On the spacings between C-nomial coefficients

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Abstract

Let $(C_n)_{n \geq 0}$ be the Lucas sequence $C_{n+2} = aC_{n+1} + bC_n$ for all $n \geq 0$, where $C_0 = 0$ and $C_1 = 1$. For $1 \leq k \leq m - 1$ let

$$\begin{bmatrix} m \\ k \end{bmatrix}_C = \frac{C_m C_{m-1} \cdots C_{m-k+1}}{C_1 \cdots C_k}$$

be the corresponding C-nomial coefficient. When $C_n = F_n$ is the Fibonacci sequence (the numbers $\begin{bmatrix} m \\ k \end{bmatrix}_F$ are called Fibonomials), or $C_n = (q^n - 1)/(q - 1)$, where $q > 1$ is an integer (the numbers $\begin{bmatrix} m \\ k \end{bmatrix}_q$ are called q-binomial, or Gaussian coefficients), we show that there are no nontrivial solutions to the Diophantine equation

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ l \end{bmatrix}_F \qquad \text{or} \qquad \begin{bmatrix} m \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ l \end{bmatrix}_q$$

with $(m, k) \neq (n, l)$ other than the obvious ones $(n, l) = (m, m - k)$. We also show that the difference

$$\left| \begin{bmatrix} m \\ k \end{bmatrix}_F - \begin{bmatrix} n \\ l \end{bmatrix}_F \right|$$

tends to infinity when (m, k, n, l) are such that $1 \leq k \leq m/2$, $1 \leq l \leq n/2$, $(m, k) \neq (n, l)$ and $\max\{m, n\}$ tends to infinity in an effective way.

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1 Introduction

A famous unsolved problem in Diophantine equations is to find all pairs of binomial coefficients having the same value. That is, to find all solutions of

$$\binom{m}{k} = \binom{n}{l}. \tag{1}$$

Here, $1 \le k \le m-1$ and $1 \le l \le n-1$. To avoid the obvious symmetry of the Pascal triangle, we may assume that $k \le m/2$ and $l \le n/2$. As of the time of this writing, the above problem has not yet been solved in its full generality. The only nontrivial solutions known at this time are

$$\binom{16}{2} = \binom{10}{3} = 120, \quad \binom{21}{2} = \binom{10}{4} = 210, \quad \binom{56}{2} = \binom{22}{3} = 1540,$$

$$\binom{120}{2} = \binom{36}{3} = 7140, \qquad \binom{153}{2} = \binom{19}{5} = 11638,$$

$$\binom{221}{2} = \binom{17}{8} = 24310, \qquad \binom{78}{2} = \binom{15}{5} = \binom{14}{6} = 3003,$$

$$\text{and} \quad \binom{F_{2i+2}F_{2i+3}}{F_{2i}F_{2i+3}} = \binom{F_{2i+2}F_{2i+3} - 1}{F_{2i}F_{2i+3} + 1} \qquad \text{for } i = 1, 2, \ldots, \tag{2}$$

where F_n is the nth Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$. See [5] for a proof of the fact that the above list contains all the nontrivial solutions of the Diophantine equation (1) with $k \le l$ and $(k, l) \in \{(2,3), (2,4), (2,6), (2,8), (3,4), (3,6), (4,6)\}$ and the recent paper [?] for the case $(k, l) = (2, 5)$.

Let us now look at the sequence of *Fibonomial coefficients*, which are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_1 F_2 \cdots F_m}{(F_1 \cdots F_k)(F_1 \cdots F_{m-k})} = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}$$

for $1 \le k \le m - 1$. These numbers are always integers as first proved by E. Lucas in [9]. Various parts of this sequence with fixed small values of k appear in Sloane's *On Line Encyclopedia of Integer Sequences* [11] (see, for example, A001655, A056565, A001658, etc.). More generally, given any sequence $C = (C_n)_{n \ge 0}$ of nonzero real numbers, one can define the *C-nomial coefficients* as

$$\begin{bmatrix} m \\ k \end{bmatrix}_C = \frac{C_m C_{m-1} \cdots C_{m-k+1}}{C_1 \cdots C_k}.$$

Bachmann [1, p. 81], Carmichael [4, p. 40], and Jarden and Motzkin [7], all showed that if C is a Lucas sequence; i.e., it has $C_0 = 0$, $C_1 = 1$ and satisfies

the recurrence $C_{n+2} = aC_{n+1} + bC_n$ for all nonnegative integers n with some nonzero integers a and b such that the quadratic equation $x^2 - ax - b = 0$ has two distinct roots α and β whose ratio is not a root of unity, then all the C-nomial coefficients are integers. The Fibonomial coefficients are particular cases of this instance with $a = b = 1$. When $a = q + 1$ and $b = -q$, where $q > 1$ is some fixed integer, the C-nomial coefficients become the so-called q-*binomial coefficients* given by

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m - 1) \cdots (q^{m-k+1} - 1)}{(q - 1) \cdots (q^k - 1)}.$$

In this paper, we study the analogous Diophantine equation (1) when the binomial coefficients are replaced by C-nomial coefficients, and, more generally, we study the spacings between the C-nomial coefficients. While our results can be formulated for general C-nomial coefficients when $C = (C_n)_{n\geq0}$ is a general Lucas sequence, we restrict our attention to the particular cases of the Fibonomial coefficients, for which $C = F = (F_n)_{n\geq0}$, or to the case of the q-binomial coefficients, when $C_n = (q^n - 1)/(q - 1)$ for all $n \geq 0$, where $q > 1$ is a fixed integer. Our results are the following.

Theorem 1 *None of the Diophantine equations*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ l \end{bmatrix}_F \qquad or \qquad \begin{bmatrix} m \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ l \end{bmatrix}_q \tag{3}$$

has any positive integer solutions $1 \leq k \leq m/2$, $1 \leq l \leq n/2$, $(m, k) \neq (n, l)$ and $q > 1$.

Next, let us put

$$\mathcal{F} = \left\{ \begin{bmatrix} m \\ k \end{bmatrix}_F : 1 \leq k \leq m/2 \right\} = \{f_1, f_2, \ldots\},$$

where $1 = f_1 < f_2 < f_3 \cdots$ are all the elements of $\mathcal{F}$ arranged increasingly. Note that

$$\mathcal{F} = \{1, 2, 3, 5, 6, 8, 13, 15, 21, 34, 40, 55, 60, 89, 104, \ldots\}.$$

This is sequence $A144712$ in [11]. Our next result shows that $f_{N+1} - f_N \to \infty$.

Theorem 2 *We have*

$$f_{N+1} - f_N \gg (\log f_N)^{1/2},$$

where the implied constant is effective. In particular, $f_{N+1} - f_N$ tends to infinity with N.

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Our arguments for the proof of Theorem 2 are entirely explicit. In particular, if $f_{N+1} - f_N \leq 100$, then $N \leq 26$.

