ON DIVISIBILITY PROPERTIES OF CERTAIN FIBONOMIAL COEFFICIENTS BY A PRIME p

DIEGO MARQUES, JAMES A. SELLERS, AND PAVEL TROJOVSKÝ

ABSTRACT. Let $(F_n)_{n\geq 0}$ be the FIBONACCI sequence. For $1 \leq k \leq m$, the Fibonomial coefficient is defined as

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_{m-k+1} \cdots F_{m-1}F_m}{F_1 \cdots F_k},$$

and $\begin{bmatrix} m \\ k \end{bmatrix}_F = 0$, for $k > m$. In this paper, we shall prove that if p is a prime number such that $p \equiv -2$ or $2 \pmod 5$, then $p \mid \begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$ for all $a \geq 1$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [8] together with its very extensive annotated bibliography for additional references and history).

In 1915, Fontené published a one-page note [3] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence (A_n) of real or complex numbers.

Since at least the 1960s, there has been much interest in the *Fibonomial coefficients* $\begin{bmatrix} m \\ k \end{bmatrix}_F$, which correspond to the choice $A_n = F_n$, thus are defined, for $1 \leq k \leq m$, as

$$\begin{bmatrix} m \\ k \end{bmatrix}_F := \frac{F_{m-k+1} \cdots F_{m-1}F_m}{F_1 \cdots F_k}, \tag{1.1}$$

and for $k > m$, $\begin{bmatrix} m \\ k \end{bmatrix}_F = 0$. (See, for example, Gould [4] as well as numerous papers referenced within Gould's paper.) It is surprising that this quantity will always take integer values. This can be shown by an induction argument and the recursion formula

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{k+1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k-1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F,$$

which is a consequence of the formula $F_m = F_{k+1}F_{m-k} + F_kF_{m-k-1}$.

Some authors have been interested in searching for divisibility properties of Fibonomial coefficients. For instance, in 1974, Gould [5] proved several such properties where one of them is an analogous to Hermite identity for binomial coefficients.

In recent papers, Marques and Trojovský [16, 17] proved, among other things, that $p \mid \begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$ holds for all integers $a \geq 1$ and $p \in \{2, 3\}$.

In this paper, we are interested in studying such Fibonomial coefficient divisibility properties for other prime numbers. Although such divisibilities are not true for all primes (e.g., $11 \nmid \begin{bmatrix} 11^2 \\ 11 \end{bmatrix}_F$), we desire to search for a large class of them.

Now we shall state our main result which clearly generalizes recent results of Marques and Trojovský [16, Theorem 1 for $n = 2^a$] and [17, Proposition 8].

Theorem 1.1. *Let p be a prime number such that $p \equiv -2$ or $2 \pmod 5$. Then $p \mid \left[{p^{a+1} \atop p^a}\right]_F$, for all integers $a \geq 1$.*

We organize this paper as follows. In Section 2, we will recall some useful properties of the Fibonacci numbers such as a result concerning the p-adic order of F_n. Section 3 is devoted to the proof of Theorem 1.1.

2. Auxiliary results

Before proceeding further, we recall some facts on the Fibonacci numbers for the convenience of the reader.

Lemma 2.1. *We have*

(a) $F_n \mid F_m$ *if and only if* $n \mid m$.

(b) *If* $m > k > 1$, *then*
$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m}{F_k}\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F.$$

(c) *(d'Ocagne's identity)* $(-1)^n F_{m-n} = F_m F_{n+1} - F_n F_{m+1}$.

(d) *For all primes* p, $F_{p-(\frac{5}{p})} \equiv 0 \pmod p$ *where* $(\frac{a}{q})$ *denotes the Legendre symbol of* a *with respect to a prime* $q > 2$.

Items (a) and (c) can be proved by using the well-known Binet's formula:
$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \text{ for } n \geq 0,$$
where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. The proof of item (b) follows directly from definition of $\left[{m \atop k}\right]_F$ in (1.1). Finally, a proof for (d) can be found in [18, p. 64]. We still refer the reader to [2, 7, 19, 22] for more properties and additional bibliography.

Before stating the next lemma, we recall that for a positive integer n, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$. (Some authors also call this the *order of apparition*, as it was called by Lucas, or the *Fibonacci entry point.*) There are several results on $z(n)$ in the literature, for example, very recently, Marques [11, 12, 13, 14, 15] found all fixed points of $z(n)$ as well as closed formulas for this function at some integers related to the Fibonacci sequence.

