies the condition (g_A), which says that there must exist an A-zero-sum subsequence of $\mathcal{S}$ of length $\exp(G)$.

The study of zero-sums is a classical area of additive number theory and goes back to the works of Erdös, Ginzburg and Ziv [6] and Harborth [9]. A very thorough survey up to 2006 can be found on Gao-Geroldinger [7], where applications of this theory are also given.

In [8], Grynkiewicz established a weighted version of Erdös-Ginzburg-Ziv theorem, which introduced the idea of considering certain weighted subsequence sums, and Thangadurai [13] presented many results on a weighted Davenport's constant and its relation to s_A.

For the particular weight $A = \{-1, 1\}$, the best results are due to Adhikari *et al* [1], where it is proved that $s_A(C_n) = n + \lfloor \log_2 n \rfloor$ (here C_n is a cyclic group of order n) and Adhikari *et al* [2], where it is proved that $s_A(C_n \times C_n) = 2n - 1$, when n is odd. Recently, Adhikari *et al* proved that $s_A(G) = \exp(G) + |G| \log 2 + O(|G| \log 2 \log 2)$ when $\exp(G)$ is even and $\exp(G) \to +\infty$ (see [3]).

The aim of this paper is to give estimates for $s_A(C_n^r)$, where as usual $C_n^r = C_n \times \cdots \times C_n$ (r times), and here are our results.

Theorem 1. *Let $A = \{-1, 1\}$, $n > 1$ odd and $r \geq 1$. If $n = 3$ and $r \geq 2$, or $n \geq 5$ then*

$$2^{r-1}(n-1) + 1 \leq s_A(C_n^r) \leq (n^r - 1)\left(\frac{n-1}{2}\right) + 1.$$

For the case of $n = 3$ we present a more detailed study and prove

Theorem 2. *Let $A = \{-1, 1\}$ and $r \geq 5$.*

(i) If r is odd then

$$s_A(C_3^r) \geq 2^r + 2\binom{r-1}{\frac{r-5}{2}} - 1$$

(ii) If r is even, with $m = \lfloor \frac{3r-4}{4} \rfloor$, then

(a) If $r \equiv 2 \pmod 4$, then $s_A(C_3^r) \geq 2\sum_{1\leq j\leq m} \binom{r}{j} + 2\binom{r}{\frac{r-2}{2}} + 1$, where j takes odd values.

(b) If $r \equiv 0 \pmod 4$, then $s_A(C_3^r) \geq 2\sum_{1\leq j\leq m} \binom{r}{j} + \binom{r}{\frac{r}{2}} + 1$, where j takes odd values.

It is simple to check that $s_A(C_3) = 4$, and it follows from Theorem 3 in [2] that $s_A(C_3^2) = 5$. Our next result presents both exact values of $s_A(C_3^r)$, and $r = 3, 4$ as well as estimates for $s_A(C_{3^a}^r)$, $r = 3, 4, 5$, for all $a \geq 1$.

Theorem 3. *Let $A = \{-1, 1\}$. Then*

(i) $s_A(C_3^3) = 9$, $s_A(C_3^4) = 21$, $41 \leq s_A(C_3^5) \leq 45$

(ii) $s_A(C_{3^a}^3) = 4 \times 3^a - 3$, for all $a \geq 1$

(iii) $8 \times 3^a - 7 \leq s_A(C_{3^a}^4) \leq 10 \times 3^a - 9$, for all $a \geq 1$

(iv) $16 \times 3^a - 15 \leq s_A(C_{3^a}^5) \leq 22 \times 3^a - 21$, for all $a \geq 1$

1. Relations between the invariants η_A, g_A and s_A

We start by proving the following result.

Lemma 1. *For $A = \{-1, 1\}$, we have*

(i) $\eta_A(C_3) = 2$, $g_A(C_3) = 3$ and $s_A(C_3) = 4$, and

(ii) $\eta_A(C_3^r) \geq r + 1$ for any $r \in \mathbb{N}$.