2 The proof of Theorem 1

Letting α and β be the two roots of the quadratic equation $x^2 - ax - b = 0$ of the Lucas sequence $(u_n)_{n \geq 0}$ with $u_0 = 0$, $u_1 = 1$ of recurrence $u_{n+2} = au_{n+1} + bu_n$ for all $n \geq 0$, we have

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{for } n = 0, 1, \ldots.$$

We make the convention that $|\alpha| \geq |\beta|$. We write $\Delta = (\alpha - \beta)^2$ and we call it the *discriminant* of the sequence. In the particular case of the Fibonacci sequence, we have $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, and $\Delta = 5$, while in the particular case of the Lucas sequence involved in the q-binomial coefficient we have $\alpha = q$, $\beta = 1$, and $\Delta = (q - 1)^2$.

A *Primitive Divisor* p of the nth term u_n of a Lucas sequence $(u_n)_{n \geq 0}$ is a prime factor of u_n which does not divide $\Delta \prod_{1 \leq m \leq n-1} u_m$. It is known that a primitive divisor p of u_n exists whenever $n \geq 13$ if α and β are real and 13 can be replaced by 7 if α and β are integers (see, for example, [4]). The above statement is usually referred to as the *Primitive Divisor Theorem* (see [3] for the most general version). It is also known that such a primitive divisor p satisfies $p \equiv \pm 1 \pmod{n}$.

We are now ready to deal with equation (3). We may assume that $m \neq n$ and that $l \neq k$. Since $m \neq n$, we may assume that $n > m$. If $l \geq k$, then

$$\begin{aligned}
\begin{bmatrix} n \\ l \end{bmatrix}_F &= \left(\frac{F_n}{F_1}\right)\left(\frac{F_{n-1}}{F_2}\right) \cdots \left(\frac{F_{n-l+1}}{F_l}\right) \geq \left(\frac{F_n}{F_1}\right) \cdots \left(\frac{F_{n-k+1}}{F_k}\right) \\
&> \left(\frac{F_m}{F_1}\right) \cdots \left(\frac{F_{m-k+1}}{F_k}\right) = \begin{bmatrix} m \\ k \end{bmatrix}_F,
\end{aligned}$$

where we used the fact that $F_n > F_m$ because $n \geq 3$ (which follows because $n > m \geq 2k \geq 2$). Hence, assuming that $n > m$ in equation (3) for the Fibonomial coefficients, we deduce that $l < k$. Thus, $n > m \geq 2k > 2l$. A similar argument holds for the case of the q-binomial coefficients. Now if $n \geq 13$, then by the Primitive Divisor Theorem there exists a primitive prime factor p for F_n. This prime will obviously divide $\begin{bmatrix} n \\ l \end{bmatrix}_F$, since p does not divide $F_1 \cdots F_l$, but it cannot divide $\begin{bmatrix} m \\ k \end{bmatrix}_F$, because p does not divide $F_1 \cdots F_m$. This shows that $n \leq 12$ and a quick computation reveals that there are no equal Fibonomial coefficients in the range $2 \leq 2l < 2k \leq m < n \leq 12$.

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In the case of the q-binomial coefficients, the Primitive Divisor Theorem tells us that $n \leq 6$. Since $2 \leq 2k < 2l \leq m < n \leq 6$, the only possibilities are $(k, l, m, n) = (1, 2, 4, 5), (1, 2, 4, 6), (1, 2, 5, 6)$. In the case $(k, l, m, n) = (1, 2, 4, 5)$, the Diophantine equation (3) for q-binomial coefficients leads to

$$\frac{(q^4 - 1)(q^3 - 1)}{(q - 1)(q^2 - 1)} = \frac{q^5 - 1}{q - 1},$$

which is equivalent to $q^3 + q^4 = q^2 + q^5$, which is impossible because of the uniqueness of the base q expansion of a positive integer. The cases $(k, l, m, n) = (1, 2, 4, 6)$ and $(1, 2, 5, 6)$ of the Diophantine equation (3) for q-binomial coefficients lead to

$$\frac{(q^4 - 1)(q^3 - 1)}{(q - 1)(q^2 - 1)} = \frac{q^6 - 1}{q - 1}, \qquad \text{or} \qquad q^7 + q^6 + q^2 = q^8 + q^4 + q^3,$$

and

$$\frac{(q^5 - 1)(q^4 - 1)}{(q - 1)(q^2 - 1)} = \frac{q^6 - 1}{q - 1}, \qquad \text{or} \qquad q^9 + q^6 + q^2 = q^8 + q^5 + q^4,$$

respectively, both of which are impossible again by the uniqueness of the base q expansion of a positive integer.

3   The proof of Theorem 2

We keep the notations from the previous section, and start with some estimates for $\begin{bmatrix} m \\ k \end{bmatrix}_F$. Let

$$p := \prod_{i \geq 1} \left(1 - \left(\frac{\beta}{\alpha}\right)^i\right) = \prod_{i \geq 1} \left(1 - (-\alpha^2)^{-i}\right) \sim 1.226742\cdots. \qquad (4)$$

Lemma 1 *We have*
$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{\alpha^{mk-k^2}}{p}(1 + \zeta_{m,k}),$$
where $\zeta_{m,k}$ is a real number satisfying

$$|\zeta_{m,k}| < \frac{2}{\alpha^{2k+1}}.$$

Proof We have

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 F_2 \cdots F_k}$$

$$= \alpha^{m+(m-1)+\cdots+(m-k+1)-1-2-\cdots-k} \prod_{1 \le i \le k} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right)^{-1}$$

$$= \alpha^{mk-k^2} \prod_{1 \le i \le k} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right)^{-1}.$$

Now observe that

$$\prod_{1 \le i \le k} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right)^{-1} = p^{-1} \prod_{i \ge k+1} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right) = p^{-1}(1 + \zeta_{m,k}).$$

It remains to estimate $\zeta_{m,k}$. We use the inequality

$$e^z > 1 + z > \begin{cases} e^{z/2} & \text{if } z \in (0, 1/4), \\ e^{2z} & \text{if } z \in (-1/4, 0). \end{cases} \tag{5}$$

We shall use the above inequality (4) with $z = -(\beta/\alpha)^i$ for $i \ge k + 1$. Note that the inequality $|z| \le \alpha^{-2(k+1)} \le \alpha^{-4} < 1/4$ holds for all $k \ge 1$. We get that

$$1 + \zeta_{m,k} = \prod_{i \ge k+1} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right) < \exp \left(- \sum_{i \ge k+1} \left(\frac{\beta}{\alpha} \right)^i \right)$$

$$= \exp \left(\frac{(-1)^{k+2}}{\alpha^{2k+2}(1 - \beta/\alpha)} \right) \le \exp \left(\frac{1}{\sqrt{5}\alpha^{2k+1}} \right)$$

$$\le 1 + \frac{2}{\sqrt{5}\alpha^{2k+1}} < 1 + \frac{1}{\alpha^{2k+1}}.$$

Thus,

$$\zeta_{m,k} < \frac{1}{\alpha^{2k+1}}. \tag{6}$$

By a similar calculation using the right hand side of inequalities (4), we get

$$1 + \zeta_{m,k} \ge \exp \left(-2 \sum_{i \ge k+1} \left| \frac{\beta}{\alpha} \right|^i \right) = \exp \left(-\frac{2}{\alpha^{2k+2}(1 - \alpha^{-2})} \right)$$