Lemma 2.2. (*Cf. Lemma 2.2 (c) of* [12]) *If* $n \mid F_m$, *then* $z(n) \mid m$.

Note that Lemma 2.1 (d) together with Lemma 2.2 implies that $z(p) \mid p - (\frac{5}{p})$ for all primes $p \neq 5$. Also, it is well-known that $(\frac{5}{p}) = -1$ or 1 according to the residue of p modulo 5. More precisely, we have that if $p \neq 5$ is a prime, then $z(p) \mid p+1$ if $p \equiv -2$ or $2 \pmod 5$ and $z(p) \mid p-1$ otherwise.

Lemma 2.3. (*Cf. Lemma 2.3 of* [15]) *For all primes* $p \neq 5$, *we have that* $\gcd(z(p), p) = 1$.

The p-adic order (or valuation) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r. The p-adic order of Fibonacci numbers has been completely characterized, see [6, 10, 20, 21]. For instance, from the main results of Lengyel [10], we extract the following result.

Proposition 2.4. *For* $n \geq 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, *and if* p *is prime* $\neq 2$ *or* 5, *then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

A proof of a more general result can be found in [10, p. 236-237 and Section 5].

By the way, using Lemmas 2.1, 2.3 and Proposition 2.4, we can prove the following two results.

Proposition 2.5. *The number* $\begin{bmatrix} 5^{a+1} \\ 5^a \end{bmatrix}_F$ *is divisible by* 5 *for all* $a \geq 1$.

Proof. By Lemma 2.1 (b), we have that

$$\begin{bmatrix} 5^{a+1} \\ 5^a \end{bmatrix}_F = \frac{F_{5^{a+1}}}{F_{5^a}} \begin{bmatrix} 5^{a+1} - 1 \\ 5^a - 1 \end{bmatrix}_F.$$

The result then follows from Proposition 2.4, since $\nu_5(F_{5^{a+1}}/F_{5^a}) = 1$. □

Although we will not need it to establish our main theorem, we prove the following result to illustrate the use of Proposition 2.4.

Proposition 2.6. *Let* $p \neq 5$ *be a prime and let* a, b, c, d *be non-negative integers with* $a > 0$ *and* $c > d$. *Then* p *divides* $\begin{bmatrix} z(p)^a p^c \\ z(p)^b p^d \end{bmatrix}_F$.

Proof. If $z(p)^a p^c < z(p)^b p^d$, then $\begin{bmatrix} z(p)^a p^c \\ z(p)^b p^d \end{bmatrix}_F = 0$ and the result follows. So, we may suppose that $z(p)^a p^c \geq z(p)^b p^d$. Then, we can write

$$\begin{bmatrix} z(p)^a p^c \\ z(p)^b p^d \end{bmatrix}_F = \frac{F_{z(p)^a p^c}}{F_{z(p)^b p^d}} \begin{bmatrix} z(p)^a p^c - 1 \\ z(p)^b p^d - 1 \end{bmatrix}_F,$$

where the Fibonomial coefficient in the right-hand side above is nonzero. Thus, it suffices to prove that $\nu_p(F_{z(p)^a p^c}/F_{z(p)^b p^d}) > 0$. In fact, observe that

$$\nu_p\left(\frac{F_{z(p)^a p^c}}{F_{z(p)^b p^d}}\right) = \nu_p(F_{z(p)^a p^c}) - \nu_p(F_{z(p)^b p^d})$$

and so if $b = 0$, then $\nu_p(F_{z(p)^b p^d}) = 0$ (Proposition 2.4) and the result follows because a and c are positive. Suppose now that $b > 0$, then the proof splits in two cases:

Case 1: $p = 2$. In this case, if $c \geq 2$, then Proposition 2.4 yields

$$\nu_2(F_{2^c \cdot 3^a}) = \nu_2(2^c \cdot 3^a) + 2 = c + 2 > d + 2 \geq \nu_2(F_{2^d \cdot 3^b}),$$

as desired. When $c = 1$ (and so $d = 0$), we have $\nu_2(F_{2 \cdot 3^a}) = 3$ since $2 \cdot 3^b \equiv 6 \pmod{12}$. On the other hand, $\nu_2(F_{2^d \cdot 3^b}) = \nu_2(F_{3^b}) = 1$, because $3^b \equiv 3 \pmod 6$. This completes the proof in this case.