Proof. The proof of item (i) is very simple and will be omitted. For (ii), the proof follows from the fact that the sequence $e_1 e_2 \cdots e_r$ with $e_j = (0, \ldots, 1, \ldots, 0)$, has no A-zero-sum subsequence. □

Proposition 1. *For $A = \{-1, 1\}$, we have $g_A(C_3^r) = 2\eta_A(C_3^r) - 1$.*

Proof. The case $r = 1$ follows from Lemma 1. Let $\mathcal{S} = \prod_{i=1}^{\mathfrak{m}} g_i$ of length $\mathfrak{m} = \eta_A(C_3^r) - 1$ which does not satisfy the condition (η_A). In particular $\mathcal{S}$ has no A-zero-sum subsequences of length 1 and 2, that is, all elements of $\mathcal{S}$ are nonzero and distinct. Now, let $\mathcal{S}^*$ be the sequence $\prod_{i=1}^{\mathfrak{m}} g_i \prod_{i=1}^{\mathfrak{m}} (-g_i)$. Observe that $\mathcal{S}^*$ has only distinct elements, since $\mathcal{S}$ has no A-zero-sum subsequences of length 2. It is easy to see that any A-zero-sum of $\mathcal{S}^*$ of length 3 is also an A-zero-sum of $\mathcal{S}$, for $A = \{-1, 1\}$. Hence $g_A(C_3^r) \geq 2\eta_A(C_3^r) - 1$.

Let $\mathcal{S}$ be a sequence of distinct elements and of length $\mathfrak{m} = 2\eta_A(C_3^r) - 1$, and write

$$\mathcal{S} = \prod_{i=1}^{t} g_i \prod_{i=1}^{t}(-g_i) \prod_{i=2t+1}^{\mathfrak{m}} g_i$$

where $g_r \neq -g_s$ for $2t+1 \leq r < s \leq \mathfrak{m}$. If $t = 0$, then $\mathcal{S}$ has no A-zero-sum of length 2, and 0 can appear at most once in $\mathcal{S}$. Let $\mathcal{S}^*$ be the subsequence of all nonzero elements of $\mathcal{S}$, hence $|\mathcal{S}^*| = 2\eta_A(C_3^r) - 2 > \eta_A(C_3^r)$, for $r \geq 2$ (see Lemma 1(ii)), hence it must contain an A-zero-sum of length 3.

For the case $t \geq 1$, we may assume $g_j \neq 0$, for every $j = 2t+1, \ldots, \mathfrak{m}$ since otherwise, $g_t + (-g_t) + g_{j_0}$ is A-zero-sum subsequence of length 3. But now, either $t \geq \eta_A(C_3^r)$, so that $\prod_{i=1}^{t} g_i$ has an $A-$zero-sum of length 3, or $\mathfrak{m} - t \geq \eta_A(C_3^r)$, so that $\prod_{i=1}^{t}(-g_i) \prod_{i=2t+1}^{\mathfrak{m}} g_i$ has an $A-$zero-sum subsequence of length 3. □

Here we note that by the definition of these invariants and the proposition above, we have

$$s_A(C_3^r) \geq g_A(C_3^r) = 2\eta_A(C_3^r) - 1. \tag{1}$$

Proposition 2. *For $A = \{-1, 1\}$, we have $s_A(C_3^r) = g_A(C_3^r)$, for $r \geq 2$.*

Proof. From Theorem 3 in [2] we have $s_A(C_3^2) = 5$ and, on the other hand, the sequence $(1,0)(0,1)(2,0)(0,2)$ does not satisfy the condition (g_A), hence $s_A(C_3^2) = g_A(C_3^2)$ (see (1)). From now on, let us consider $r \geq 3$.

Let $\mathcal{S}$ be a sequence of length $\mathfrak{m} = s_A(C_3^r) - 1$ which does not satisfy the condition (s_A). In particular $\mathcal{S}$ does not contain three equal elements, since $3g = 0$. If $\mathcal{S}$ contains only distinct elements, then it does not satisfy also the condition (g_A), and then $\mathfrak{m} \leq g_A(C_3^r) - 1$, which implies $s_A(C_3^r) = g_A(C_3^r)$ (see (1)). Hence, let us assume that $\mathcal{S}$ has repeated elements and write

$$\mathcal{S} = \mathcal{E}^2\mathcal{F} = \prod_{i=1}^{t} g_i^2 \prod_{j=2t+1}^{\mathfrak{m}} g_j \tag{2}$$

where $g_1, \ldots, g_t, g_{2t+1}, \ldots, g_{\mathfrak{m}}$ are distinct. If for some $1 \leq j \leq \mathfrak{m}$ we have $g_j = 0$, then the subsequence of all nonzero elements of $\mathcal{S}$ has length at least equal to $s_A(C_3^r) - 3 \geq 2\eta_A(C_3^r) - 4 \geq \eta_A(C_3^r)$ for $r \geq 3$ (see Lemma 1 (ii)). Then it must have an A-zero-sum of length 2 or 3. And if the A-zero-sum is of length 2, together with $g_j = 0$ we would have an A-zero-sum of length 3 in $\mathcal{S}$, contradicting the assumption that it does not satisfy the condition (s_A).