$$= \exp \left(-\frac{2}{\alpha^{2k+1}} \right) > 1 - \frac{2}{\alpha^{2k+1}},$$

yielding

$$\zeta_{m,k} > -\frac{2}{\alpha^{2k+1}}. \tag{7}$$

The desired inequality now follows from estimates (5) and (6). $\quad \square$

Lemma 2 *We have*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{\alpha^{mk-k(k-1)/2}}{5^{k/2}F_1\cdots F_k}(1+\zeta'_{m,k}),$$

where

$$|\zeta'_{m,k}| < \frac{2}{\alpha^{2m-2k+1}}.$$

Proof We write

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1}\cdots F_{m-k+1}}{F_1 F_2 \cdots F_k}$$

$$= \frac{\alpha^{m+(m-1)+\cdots+(m-k+1)}}{5^{k/2}F_1 F_2 \cdots F_k}\prod_{i=0}^{k-1}\left(1-\left(\frac{\beta}{\alpha}\right)^{m-i}\right).$$

We now study the last product above. When $k = 1$, then the above product is

$$1+\zeta'_{m,1} = 1 - \frac{(-1)^m}{\alpha^{2m}}.$$

Thus,

$$|\zeta'_{m,1}| = \frac{1}{\alpha^{2m}} < \frac{2}{\alpha^{2m-1}},$$

so the desired inequality holds in this case. Assume now that $k \geq 2$. We then have, again by inequalities (4), that

$$1+\zeta'_{m,k} = \prod_{i=0}^{k-1}\left(1-\left(\frac{\beta}{\alpha}\right)^{m-i}\right) < \exp\left(\sum_{i=0}^{k-1}\frac{1}{\alpha^{2(m-i)}}\right)$$

$$\leq \exp\left(\frac{1}{\alpha^{2(m-k+1)}(1-\alpha^{-2})}\right)$$

$$= \exp\left(\frac{1}{\alpha^{2m-2k+1}}\right) < 1 + \frac{2}{\alpha^{2m-2k+1}}.$$

In the above inequality, we used again inequality (4) with $z = 2/\alpha^{2m-2k+1}$ together with the fact that $z \leq 2/\alpha^{2k+1} \leq 2/\alpha^5 < 1/4$ holds for all integers $k \geq 2$. Thus,

$$\zeta'_{m,k} < \frac{2}{\alpha^{2m-2k+1}}.$$

Using now the right hand side of inequality (4), we get

$$1+\zeta'_{m,k} \geq \exp\left(-2\sum_{i=0}^{k-1}\frac{1}{\alpha^{2(m-i)}}\right) \geq \exp\left(-\frac{2}{\alpha^{2m-2k+1}}\right) \geq 1 - \frac{2}{\alpha^{2m-2k+1}},$$

leading to

$$\zeta'_{m,k} > -\frac{2}{\alpha^{2m-2k+1}}.$$

This completes the proof of this lemma. $\square$

Lemma 3 *The inequality*

$$\left| \frac{\alpha^{k(k+1)/2-l(l+1)/2}}{5^{(k-l)/2}F_{l+1}\cdots F_k} - 1 \right| > \frac{1}{\alpha^{2l+5}} \tag{8}$$

holds for all $k > l \geq 1$.

Proof Clearly,

$$\left| \frac{\alpha^{k(k+1)/2-l(l+1)/2}}{5^{(k-l)/2}F_{l+1}\cdots F_k} - 1 \right| = \left| \prod_{i=l+1}^{k}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)^{-1} - 1 \right|.$$

If $k = l + 1$, then the above expression becomes

$$\left| \frac{1}{1 \pm \alpha^{-2l-2}} - 1 \right| = \frac{1}{\alpha^{2l+2}(1 \pm \alpha^{-2l-2})} > \frac{1}{\alpha^{2l+2}(1 + \alpha^{-1})} = \frac{1}{\alpha^{2l+3}}.$$

Assume now that $k \geq l + 2$. Observe that if i is odd, then

$$\left(1-\left(\frac{\beta}{\alpha}\right)^{i+1}\right)\left(1-\left(\frac{\beta}{\alpha}\right)^{i+2}\right) = \left(1-\frac{1}{\alpha^{2i+2}}\right)\left(1+\frac{1}{\alpha^{2i+4}}\right)$$

$$= 1 - \frac{1}{\alpha^{2i+2}} + \frac{1}{\alpha^{2i+4}} - \frac{1}{\alpha^{4i+6}}$$

$$= 1 - \frac{1}{\alpha^{2i+3}} - \frac{1}{\alpha^{4i+6}} < 1 - \frac{1}{\alpha^{2i+3}}.$$

In particular, if l is odd, then

$$\prod_{i=l+1}^{k}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right) < 1 - \frac{1}{\alpha^{2l+3}},$$

so that

$$\left| \prod_{i=l+1}^{k}\left(1-\left(\frac{\beta}{\alpha}\right)^i\right)^{-1} - 1 \right| > \frac{1}{1 - 1/\alpha^{2l+3}} - 1 > \frac{1}{\alpha^{2l+3}(1 - \alpha^{-2})} > \frac{1}{\alpha^{2l+3}}.$$

However, if i is even, then

$$\left(1-\left(\frac{\beta}{\alpha}\right)^{i+1}\right)\left(1-\left(\frac{\beta}{\alpha}\right)^{i+2}\right) = \left(1+\frac{1}{\alpha^{2i+2}}\right)\left(1-\frac{1}{\alpha^{2i+4}}\right)$$

$$= 1 + \frac{1}{\alpha^{2i+2}} - \frac{1}{\alpha^{2i+4}} - \frac{1}{\alpha^{4i+6}}$$

$$> 1 + \frac{1}{\alpha^{2i+3}} - \frac{1}{\alpha^{2i+5}} = 1 + \frac{1}{\alpha^{2i+4}},$$

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so that, if l is even, then

$$\prod_{i=l+1}^{k} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right) > 1 + \frac{1}{\alpha^{2l+4}},$$

leading to

$$\left| \prod_{i=l+1}^{k} \left(1 - \left(\frac{\beta}{\alpha} \right)^i \right)^{-1} - 1 \right| \geq 1 - \frac{1}{1 + 1/\alpha^{2l+4}} > \frac{1}{\alpha^{2l+5}},$$

which completes the proof of this lemma. □

It is known that the number p appearing at (??) is transcendental (apply, for example, Lemma 1 and Lemma 2 from [6] to the function $f(z) = \eta(z)$ and the algebraic number $q = \beta/\alpha$). In particular, it cannot be of the form $5^{l/2}/\alpha^s$ for any positive integers l and s The next lemma tells us even more, namely that the number p cannot be approximated too well by numbers of the form $5^{l/2}/\alpha^s$ for positive integers l and s.