Case 2: $p \neq 2$. We use again Proposition 2.4 to obtain

$$\nu_p\left(\frac{F_{z(p)^a p^c}}{F_{z(p)^b p^d}}\right) = (\nu_p(z(p)^a p^c) + \nu_p(F_{z(p)})) - (\nu_p(z(p)^b p^d) + \nu_p(F_{z(p)})) = c - d > 0,$$

where we used that $\gcd(p, z(p)) = 1$ by Lemma 2.3. The proof is complete. $\square$

Note that the previous proposition implies that if p is a prime satisfying the hypothesis of Theorem 1.1, then $p \mid \left[{(p+1)p^{a+1} \atop p^a}\right]_F$.

We require one last fact about ν_p in order to complete our proof of Theorem 1.1.

Lemma 2.7. *For any integer $k \geq 1$ and p prime, we have*

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_p(k!) \leq \frac{k-1}{p-1}, \tag{2.1}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

We refer the reader to [13, Lemma 2.4] for a proof of this result.

3. Proof of Theorem 1.1

With all of the above tools in hand, we now move to the proof of Theorem 1.1.

Let us suppose, without loss of generality, that a is even (the case of a odd can be handled in much the same way). By the definition of the Fibonomial coefficient (1.1), we have

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F = \frac{F_{(p-1)p^a+1} \cdots F_{p^{a+1}}}{F_1 \cdots F_{p^a}}. \tag{3.1}$$

Our goal now is to compare the p-adic order of the numerator and denominator in (3.1). Since $p \mid F_n$ if and only if $z(p) \mid n$ (by Proposition 2.4), we need only to consider the p-adic order of the $(p^a-1)/z(p)$ numbers $F_{z(p)}, F_{2z(p)}, F_{3z(p)}, \ldots, F_{p^a-1}$ in the denominator and $F_{(p-1)p^a+2}, F_{(p-1)p^a+2+z(p)}, F_{(p-1)p^a+2+2z(p)}, \ldots, F_{p^{a+1}-z(p)+1}$ in the numerator. So, in the first case, we use Proposition 2.4 to obtain

$$\begin{aligned} S_1 &:= \nu_p(F_1 \cdots F_{p^a}) \\ &= \nu_p(F_{z(p)}) + \nu_p(F_{2z(p)}) + \cdots + \nu_p(F_{p^a-1}) \\ &= (\nu_p(z(p)) + \nu_p(F_{z(p)})) + (\nu_p(2z(p)) + \nu_p(F_{z(p)})) + \cdots + (\nu_p(p^a-1) + \nu_p(F_{z(p)})) \\ &= \nu_p(z(p)) + \nu_p(2z(p)) + \cdots + \nu_p(p^a-1) + \left(\frac{p^a-1}{z(p)}\right)\nu_p(F_{z(p)}). \end{aligned} \tag{3.2}$$

Note that a is even and $z(p) \mid p+1$ (since $p \equiv -2$ or $2 \pmod 5$). Thus $(p^a-1)/z(p)$ is an integer. For the p-adic order of the numerator, we proceed as before to get

$$\begin{aligned} S_2 &:= \nu_p(F_{(p-1)p^a+1} \cdots F_{p^{a+1}}) \\ &= \nu_p(F_{(p-1)p^a+2}) + \cdots + \nu_p(F_{p^{a+1}-z(p)+1}) \\ &= \nu_p((p-1)p^a+2) + \cdots + \nu_p(p^{a+1} - z(p) + 1) + \left(\frac{p^a-1}{z(p)}\right)\nu_p(F_{z(p)}) \end{aligned}$$

Now, the proof splits into two cases:

Case 1: $z(p) = p+1$. In this case, $p(p^a - (p^{a-1} - p + 1 + z(p)))$ is the small multiple of $pz(p)$ greater than $(p-1)p^a + 2$. Thus

$$\begin{aligned}
S_2 &= \nu_p(p(p^a-(p^{a-1}-p+1+z(p))))+\cdots+\nu_p(p(p^a-1))+\binom{p^a-1}{z(p)}\nu_p(F_{z(p)})\\
&= \nu_p(p^a-(p^{a-1}-p+1+z(p)))+\cdots+\nu_p(p^a-1)+\frac{p^{a-1}+1}{z(p)}+\\
&\quad +\binom{p^a-1}{z(p)}\nu_p(F_{z(p)}). \qquad (3.3)
\end{aligned}$$