Hence let us assume that all elements of $\mathcal{S}$ are nonzero. Observe that we can not have g in $\mathcal{E}$ and h in $\mathcal{F}$ (see (2)) such that $h = -g$, for $g + g - h = 3g = 0$, an A-zero-sum of length 3. Therefore the new sequence

$$\mathcal{R} = \prod_{i=1}^{t} g_i \prod_{i=1}^{t}(-g_i) \prod_{i=2t+1}^{\mathfrak{m}} g_i$$

has only distinct elements, length $\mathfrak{m} = s_A(C_3^r) - 1$, and does not satisfy the condition (g_A). Hence $\mathfrak{m} \leq g_A(C_3^r) - 1$, and this concludes the proof according to (1). □

2. Proof of Theorem 1

2.1. The lower bound for $s_A(C_n^r)$

Let $e_1, \dots, e_r$ be the elements of C_n^r defined as $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, and for every subset $I \subset \{1, \dots, r\}$, of *odd cardinality*, define $\mathfrak{e}_I = \sum_{i \in I} e_i$ (e.g., taking $I = \{1, 3, r\}$, we have $\mathfrak{e}_I = (1, 0, 1, 0, \dots, 0, 1)$), and let $\mathscr{I}_m$ be the collection of all subsets of $\{1, \dots, r\}$ of cardinality odd and at most equal to m.

There is a natural isomorphism between the cyclic groups $C_n^r \cong (\mathbb{Z}/n\mathbb{Z})^r$, and this result here will be proved for $(\mathbb{Z}/n\mathbb{Z})^r$. Let $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical group epimorphism, and define $\varphi : \mathbb{Z}^r \to (\mathbb{Z}/n\mathbb{Z})^r$ as $\varphi(a_1, \cdots, a_r) = (\phi(a_1), \cdots, \phi(a_r))$. If $\mathcal{S} = g_1 \cdots g_m$ is a sequence over the group $\mathbb{Z}^r$, let us denote by $\varphi(\mathcal{S})$ the sequence $\varphi(\mathcal{S}) = \varphi(g_1) \cdots \varphi(g_m)$ of same length over the group $(\mathbb{Z}/n\mathbb{Z})^r$.

Let $e_1^*, \dots, e_r^*$ be the canonical basis (i.e.,$e_j^* = (0, \dots, 0, 1, 0, \dots, 0)$) of the group $\mathbb{Z}^r$, and define, as above

$$\mathfrak{e}_I^* = \sum_{i \in I} e_i^*$$

Now consider the sequence

$$\mathcal{S} = \prod_{I \in \mathscr{I}_r} (\mathfrak{e}_I^*)^{n-1},$$

of length $2^{r-1}(n-1)$. We will prove that the corresponding sequence

$$\varphi(\mathcal{S}) = \prod_{I \in \mathscr{I}_r} \mathfrak{e}_I^{n-1},$$

has no A-zero-sum subsequences of length n, which is equivalent to prove that given $A = \{-1, 1\}$ and any subsequence $\mathcal{R} = g_1 \cdots g_n$ of $\mathcal{S}$, it is not possible to find $\epsilon_1, \ldots, \epsilon_s \in A$ such that (with an abuse of notation)

$$\epsilon_1 g_1 + \cdots + \epsilon_n g_n \equiv (0, \ldots, 0) \pmod n. \tag{3}$$

Writing $g_k = (c_1^{(k)}, \ldots, c_r^{(k)})$, for $1 \le k \le n$, it follows from (3) that, for every $j \in \{1, \ldots, r\}$, we have

$$\sum_{k=1}^{n} \epsilon_k c_j^{(k)} \equiv 0 \ (\bmod\ n). \tag{4}$$

For every $1 \le j \le r$, let us define the sets

$$A_j = \{\ell \mid c_j^{(\ell)} = 1\}.$$

Since $c_j^{(\ell)} \in \{0, 1\}$ and $\epsilon_j \in \{-1, 1\}$ for any j and any ℓ, we must have, according to (4), that either