Lemma 4 *The inequality*

$$\left| \frac{\alpha^s}{5^{l/2}} - \frac{1}{p} \right| > \frac{1}{5^{3l/2}}$$

holds for all positive integers s and l with finitely many exceptions.

Proof Assume that s is large and that

$$\left| \frac{\alpha^s}{5^{l/2}} - \frac{1}{p} \right| < \frac{1}{5^{3l/2}}$$

holds with some positive integer l. Then also the inequality

$$\left| \frac{5^{l/2}}{\alpha^s} - p \right| \ll \frac{1}{\alpha^{3s}}$$

holds, where we can take the above implied constant as $2/p^2$ once s is sufficiently large. By Euler's pentagonal formula,

$$p = \sum_{n \geq 1} (-1)^n \left(\left(\frac{\beta}{\alpha} \right)^{n(3n-1)/2} + \left(\frac{\beta}{\alpha} \right)^{n(3n+1)/2} \right)$$
$$= \sum_{n \geq 1} \left(\frac{\varepsilon_{n,1}}{\alpha^{n(3n-1)}} + \frac{\varepsilon_{n,2}}{\alpha^{n(3n+1)}} \right),$$

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for some signs $\varepsilon_{n,1}$, $\varepsilon_{n,2} \in \{\pm 1\}$. Let $N := N_s$ be the minimal positive integer such that $N(3N + 1) > 2s + 10$. Then both estimates

$$N(3N + 1) = 2s + O(s^{1/2}),$$

and

$$(N + 1)(3(N + 1) - 1) - N(3N + 1) = 4N + 2 \in (c_0 s^{1/2}, c_1 s^{1/2})$$

hold for large s with some positive constants c_0 and c_1. Thus,

$$\left| \frac{5^{l/2}}{\alpha^s} - \sum_{n=1}^{N} \left(\frac{\varepsilon_{n,1}}{\alpha^{n(3n-1)}} + \frac{\varepsilon_{n,2}}{\alpha^{n(3n+1)}} \right) \right| = O\left(\frac{1}{\alpha^{3s}} + \frac{1}{\alpha^{(N+1)(3(N+1)-1)}} \right).$$

Multiplying both sides of the above approximation by $\alpha^{N(3N+1)}$ we get

$$|5^{l/2}\alpha^{N(3N+1)-s} - \sum_{n=1}^{N} \left(\varepsilon_{n,1}\alpha^{u_{n,1}} + \varepsilon_{n,2}\alpha^{u_{n,2}} \right)| \ll \frac{1}{\alpha^{s+O(s^{1/2})}} + \frac{1}{\alpha^{4N}} \qquad (9)$$
$$\ll \frac{1}{\alpha^{c_0 s^{1/2}}}$$

provided that s is sufficiently large. Here, $u_{n,1}$ and $u_{n,2}$ stand for the nonnegative exponents of the form $N(3N+1) - n(3n-1)$ and $N(3N+1) - n(3n+1)$, respectively, for all positive integers $n = 1, \ldots, N$. Put now $u = N(3N+1) - s > s + 10$ and observe that the number

$$\gamma = 5^{l/2}\alpha^u - \sum_{n=1}^{N} \left(\varepsilon_{n,1}\alpha^{u_{n,1}} + \varepsilon_{n,2}\alpha^{u_{n,2}} \right)$$

is an algebraic integer in $\mathbb{K} = \mathbb{Q}[\sqrt{5}]$. Its absolute value is, by (8), bounded above by $O(\alpha^{-c_0\sqrt{s}})$. Its conjugate in $\mathbb{K}$ is

$$\sigma(\gamma) = \pm 5^{l/2}\beta^u - \sum_{n=1}^{N} \left(\varepsilon_{n,1}\beta^{u_{n,1}} + \varepsilon_{n,2}\beta^{u_{n,2}} \right).$$

Since s is assumed to be large, it follows that $\alpha^s \asymp 5^{l/2}$, therefore $|5^{l/2}\beta^u| = 5^{l/2}\alpha^{-u} < 5^{l/2}\alpha^{-s} \ll 1$. Since also

$$\left| \sum_{n \leq N} \left(\varepsilon_{n,1}\beta^{u_{n,1}} + \varepsilon_{n,2}\beta^{u_{n,2}} \right) \right| < \sum_{n \geq 0} |\beta|^n = \frac{1}{1 - |\beta|} \ll 1,$$

it follows that $\sigma(\gamma) \ll 1$. Thus,

$$|N_{\mathbb{K}/\mathbb{Q}}(\gamma)| = |\gamma||\sigma(\gamma)| \ll \frac{1}{\alpha^{c_0\sqrt{s}}}.$$

However, $|N_{K/\mathbb{Q}}(\gamma)|$ is an integer. For large s, the above inequality is possible only when $\gamma = 0$.

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We now study this last condition and show that it is impossible, which will complete the proof of the lemma. Assume that $\gamma = 0$. Then we get

$$5^{l/2} = \sum_{n \leq N} \left(\varepsilon_{n,1} \alpha^{u_{n,1}-u} + \varepsilon_{n,2} \alpha^{u_{n,2}-u} \right). \tag{10}$$

Conjugating in $\mathbb{K}$, we get that

$$\varepsilon 5^{l/2} = \sum_{n \leq N} \left(\varepsilon_{n,1} \beta^{u_{n,1}-u} + \varepsilon_{n,2} \beta^{u_{n,2}-u} \right), \tag{11}$$

where $\varepsilon \in \{\pm 1\}$ according to whether l is even or odd. Suppose say that $\varepsilon = 1$. Then subtracting the above relations (9) and (10) and dividing both sides of the resulting relation by $\sqrt{5}$, we get

$$\sum_{n \leq N} \left(\varepsilon_{n,1} F_{u_{n,1}-u} + \varepsilon_{n,2} F_{u_{n,2}-u} \right) = 0. \tag{12}$$

Observe that the indices in the above relation (11) satisfy

$$u_{1,1} - u < u_{1,2} - u < u_{2,1} - u < u_{2,2} - u < \cdots < u_{N,1} - u < u_{N,2} - u. \tag{13}$$

Let

$$\mathcal{M} = \{|u_{n,i} - u| : 1 \leq n \leq N, \ i = 1, 2\} = \{m_1, m_2, \ldots, m_t\},$$

where $m_1 < m_2 < \cdots < m_t := M$. Observe that all members of $\mathcal{M}$ are nonnegative. Furthermore, because of inequalities (12) for each $m_j \in \mathcal{M}$, there are at most two numbers $u_{n,i} - u$ in the string (12) such their absolute values is m_j, and if there are two then one of them is the negative of the other. Observe also that since $u_{n,i}$ is always even, it follows that all the numbers in $\mathcal{M}$ have the same parity. Finally, let us observe that the first three large values of $\mathcal{M}$ appear only once. Indeed, the largest positive member in (12) is $u_{N,2} - u = N(3N + 1) - (N(3N + 1) - s) = s$, whereas the first three negative ones in (12) are

$$u_{1,1} - u = -(N(3N + 1) - (s + 2)) < -(s + 8),$$
$$u_{1,2} - u = -(N(3N + 1) - (s + 4)) < -(s + 6),$$
$$u_{2,1} - u = -(N(3N + 1) - (s + 10) < -s.$$