Observe that there exist several common terms in sums (3.2) and (3.3), so combining them we get

$$\begin{aligned}
S_2-S_1 &= \frac{p^{a-1}+1}{z(p)}-(\nu_p(z(p))+\nu_p(2z(p))+\cdots+\nu_p(p^a-p^{a-1}+p-z(p)-1))\\
&= \frac{p^{a-1}+1}{z(p)}-\left(\nu_p(pz(p))+\cdots+\nu_p\left(pz(p)\left\lfloor\frac{p^a-p^{a-1}+p-z(p)-1}{pz(p)}\right\rfloor\right)\right)\\
&= \frac{p^{a-1}+1}{z(p)}-\left\lfloor\frac{p^a-p^{a-1}+p-z(p)-1}{pz(p)}\right\rfloor-\nu_p\left(\left\lfloor\frac{p^a-p^{a-1}+p-z(p)-1}{pz(p)}\right\rfloor!\right).
\end{aligned}$$

The fact that $\lfloor x\rfloor\le x$ yields the estimate

$$\nu_p\left(\begin{bmatrix}p^{a+1}\\p^a\end{bmatrix}_F\right)=S_2-S_1\ge\frac{p^{a-1}+z(p)+1}{pz(p)}-\nu_p\left(\left\lfloor\frac{p^a-p^{a-1}+p-z(p)-1}{pz(p)}\right\rfloor!\right). \qquad (3.4)$$

By applying Lemma 2.7 to the p-adic order which appears in the right-hand side of (3.4), we obtain

$$\nu_p\left(\left\lfloor\frac{p^a-p^{a-1}+p-z(p)-1}{pz(p)}\right\rfloor!\right)\le\frac{p^a-p^{a-1}+p-z(p)-1-pz(p)}{(p-1)pz(p)}. \qquad (3.5)$$

Now, we combine (3.4) and (3.5) to obtain

$$\nu_p\left(\begin{bmatrix}p^{a+1}\\p^a\end{bmatrix}_F\right)\ge\frac{p^{a-1}+z(p)+1}{pz(p)}-\frac{p^a-p^{a-1}+p-z(p)-1-pz(p)}{(p-1)pz(p)}=\frac{2}{p-1}>0.$$

Case 2: $z(p)\le(p+1)/2$. In this case, $p(p^a-p^{a-1}+z(p)-2)$ is the small multiple of $pz(p)$ greater than $(p-1)p^a+2$. Therefore

$$\begin{aligned}
S_2 &= \nu_p(p(p^a-p^{a-1}+z(p)-2))+\cdots+\nu_p(p(p^a-1))+\binom{p^a-1}{z(p)}\nu_p(F_{z(p)})\\
&= \nu_p(p^a-p^{a-1}+z(p)-2)+\cdots+\nu_p(p^a-1)+\frac{p^{a-1}+1+z(p)}{z(p)}+\\
&\quad +\binom{p^a-1}{z(p)}\nu_p(F_{z(p)}). \qquad (3.6)
\end{aligned}$$

Proceeding as before, that is, combining (3.2), (3.6) and using that $\lfloor x\rfloor\le x$ together with Lemma 2.7, we get

$$\nu_p\left(\begin{bmatrix}p^{a+1}\\p^a\end{bmatrix}_F\right)\ge\frac{p^{a-1}+(p+1)z(p)+1}{pz(p)}-\frac{p^a-p^{a-1}+p-z(p)-1-pz(p)}{(p-1)pz(p)}=\frac{p+1}{p-1}.$$

Theorem 1.1 follows because in any case, the numbers $2/(p-1)$ and $(p+1)/(p-1)$ are positive $\square$

We finish by stating a conjecture which, in particular, asserts that a kind of reciprocal of Theorem 1.1 is also true.

Conjecture 3.1. *If $p \equiv -1$ or $1 \pmod 5$, then $\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$ is not divisible by p, for any integer $a \geq 1$.*

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Departamento de Matemática, Universidade de Brasília, Brasília, 70910-900, Brazil
E-mail address: diego@mat.unb.br

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802
E-mail address: sellersj@math.psu.edu

Department of Mathematics, Faculty of Science, University of Hradec Králové, Rokitanského 62, Hradec Králové, 50003, Czech Republic
E-mail address: pavel.trojovsky@uhk.cz