$$|A_j| = n \quad \text{or} \quad |A_j| \ \text{ is even.} \tag{5}$$

Since $g_\ell = \mathfrak{e}_{I_\ell}$, for some I, by the definition we have $\sum_{j=1}^{r} c_j^{(\ell)} = |I|$ for all ℓ, then

$$\sum_{j=1}^{r} |A_j| = \sum_{j=1}^{r} \sum_{\ell=1}^{n} c_j^{(\ell)} = \sum_{\ell=1}^{n} \sum_{j=1}^{r} c_j^{(\ell)} = |I_1| + \cdots + |I_n|,$$

an odd sum of odd numbers. Hence there exists a $\jmath_0$, such that $|A_{\jmath_0}| = n$ (see (5)), but then, it follows from (4) that $\sum_{k=1}^{n} \epsilon_k c_{\jmath_0}^{(k)} = n$ and therefore $\epsilon_1 = \cdots = \epsilon_n = 1$. And the important consequence is that we must have $g_1 = \cdots = g_n$, which is impossible since in the sequence $\mathcal{S}$ no element appears more than $n-1$ times.

Remark 1. If we consider the sequence $\varphi(\mathcal{S}) = \prod_{I \in \mathscr{I}_r} \mathfrak{e}_I$, for $n = 3$, we see that this does not satisfy the condition (η_A). So $\eta_A(C_3^r) \ge 2^{r-1} + 1$ for any $r \in \mathbb{N}$, which is an improvement of the item (ii) of the Lemma 1.

2.2. The upper bound for $s_A(C_n^r)$

Let us consider the set of elements of the group C_n^r as the union $\{0\} \cup G^+ \cup G^-$, where if $g \in G^+$ then $-g \in G^-$. And write the sequence $\mathcal{S}$ as

$$\mathcal{S} = 0^m \prod_{g \in G^+} (g^{v_g(\mathcal{S})} (-g)^{v_{-g}(\mathcal{S})}).$$

First observe that if for some g, $v_g(\mathcal{S}) + v_{-g}(\mathcal{S}) \ge n$, then we can find a subsequence $\mathcal{R} = c_1 \cdots c_n$ of $\mathcal{S}$, which is an A-zero-sum, for $A = \{-1, 1\}$, and any sum of n equal elements is equal to zero in C_n^r. Now consider $m \geq 1$ and $m + v_g(\mathcal{S}) + v_{-g}(\mathcal{S}) > n$, then we can find a subsequence $\mathcal{R} = h_1 \cdots h_t$ of $\mathcal{S}$ of *even* length $t \geq n - m$ with $h_j \in \{-g, g\}$. Since $A = \{-1, 1\}$, this is an A-zero-sum. Hence, the subsequence $T = 0^{m^*}\mathcal{R}$ ($m^* \leq m$) of $\mathcal{S}$ is an A-zero-sum of length n.

Thus assume that, for every g in $\mathcal{S}$ we have $v_g(\mathcal{S}) + v_{-g}(\mathcal{S}) \leq n - m$, which gives

$$|\mathcal{S}| \leq \begin{cases} m + \frac{n^r-1}{2}(n-m) & \text{if } \; m > 0 \; \text{ even} \\ m - 1 + \frac{n^r-1}{2}(n-m) & \text{if } \; m > 0 \; \text{ odd} \\ \frac{n^r-1}{2}(n-1) & \text{if } \; m = 0, \end{cases}$$

for $|G^+| = \frac{n^r-1}{2}$. We observe than in the case m even $m + \frac{n^r-1}{2}(n-m) \leq 2 + \frac{n^r-1}{2}(n-2) \leq 2 + \frac{n^r-1}{2}(n-2) + \frac{n^r-1}{2} - 1$ and the equality only happens when $n = 3$ and $r = 1$. In any case, if $|\mathcal{S}| \geq \frac{n^r-1}{2}(n-1) + 1$, it has a subsequence of length n which is an A-zero-sum.

Remark 2. For $n = 3$, the upper bound for $s_A(C_3^r)$ can be improved using the result of Meshulam[12] as follows. According to Proposition 2, $s_A(C_3^r) = g_A(C_3^r)$ for $r \geq 2$, and it follows from the definition that $g_A(C_3^r) \leq g(C_3^r)$, where $g(C_3^r)$ is the invariant $g_A(C_3^r)$ with $A = \{1\}$. Now we use the Theorem 1.2 of [12] to obtain $s_A(C_3^r) = g_A(C_3^r) \leq g(C_3^r) \leq 2 \times 3^r/r$.