Thus,

$$M = m_t = N(3N + 1) - (s + 2),$$
$$m_{t-1} = N(3N + 1) - (s + 4) = M - 2,$$
$$m_{t-2} = N(3N + 1) - (s + 10) = M - 8,$$

are the first three largest elements in $\mathcal{M}$, and for each of them, there is only one element (namely the corresponding negative one) in the string (12) whose

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absolute value is this given element. From the above discussion, and using also the fact that $F_{-m} = (-1)^{m-1}F_m$, we deduce that relation (11) leads to the inequality

$$F_M \le F_{M-2} + F_{M-8} + 2 \sum_{5 \le k \le M/2} F_{M-2k}. \tag{14}$$

Using the fact that $F_m = \alpha^m/\sqrt{5} + O(1)$, we get that

$$\sum_{5 \le k \le M/2} F_{M-2k} < F_{M-10} + F_{M-9} + \cdots + F_1$$

$$= \frac{1}{\sqrt{5}}(\alpha^{M-10} + \alpha^{M-9} + \cdots + 1) + O(M)$$

$$= \frac{1}{\sqrt{5}(\alpha - 1)}\alpha^{M-9} + O(M).$$

Thus, relation (13) leads to

$$\frac{\alpha^M}{\sqrt{5}} \le \frac{\alpha^{M-2}}{\sqrt{5}} + \frac{\alpha^{M-8}}{\sqrt{5}} + \frac{2\alpha^{M-9}}{\sqrt{5}(\alpha - 1)} + O(M),$$

or

$$\alpha^9 \le \alpha^7 + \alpha + \frac{2}{\alpha - 1} + O\left(\frac{M}{\alpha^M}\right), \tag{15}$$

and this is false for large M; hence, for large s.

The case when $\varepsilon = -1$ is similar. In this case, we sum up relations (9) and (10) and get a relation similar to (11), except that the Fibonacci numbers are replaced by the Lucas numbers $(L_n)_{n \ge 0}$, where $L_0 = 0$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \ge 0$. The general term of the Lucas sequence $(L_n)_{n \ge 0}$ is $L_n = \alpha^n + \beta^n$. A similar argument leads to inequality (14) from which the same contradiction as in the case $\varepsilon = 1$ is derived. This shows that γ cannot be zero and completes the proof of the lemma. $\square$

The proof of Theorem 2.

In parallel with the proof of this theorem, we also show that $f_{N+1} - f_N > 100$ for $N > 26$.

Let N be sufficiently large, say at least such that $\log \log \log f_N > 1$. Let K be tending to infinity with N. We need to show that if K tends to infinity with N sufficiently slowly, say, if $K < c_2(\log f_N)^{1/2}$ with a sufficiently small positive constant c_2, then the Diophantine inequality

$$\left| \begin{bmatrix} m \\ k \end{bmatrix}_F - \begin{bmatrix} n \\ l \end{bmatrix}_F \right| \le K \tag{16}$$

has only finitely many positive integer solutions (m, k, n, l) with $1 \leq k \leq m/2$, $1 \leq l \leq n/2$, $(m, k) \neq (n, l)$. We assume that $n \geq m$. Observe that by Lemma 1, we have that for N large,

$$f_N = \exp(mk - k^2 + O(1)) = \exp(nl - l^2 + O(1)),$$

so that both inequalities $mk - k^2 \gg \log f_N$ and $nl - l^2 \gg \log f_N$ hold. Since $mk - k^2 = k(m - k) \leq (m/2)^2$, we get that $m \gg (\log f_N)^{1/2}$. Let c_3 be the implied constant above. Assume that $K \leq c_3(\log f_N)^{1/2} - 3$. Then $m > K + 2$. For large N; hence, for large m, F_m has a primitive divisor which is at least as large as $m - 1 \geq K + 1$.

In the particular case that we take $K = 100$, we also assume that $m > 100$ since the remaining cases can be checked using Mathematica.

Assume first that $n = m$. Let P be some primitive prime factor of F_n. Then certainly P divides both $\begin{bmatrix} m \\ k \end{bmatrix}_F$ and $\begin{bmatrix} n \\ l \end{bmatrix}_F$, therefore it divides their difference which is $\leq K$ by inequality (15). Since $P \equiv \pm 1 \pmod{n}$, we get that $P \geq n - 1 \geq K + 1$. Thus, $P \geq K + 2$, so the only possibility is therefore that the difference appearing in the left hand side of inequality (15) is zero. However, this is impossible for $(m, k) \neq (n, l)$ by Theorem 1.

Thus, we may assume that $n > m$. Assume next that $l \geq k$. Then

$$\begin{bmatrix} n \\ l \end{bmatrix}_F - \begin{bmatrix} m \\ k \end{bmatrix}_F \geq \begin{bmatrix} n \\ k \end{bmatrix}_F - \begin{bmatrix} n-1 \\ k \end{bmatrix}_F = \frac{F_{n-1} \cdots F_{n-k+1}(F_n - F_{n-k})}{F_1 \cdots F_k}$$
$$\geq (F_n - F_{n-1}) \left(\frac{F_{n-1}}{F_2} \right) \cdots \left(\frac{F_{n-(k-1)}}{F_k} \right) > F_{n-2} > \exp(c_3 (\log f_N)^{1/2}),$$
$$(17)$$

where we used the fact that $n - i \geq i + 1$ for all $i = 1, \ldots, k - 1$. So, in particular, the above difference exceeds $c_3(\log f_N)^{1/2} > K$ for N sufficiently large. Thus, we may assume that $n > m \geq 2k > 2l$. In particular, $n \geq 2l + 3$.

Next, let us notice that $m \leq n - l$. Indeed, assume that this is not so. Let P be a primitive prime factor of F_m. Then P divides both $\begin{bmatrix} m \\ k \end{bmatrix}_F$ and $\begin{bmatrix} n \\ l \end{bmatrix}_F$, so again P divides their difference which is $\leq K$. Since $P \geq K + 2$, we get again a contradiction. Thus, $m \leq n - l$.

Next, we write

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{\alpha^{mk-k^2}}{p}(1+\zeta_{m,k}) \qquad \text{and} \qquad \begin{bmatrix} n \\ l \end{bmatrix}_F = \frac{\alpha^{nl-l^2}}{p}(1+\zeta_{n,l}).$$

Put $u = \max\{nl - l^2, mk - k^2\}$. By inequality (15) together with Lemma 1, we have

$$|\alpha^{mk-k^2} - \alpha^{nl-l^2}| < Kp + \alpha^u p(|\zeta_{m,k}| + |\zeta_{n,l}|) < Kp + \frac{4p\alpha^u}{\alpha^{2l+1}}.$$

Dividing both sides of the above inequality by α^u, putting $\lambda = (mk - k^2) - (nl - l^2)$, and observing that

$$u \geq l(n-l) \geq l(l+3),$$

we get that

$$|1 - \alpha^{-|\lambda|}| \leq \frac{Kp}{\alpha^{l(l+3)}} + \frac{4p}{\alpha^{2l+1}}. \tag{18}$$

We now distinguish two cases according to whether $\lambda \neq 0$ or $\lambda = 0$, respectively.

The Case $\lambda \neq 0$.

This is the heart of the proof. We find it easier to explain it in the particular case when $K = 100$, and to treat the general case later.

The case $K = 100$.

In this case, the left hand side of the above inequality (17) is > 0.38, and we therefore get the inequality

$$0.38 < \frac{123}{\alpha^{l(l+3)}} + \frac{5}{\alpha^{2l+1}},$$

leading to $l = 1, 2$. Since $l = 1, 2$, by Lemma 2, we have

$$\begin{bmatrix} n \\ l \end{bmatrix}_F = \frac{\alpha^{nl-l(l-1)/2}}{5^{l/2}}(1+\zeta'_{n,l}),$$

where

$$|\zeta'_{n,l}| \leq \frac{2}{\alpha^{2n-3}}.$$

Thus, inequality (15) together with Lemma 2 now lead to

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$$\left| \frac{\alpha^{nl-l^2+l(l+1)/2}}{5^{l/2}} - \frac{\alpha^{mk-k^2}}{p} \right| < 100 + \alpha^u \left(\frac{\alpha^{l(l+1)/2}}{5^{l/2}} |\zeta'_{n,l}| + \frac{|\zeta_{m,k}|}{p} \right)$$

$$< 100 + \alpha^u \left(\frac{2\alpha^3}{\sqrt{5}\alpha^{2n-3}} + \frac{2}{p\alpha^{2k+1}} \right)$$

$$< 100 + \alpha^u \left(\frac{1}{\alpha^{2n-6}} + \frac{2}{p\alpha^{2k+1}} \right).$$

Observe that since $n > m \geq 2k$, we have that $2n - 6 \geq 2(2k+1) - 6 = 4k - 4 \geq 2k + 2$ provided that $k \geq 3$. The inequality $2n - 6 \geq 2k + 2$ holds also when $k = 2$ since $n > 100$. We thus get that the inequality $2n - 6 \geq 2k + 2$ holds always in our range, which leads to

$$\left| \frac{\alpha^{nl-l^2+l(l+1)/2}}{5^{l/2}} - \frac{\alpha^{mk-k^2}}{p} \right| < 100 + \frac{\alpha^u}{\alpha^{2k+1}} \left(\frac{1}{\alpha} + \frac{3}{p} \right)$$

$$< 100 + \frac{\alpha^u}{\alpha^{2k-2}},$$

where we also used the fact that $1/\alpha + 3/p < \alpha^3$. We now divide both sides of the above inequality again by α^u getting

$$\left| \frac{\alpha^{nl-l^2+l(l+1)/2-u}}{5^{l/2}} - \frac{\alpha^{mk-k^2-u}}{p} \right| < \frac{100}{\alpha^{k^2}} + \frac{1}{\alpha^{2k-2}}. \tag{19}$$

In the last inequality above, we also used the fact that $u \geq k(m-k) \geq k^2$. Note that the left side of the above inequality is either

$$\left| \frac{\alpha^{l(l+1)/2}}{5^{l/2}} - \frac{\alpha^{-|\lambda|}}{p} \right|, \qquad \text{or} \qquad \left| \frac{\alpha^{l(l+1)/2-|\lambda|}}{5^{l/2}} - \frac{1}{p} \right|, \tag{20}$$

according to whether $u = nl - l^2$ or $mk - k^2$. Since $\lambda \neq 0$, in the first case we get that $\alpha^{-|\lambda|}/p < \alpha^{-1}/1.22 < 0.51$, while $\alpha^{l(l+1)/2}/5^{l/2} \geq \alpha/\sqrt{5} \geq 0.71$. Thus, in the first case the left hand side of inequality (18) exceeds 0.2. In the second case, again since $\lambda \neq 0$, it follows that the inequality $|\lambda| \geq l(l+1)/2$ holds in all cases except when $|\lambda| = 1$, 2 and $l = 2$. Hence,

$$\frac{\alpha^{l(l+1)/2-|\lambda|}}{5^{l/2}} \leq \max \left\{ \frac{1}{5^{1/2}}, \frac{\alpha^2}{5} \right\} = \frac{\alpha^2}{5}.$$

This shows that in this second case the left hand side of inequality (18) is at least $1/p - \alpha^2/5 > 0.2$. Thus, in both cases we have that

$$0.2 < \frac{100}{\alpha^{k^2}} + \frac{1}{\alpha^{2k-2}}, \tag{21}$$

giving $k \leq 3$.

Hence, $(l, k) \in \{(1, 2), (1, 3), (2, 3)\}$. Next, we use again inequality (15) and Lemma 2 together with the obvious fact that

$$\max\left\{\frac{\alpha^{mk-k(k-1)/2}}{5^{k/2}F_1 \cdots F_k}, \frac{\alpha^{nl-l(l-1)/2}}{5^{l/2}F_1 \cdots F_l}\right\} \le \frac{\alpha^u}{5^{1/2}},$$

to get that

$$\left|\frac{\alpha^{mk-k(k-1)/2}}{5^{k/2}F_1 \cdots F_k} - \frac{\alpha^{nl-l(l-1)/2}}{5^{l/2}F_1 \cdots F_l}\right| < 100 + \frac{\alpha^u}{\sqrt{5}}\left(\frac{2}{\alpha^{2n-3}} + \frac{2}{\alpha^{2m-5}}\right)$$
$$< 100 + \frac{\alpha^u}{\alpha^{2m-7}}, \qquad (22)$$

where we used the fact that

$$\frac{2}{\alpha^{2n-3}} + \frac{2}{\alpha^{2m-5}} \le \frac{1}{\alpha^{2m-5}}\left(\frac{2}{\alpha^{2n-2m+2}} + 2\right) < \frac{1}{\alpha^{2m-7}},$$

since

$$\frac{2}{\alpha^{2n-2m+2}} + 2 \le \frac{2}{\alpha^4} + 2 < \alpha^2.$$

Dividing again both sides of the above inequality (21) by α^u and using the fact that $u \ge k(m - k) \ge 2(m - 2)$, we get that

$$\left|\frac{\alpha^{mk-k^2-u+k(k+1)/2}}{5^{k/2}F_1 \cdots F_k} - \frac{\alpha^{nl-l^2-u+l(l+1)/2}}{5^{l/2}F_1 \cdots F_l}\right| < \frac{100}{\alpha^{2m-4}} + \frac{1}{\alpha^{2m-7}}. \qquad (23)$$