3. Proof of Theorem 2.

Now we turn our attention to prove the following proposition.

Proposition 3. *If $r > 3$ is odd and $A = \{-1, 1\}$ then $\eta_A(C_3^r) \geq 2^{r-1} + \binom{r-1}{\delta}$, where*

$$\delta = \delta(r) = \begin{cases} \frac{(r-3)}{2} & \text{if } r \equiv 1 \pmod 4 \\ \frac{(r-5)}{2} & \text{if } r \equiv 3 \pmod 4. \end{cases} \tag{6}$$

Proof. We will prove this proposition by presenting an example of a sequence of length $2^{r-1} + \binom{r-1}{\delta} - 1$ with no A-zero-sum subsequences of length smaller or equal to 3. Let $\ell = \binom{r-1}{\delta}$, and consider the sequence

$$\mathcal{S} = \mathcal{E}.\mathcal{G} = \left(\prod_{I \in \mathscr{I}_{r-2}} \mathfrak{e}_I \right) \cdot g_1 \cdots g_\ell,$$

with

$$\begin{aligned} g_1 &= (-1,\underbrace{-1,\ldots,-1}_{\delta},1,1,\ldots,1) \\ &\vdots \\ g_\ell &= (-1,1,\ldots,1,\underbrace{-1,\ldots,-1}_{\delta}), \end{aligned}$$

where $\mathfrak{e}_I$ and $\mathscr{I}_{r-2}$ are defined in the beginning of section 2. Clearly $\mathcal{S}$ has no A-zero-sum subsequences of length 1 or 2 and also sum or difference of two elements of $\mathcal{G}$ will never give another element of $\mathcal{G}$, for no element of $\mathcal{G}$ has zero as one of its coordinates. Now we will consider $\mathfrak{e}_s - \mathfrak{e}_t$, where $\mathfrak{e}_s$ and $\mathfrak{e}_t$ represent the $\mathfrak{e}_I$'s for which s coordinates are equal to 1 and t coordinates are equal to 1 respectively. Thus, we see that $\mathfrak{e}_s - \mathfrak{e}_t$ will never be an element of $\mathcal{G}$ since it necessarily has either a zero coordinate or it has an odd number of 1's and -1's (and $\delta + 1$ is even).

Now, if for some s, t we would have

$$\mathfrak{e}_s + \mathfrak{e}_t = g_i,$$

Then $\mathfrak{e}_t, \mathfrak{e}_s$ would have $\delta + 1$ nonzero coordinates at the same positions (to obtain $\delta + 1$ coordinates -1's). Hence we would need to have

$$r + (\delta + 1) = s + t$$

Which is impossible since $s + t$ is even and $r + (\delta + 1)$ is odd, for δ is odd in any of the two cases.

Thus, the only possible A-zero-sum subsequence of length 3 would necessarily include one element of $\mathcal{E}$ and two elements of $\mathcal{G}$.

Let v, w be elements of $\mathcal{G}$. Now it simple to verify that (the calculations are modulo 3) either $v + w$ or $v - w$ have two of their entries with opposite signs (for $\delta(r) < (r - 1)/2$) and hence either of them can not be added to an $\pm\mathfrak{e}_I$ to obtain an A-zero-sum, since all its nonzero entries have the same sign. □

Proposition 4. *Let $r > 4$ be even, $m = \left\lfloor \frac{3r-4}{4} \right\rfloor$ and $A = \{-1, 1\}$. Then*

$$\eta_A(C_3^r) \geq \sum_{\substack{j=1 \\ j\ odd}}^{m} \binom{r}{j} + \ell(r) + 1,$$

where

$$\ell(r) = \begin{cases} \binom{r}{\frac{r-2}{2}} & \text{if } r \equiv 2 \pmod 4 \\ \binom{r}{\frac{r}{2}}/2 & \text{if } r \equiv 0 \pmod 4), \end{cases}$$

Proof.