The left hand side above is either

$$\left|\frac{\alpha^{k(k+1)/2}}{5^{k/2}F_1 \cdots F_k} - \frac{\alpha^{l(l+1)/2-|\lambda|}}{5^{l/2}F_1 \cdots F_l}\right|, \quad \text{or} \quad \left|\frac{\alpha^{k(k+1)/2-|\lambda|}}{5^{k/2}F_1 \cdots F_k} - \frac{\alpha^{l(l+1)/2}}{5^{l/2}F_1 \cdots F_l}\right|, \qquad (24)$$

according to whether $u = nl - l^2$ or $mk - k^2$. In the first case, we have that

$$\alpha^{k(k+1)/2}/(5^{k/2}F_1 \cdots F_k) \ge 0.8$$

for $k = 2, 3$, while $\alpha^{l(l+1)/2-|\lambda|}/(5^{l/2}F_1 \cdots F_l) \le 1/5^{1/2} < 0.5$ by an argument already used. Thus, in this case the left hand side of inequality (22) is at least 0.3. In the second case, we have that

$$\alpha^{l(l+1)/2}/(5^{l/2}F_1 \cdots F_l) \ge 0.72$$

for $l = 1, 2$, while

$$\frac{\alpha^{k(k+1)/2-|\lambda|}}{5^{k/2}F_1 \cdots F_k} \le \max\left\{\frac{\alpha^2}{5}, \frac{\alpha^5}{2 \cdot 5^{3/2}}\right\} < .53$$

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for $k = 2, 3$ and $\lambda \neq 0$, therefore the left hand side of inequality (22) exceeds 0.19 in this case. Thus, in both cases we have

$$0.19 < \frac{100}{\alpha^{2m-4}} + \frac{1}{\alpha^{2m-7}}, \qquad (25)$$

leading to $m < 9$, which is false.

This takes care of the case $\lambda \neq 0$ when $K = 100$.

The case of the general K.

In the case of the general K and when $\lambda \neq 0$, the inequality (17) shows that $l = O((\log K)^{1/2})$. Then the arguments from the case $K = 100$ show that the analog inequality (18) holds with 100 replaced by K and with the exponent $2k - 2$ replaced by $2k + O(\log K)$. Indeed, the only inequality to justify is the fact that the difference $n - k$ grows faster than any fixed multiple of $\log K$ once N is sufficiently large.

Well, assuming that this were not so, we would get that infinitely often $n - k < c_4 \log K$ holds with some positive constant c_4. Thus, $k \leq m - k < n - k \ll \log K$ implying $k \ll \log K$, and later that $m < n \leq k + O(\log K) \ll \log K$. Thus, $k(m - k) \ll (\log K)^2 \ll (\log\log f_N)^2$, which combined with the fact that $k(m - k) \gg \log f_N$ gives only finitely many possibilities for N. Thus, if N is sufficiently large, we get to inequality (18) with the right hand side replaced by $\alpha^{-2k+O(\log K)}$.

Now inequality (18) with 100 replaced by K together with the lower bound given by Lemma 4 on the expression appearing in the right hand side in display (19) lead to an inequality of the form

$$\frac{1}{5^{3l/2}} \ll \frac{1}{\alpha^{2k+O(\log K)}},$$

implying $k \ll l + O(\log K)$. Since also $l \ll (\log K)^{1/2}$, we get that $k \ll \log K$.

We thus bounded both l and k in terms of K. Following through the arguments from the case when $K = 100$, we arrive at the analog of inequality (22) with 100 replaced by K, which implies that

$$\left| 1 - \alpha^{k(k+1)/2 - l(l+1)/2 \pm |\lambda|} 5^{-(k-l)/2} (F_{l+1} \cdots F_k)^{-1} \right| \ll \frac{1}{\alpha^{2m+O(\log K)}}. \qquad (26)$$

The right hand side above is not zero. Indeed, if it were, then since no power of α of nonzero exponent can be an integer, we get that the exponent of α in the left hand side of equation (25) is zero, and further that $F_{l+1} \cdots F_k = 5^{(k-l)/2}$. By the Primitive Divisor Theorem, the above relation is false for $k > 12$,

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and it can be checked that it does not hold for any $1 \leq l < k \leq 12$ either. Thus, the left hand side of equation (25) is indeed nonzero. Furthermore, since $\alpha^{k(k+1)/2} \sim p5^{k/2}F_1 \cdots F_k$ as $k \to \infty$, it follows that the only chance the above expression from the right hand side has of being small is when $\lambda = O(1)$. We now use a linear form in logarithms á la Baker [2], which states that if α_1, α_2, α_3 are algebraic numbers of heights H_1, H_2, H_3, respectively, and b_1, b_2, b_3 are integers of absolute value at most B such that

$$\Lambda = \alpha_1^{b_1} \alpha_2^{b_2} \alpha_3^{b_3} - 1 \neq 0,$$

then

$$|\Lambda| > \exp(-c_5 \log(H_1 + 2) \log(H_2 + 2) \log(H_3 + 2) \log(B + 2))$$

for some positive constant c_5 depending on the degree of the field $\mathbb{Q}[\alpha_1, \alpha_2, \alpha_3]$ over $\mathbb{Q}$. Recall that the height of an algebraic number is the maximum absolute value of the coefficients of its minimal polynomial over the integers. We apply this with

$$\alpha_1 = \alpha, \qquad \alpha_2 = \sqrt{5}, \qquad \alpha_3 = F_{l+1} \cdots F_k,$$

and

$$b_1 = k(k+1)/2 - l(l+1)/2 \pm |\lambda|, \qquad b_2 = -(k-l), \qquad b_3 = -1.$$

Note that $H_1 = O(1)$, $H_2 = O(1)$, $H_3 = \alpha^{O(k^2)}$ and $B = O(k^2)$. We thus get that the left hand side of (25) is bounded below by $\exp(-c_6 k^2 \log k)$, where c_6 is some absolute constant. Thus, we get the inequality

$$m + O(\log K) \ll k^2 \log k \ll (\log K)^2 \log \log K,$$

yielding

$$m \ll (\log K)^2 \log \log K.$$

Hence,

$$mk \ll (\log K)^3 \log \log K \ll (\log \log f_N)^3 \log \log \log f_N.$$

However, $mk > k(m - k) \gg \log f_N$, which gives

$$\log f_N \ll (\log \log f_N)^3 (\log \log \log f_N),$$

and this has only finitely many solutions N. This takes care of the case $\lambda \neq 0$.

The case $\lambda = 0$.

Here too we distinguish between the instance when $K = 100$ and the general K.

The case when $K = 100$.

We use inequality (15) with $K = 100$ together with Lemma 2 to get

$$\left| \frac{\alpha^{mk-k(k-1)/2}}{5^{k/2}F_1 \cdots F_k} - \frac{\alpha^{nl-l(l-1)/2}}{5^{l/2}F_1 \cdots F_l} \right| \le 100 + \frac{2\alpha^{mk-k(k-1)/2}}{5^{k/2}F_1 \cdots F_k \alpha^{2m-2k+1}} + \frac{2\alpha^{nl-l(l-1)/2}}{5^{l/2}F_1 \cdots F_l \alpha^{2n-2l+1}}.$$

We divide both sides of the above inequality by $\alpha^{nl-l(l-1)/2}/\left(5^{l/2}F_1 \cdots F_l\right)$, and recalling that $mk - k^2 = nl - l^2$ we obtain

$$\left| \frac{\alpha^{k(k+1)/2-l(l+1)/2}}{5^{(k-l)/2}F_{l+1} \cdots F_k} - 1 \right| < \frac{100 \cdot 5^{l/2}F_1 \cdots F_l}{\alpha^{nl-l(l-1)/2}} + \frac{2\alpha^{k(k+1)/2-l(l+1)/2}}{5^{(k-l)/2}F_{l+1} \cdots F_k \alpha^{2m-2k+1}} + \frac{2}{\alpha^{2n-2l+1}}. \tag{27}$$

Next, we estimate the terms involved in the right hand side of (26). Observe that

$$5^{l/2}F_1 \cdots F_l < \alpha^{1+\cdots+l} \prod_{i=1}^{l} \left(1 + \frac{1}{\alpha^{2i}} \right) < 1.8\alpha^{l(l+1)/2},$$

while

$$\frac{\alpha^{k(k+1)/2-l(l+1)/2}}{5^{(k-l)/2}F_{l+1} \cdots F_k} \le \prod_{i=l+1}^{k} \left(1 - \frac{1}{\alpha^{2i}} \right) < \prod_{i \ge 1} \left(1 - \frac{1}{\alpha^{2i}} \right)^{-1} < 2.1.$$

Thus, using also the facts that $l < k$, $n - l \ge m - k + 1$, and $nl - l^2 = k(m - k) \ge 2(m - k)$, we get that the right hand side of inequality (26) is

$$< \frac{1}{\alpha^{2m-2k}} \left(180 + \frac{4.2}{\alpha} + \frac{2}{\alpha^3} \right) < \frac{184}{\alpha^{2m-2k}} < \frac{1}{\alpha^{2m-2k-11}}. \tag{28}$$

As for the left hand side of inequality (26), we use inequality (7) of Lemma 3, to arrive at

$$\frac{1}{\alpha^{2l+5}} < \frac{1}{\alpha^{2m-2k-11}}, \tag{29}$$

which leads to $2m < 2k + 2l + 16$.

Thus, $m \le k + l + 7$. Since $m \ge 2k$, we get that $k \le l + 7$. Hence, $mk - k^2 = k(m-k) \le (l+7)^2 = l^2 + 14l + 49$. Since $l(n-l) = k(m-k) \le l^2 + 14l + 49$, we get that $n - l \le l + 14 + 49/l$, therefore $n \le 2l + 14 + 49/l$. Thus, $n - l \le l + 14 + 49/l$. Since $m \le n - l$, we get that $2k \le m \le n - l \le l + 14 + 49/l$. Since $k \ge l + 1$, we get that $2l + 2 \le l + 14 + 49/l$, or $l \le 12 + 49/l$, leading to $l \le 15$. Thus, $k \le l + 7 \le 22$ and $m \le k + l + 7 \le 22 + 15 + 7 \le 44$, which is a contradiction. Hence, inequality (15) has no solutions with $n \ge m > 100$.

The case of the general K.

Arguments identical to the ones used in when $K = 100$ lead to the analogue of inequality (28) with $2m - 2k - 11$ replaced by $2m - 2k + O(\log K)$, which leads to $m \leq k + l + O(\log K)$. The argument used in the final step of the proof of the argument for $K = 100$ shows first that $k = l + O(\log K)$, then that $n = 2l + O(\log K)$, so that $n - l = l + O(\log K)$. Since $n - l \geq 2k = 2l + O(1)$, we get $2l + O(\log K) \leq l + O(\log K)$; thus, $l = O(\log K)$, therefore $n = O(\log K)$. Thus, $mk = O((\log K)^2) = O((\log \log f_N)^2)$. Since also $mk \gg \log f_N$, we get that $\log f_N \ll (\log \log f_N)^2$, so again only finitely many possibilities for N. This finishes the proof of the theorem.

As for the computational case when $K = 100$ and $m \leq 100$, by going through the previous bounds, a very rough estimate renders the bound $n \leq 5000$. A computer program exhausted easily this entire range, revealing that $f_{N+1} - f_N \leq 100$, only if $N \leq 18$, or $N = 26$.

4   Further comments and a generalization

A similar result as Theorem 2 holds when instead of the Fibonomials, we arrange all the q-binomial coefficients in increasing order, or, more generally, all the C-nomial coefficients, when $C = (C_n)_{n \geq 0}$ is a Lucas sequence of integers satisfying the property that $|\alpha| > |\beta|$ (in particular, its roots are real). In order to prove this, at least for q-binomial coefficients, one would need to prove obvious analogues of Lemmas 1, 2, 3 and 4. For example, the analogue of Lemma 4 would state that the inequality

$$\left| \frac{q^s}{(q-1)^l} - \frac{1}{p_q} \right| > \frac{1}{(q-1)^{3l}}$$

holds for all but finitely many pairs (l, s), once q is fixed, where

$$p_q = \prod_{n \geq 1} \left(1 - \frac{1}{q^n} \right).$$

It would be of interest to study mixed Diophantine equations of the type

$$\begin{bmatrix} n \\ l \end{bmatrix}_{C_1} = \begin{bmatrix} m \\ k \end{bmatrix}_{C_2},$$

where C_1 and C_2 are two distinct Lucas sequences. For example, what can one say about the number of solutions of the Diophantine equation

$$\begin{bmatrix} n \\ l \end{bmatrix}_{F} = \begin{bmatrix} m \\ k \end{bmatrix}_{q},$$

where $1 \le l \le n/2$, $1 \le k \le m/2$ in unknowns (k, l, m, n) once $q > 1$ is a fixed integer? Does this have finitely or infinitely many solutions? When k and l are fixed, then the two sides of the above equation become linear recurrent sequences (of orders depending on k and l) with dominant roots (powers of α and q, respectively), where these dominant roots are multiplicatively independent. Standard results from the theory of Diophantine equations (see [10], for example), will then lead to the conclusion that there are only finitely many possibilities for the pair (m, n) once the pair (k, l) is fixed. We do not have an argument to the effect that the above Diophantine equation has only finitely many solutions in all four variables (k, l, m, n). Of a somewhat related form is the main result from [8], where it is shown that there is no non-abelian finite simple group whose order is a Fibonacci number.

Finally, let us look at the series

$$\sum_{N \ge 1} \frac{1}{f_N}. \tag{30}$$

The fact that it is convergent follows because for each $n \ge 1$, row n contains $\lfloor (n+1)/2 \rfloor$ Fibonomial coefficients the smallest one being $\begin{bmatrix} n \\ 1 \end{bmatrix}_F = F_n$. This shows that series (29) is bounded above by

$$\ll \sum_{n \ge 1} \frac{n}{F_n},$$

and this last series is certainly convergent. What is the nature of the number (29)? Is it algebraic or transcendental? We recall that it is known that the sum of the reciprocals of the odd indexed Fibonacci numbers is transcendental (see [6]).

Acknowledgments

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