Consider the sequence $\mathcal{K} = g_1 \cdots g_\tau$ with

$$\begin{aligned} g_1 &= (\underbrace{-1,\ldots,-1}_{\delta},1,1,\ldots,1) \\ &\vdots \\ g_\tau &= (1,1,\ldots,1,\underbrace{-1,\ldots,-1}_{\delta}) \end{aligned}$$

where

$$\tau = \begin{cases} \ell(r) & \text{if } r \equiv 2 \pmod 4 \\ 2\ell(r) & \text{if } r \equiv 0 \pmod 4, \end{cases} \quad \text{and} \quad \delta = \begin{cases} \frac{r-2}{2} & \text{if } r \equiv 2 \pmod 4 \\ \frac{r}{2} & \text{if } r \equiv 0 \pmod 4, \end{cases}$$

and rearrange the elements of the sequence $\mathcal{K}$, and write it as

$$\mathcal{K} = \prod_{i=1}^{\tau/2} g_i \prod_{i=1}^{\tau/2} (-g_i) = \mathcal{K}^+\mathcal{K}^-.$$

It is simple to observe that if $r \equiv 2 \pmod 4$, then $\tau = \ell$ and $\mathcal{K}^- = \emptyset$.

Now define the sequence

$$\mathcal{S} = \left(\prod_{I \in \mathscr{I}_m} \mathfrak{e}_I \right) \mathcal{G},$$

where $\mathcal{G} = \mathcal{K}$ if $r \equiv 2 \pmod 4$ or $\mathcal{G} = \mathcal{K}^+$ if $r \equiv 0 \pmod 4$, and $m = \left\lfloor \frac{3r-4}{4} \right\rfloor$, a sequence of length $|\mathcal{S}| = \sum_{\substack{j=1 \\ j \; odd}}^{m} \binom{r}{j} + \ell(r) + 1$.

The first important observation is that $\mathcal{S}$ has no A-zero-sum subsequences of length 1 or 2. And also sum or difference of two elements of $\mathcal{G}$ will never be another element of $\mathcal{G}$, for it necessarily will have a zero as coordinate. Also $\mathfrak{e}_I - \mathfrak{e}_J$ will never be an element of $\mathcal{G}$ since it necessarily has either a zero coordinate or it has an odd number of 1's and -1's (and δ is even). Now, if for some s, t (both defined as in the proof of the Proposition 3) we would have

$$\mathfrak{e}_s + \mathfrak{e}_t = \pm g_j, \quad \text{for some } \jmath$$

then $\mathfrak{e}_t, \mathfrak{e}_s$ would necessarily have δ nonzero coordinates at the same positions (to obtain δ coordinates -1's). But then

$$s + t = r + \delta \geq \frac{3r-2}{2}, \quad \text{for any value of } \delta$$

which is impossible since

$$s + t \leq 2m \leq \frac{3r-4}{2}.$$

Thus the only A-zero-sum subsequence of length 3 possible necessarily includes an element $\mathfrak{e}_t$ and two elements of $\mathcal{G}$.

Let v, w elements of $\mathcal{G}$. First, observe that if they do not have -1's in common positions, then $v + w$ has an even amount of zeros and an even amount of -1's (since r and δ are both even), i.e., $v + w \neq \pm\mathfrak{e}_I$. If we make $v - w$ also have an even amount of nonzero coordinates, i.e., we haven't $\pm\mathfrak{e}_I$. Now, assuming that v, w have at last a -1 in same position, it simple to verify that (the calculations are modulo 3) either $v + w$ or $v - w$ have two or more of their entries with opposite signs and hence either of them can not be added to an $\pm\mathfrak{e}_I$ to obtain an A-zero-sum, since all its nonzero entries have the same sign. □

Theorem 2 now follows from propositions 1, 2, 3 and 4.

4. Proof of Theorem 3.

We start by proving the following proposition.

Proposition 5. *For $A = \{-1, 1\}$, we have*

(i) $\eta_A(C_3^2) = 3$;

(ii) $\eta_A(C_3^3) = 5$;

(iii) $\eta_A(C_3^4) = 11$.

(iv) $21 \leq \eta_A(C_3^5) \leq 23$.

Proof. By Propositions 1 and 2, we have that $s_A(C_3^r) = g_A(C_3^r) = 2\eta_A(C_3^r) - 1$, for $r > 1$, and by definition, we have $g_A(C_3^r) \leq g(C_3^r)$ resulting in $\eta_A(C_3^r) \leq \frac{g(C_3^r)+1}{2}$, for $r > 1$. It follows from

$$g(C_3^2) = 5 \text{ ([10])}, g(C_3^3) = 10, g(C_3^4) = 21 \text{ ([11])}, g(C_3^5) = 46 \text{ ([5])},$$

that $\eta_A(C_3^2) \leq 3$, $\eta_A(C_3^3) \leq 5$, $\eta_A(C_3^4) \leq 11$ and $\eta_A(C_3^5) \leq 23$. It is easy to see that the sequences $(1,0)(0,1)$ and $(1,0,0)(0,1,0)(0,0,1)(1,1,1)$ has no A-zero-sum of length at most three, so $\eta_A(C_3^2) = 3$ and $\eta_A(C_3^3) = 5$. It is also simple to check that following sequences of lengths 10 and 20 respectively do not satisfy the condition (η_A):

$$\begin{gathered}(1,1,0,0) \cdots (0,0,1,1)(1,1,1,0) \cdots (0,1,1,1) \\ \text{and} \\ (1,1,0,0,0) \cdots (0,0,0,1,1)(1,1,1,0,0) \cdots (0,0,1,1,1),\end{gathered} \tag{7}$$

hence $\eta_A(C_3^4) = 11$ and $\eta_A(C_3^5) \geq 21$.

□

Proposition 5 together with propositions 1 and 2 gives the proof of item (i) of Theorem 3. The proof of the remaining three items is given in Proposition 7 below.

Before going further, we need a slight modification of a result due to Gao *et al* for $A = \{1\}$ in [4]. Here we shall use it in the case $A = \{-1, 1\}$. The proof in this case is analogous to the original one, and shall be omit it.

Proposition 6. *Let G be a finite abelian group, $A = \{-1, 1\}$ and $H \leq G$. Let $\mathcal{S}$ be a sequence in G of length*

$$\mathfrak{m} \geq (s_A(H) - 1)\exp(G/H) + s_A(G/H).$$

Then $\mathcal{S}$ has an A-zero-sum subsequence of length $\exp(H)\exp(G/H)$. In particular, if $\exp(G) = \exp(H)\exp(G/H)$, then

$$s_A(G) \leq (s_A(H) - 1)\exp(G/H) + s_A(G/H).$$

Proposition 7. *For $A = \{-1, 1\}$, we have*

(i) $s_A(C_{3^a}^3) = 4 \times 3^a - 3$, *for all* $a \geq 1$

(ii) $8 \times 3^a - 7 \leq s_A(C_{3^a}^4) \leq 10 \times 3^a - 9$, *for all* $a \geq 1$

(iii) $16 \times 3^a - 15 \leq s_A(C_{3^a}^5) \leq 22 \times 3^a - 21$, *for all* $a \geq 1$

Proof. It follows of (i) from Theorem 3 that $s_A(C_3^3) = 4 \times 3 - 3 = 9$. Now assume that $s_A(C_{3^{a-1}}^3) = 4 \cdot 3^{a-1} - 3$. Thus, Proposition 6 yields

$$\begin{aligned} s_A(C_{3^a}^3) &\leq 3 \times (s_A(C_{3^{a-1}}^3) - 1) + s_A(C_3^3) \\ &\leq 4 \times 3^a - 3 \end{aligned}$$

On the other hand, Theorem 1 gives $s_A(C_{3^a}^3) \geq 4 \times 3^a - 3$, concluding the proof of (i).

Again by (i) from Theorem 3, we have that $s_A(C_3^4) = 10 \times 3 - 9 = 21$. Now, assume that $s_A(C_{3^{a-1}}^4) \leq 10 \cdot 3^{a-1} - 9$. It follows from Proposition 6 that

$$\begin{aligned} s_A(C_{3^a}^4) &\leq 3 \times (s_A(C_{3^{a-1}}^4) - 1) + s_A(C_3^4) \\ &\leq 10 \times 3^a - 9 \end{aligned}$$

On the other hand, Theorem 1 gives the lower bound $s_A(C_{3^a}^4) \geq 8 \times 3^a - 7$, concluding the proof of (ii). The proof of item (iii) is analogous to the proof of item (ii), again using (i) of the Theorem 3 and Theorem 1. □

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CONTACT INFORMATION

H. Godinho Departamento de Matemática, Universidade de Brasília, Brasília-DF, Brazil
E-Mail: hemar@mat.unb.br
URL:

A. Lemos Departamento de Matemática, Universidade Federal de Viçosa, Viçosa-MG, Brazil
E-Mail: abiliolemos@ufv.br
URL:

D. Marques Departamento de Matemática, Universidade de Brasília, Brasília-DF, Brazil
E-Mail: diego@mat.unb.br
URL: