ON TERMS OF LINEAR RECURRENCE SEQUENCES WITH ONLY ONE DISTINCT BLOCK OF DIGITS

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Abstract. In 2000, Florian Luca proved that $F_{10} = 55$ and $L_5 = 11$ are the largest numbers with only one distinct digit in the Fibonacci and Lucas sequence, respectively. In this paper, we find terms of a linear recurrence sequence with only one block of digits in its expansion in base $g \geq 2$. As an application, we generalize Luca's result by finding the Fibonacci and Lucas numbers with only one distinct block of digits of length up to 10 in its decimal expansion.

1. Introduction

A sequence $(G_n)_{n\geq 1}$ is *a linear recurrence sequence* with coefficients $c_0, c_1, ..., c_{k-1}$, with $c_0 \neq 0$, if

$$G_{n+k} = c_{k-1}G_{n+k-1} + \cdots + c_1G_{n+1} + c_0G_n, \tag{1.1}$$

for all positive integer n. A recurrence sequence is therefore completely determined by the *initial values* $G_0, ..., G_{k-1}$, and by the coefficients $c_0, c_1, ..., c_{k-1}$. The integer k is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(G_n)_{n\geq 0}$ is given by

$$G(x) = x^k - c_{k-1}x^{k-1} - \cdots - c_1x - c_0.$$

It is well-known that for all n

$$G_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n,$$

where r_j is a root of $G(x)$ and $g_j(x)$ is a polynomial over a certain number field, for $j = 1, ..., \ell$. A root r_j of the recurrence is called a *dominant root* if $|r_j| > |r_i|$, for all $j \neq i \in \{1, ..., \ell\}$. The corresponding polynomial $g_j(n)$ is named the *dominant polynomial* of the recurrence. In this paper, we consider only integer recurrence sequences, i.e. recurrence sequences whose coefficients and initial values are integers. Hence, $g_j(n)$ is an algebraic number, for all $j = 1, ..., \ell$, and $n \in \mathbb{Z}$.

A general Lucas sequence $(C_n)_{n\geq 1}$ given by $C_{n+2} = C_{n+1} + C_n$, for $n \geq 1$, where the values C_0 and C_1 are previously fixed, is an example of a linear recurrence of order 2 (also called *binary*). For instance, if $C_0 = 0$ and $C_1 = 1$, then $(C_n)_{n\geq 1} = (F_n)_{n\geq 1}$ is the well-known *Fibonacci sequence*:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

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Also, if $C_0 = 2$ and $C_1 = 1$, the sequence $C_n = L_n$ gives the *Lucas numbers*:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

In 2000, F. Luca [2] proved that $F_{10} = 55$ and $L_5 = 11$ are the largest numbers with only one distinct digit in the Fibonacci and Lucas sequence, respectively. A question arises: is there any Fibonacci or Lucas number of the form 1212...12? And of the form 175175...175? And so on? More generally, let B be a natural number with ℓ digits. One can think in a string of B's, that is, $B \cdot \left(\frac{10^{\ell m}-1}{10^{\ell}-1}\right) = B \cdots B$ (m times). Therefore, Luca's result concerns the case $\ell = 1$. Moreover, it seems to be harder to answer the previous questions when we replace Fibonacci and Lucas numbers by a term of a general linear recurrence sequence.

The aim of this paper is to determine terms of an integer linear recurrence sequence with only B in its expansion in a base $g \geq 2$. More precisely, our main result is the following.

Theorem 1. *Let $(G_n)_{n\geq 1}$ be an integer linear recurrence sequence such that its characteristic polynomial has a positive dominant root. Let $g \geq 2$ and $\ell \geq 1$ be integers. Then, there exists an effectively computable constant C such that if n, m, B are solutions of the Diophantine equation*

$$G_n = B \cdot \left(\frac{g^{\ell m} - 1}{g^{\ell} - 1}\right) \tag{1.2}$$

such that $0 < B < g^{\ell}$, then $n, m \leq C$. The constant C depends only on g, ℓ and the parameters of G_n.

As an application, we use our method to find Fibonacci and Lucas numbers with only B in its decimal expansion, where the number B has at most 10 digits.

Theorem 2. *Let B be a natural number with ℓ digits. The only solutions of the Diophantine equations*

$$F_n = B \cdot \left(\frac{10^{\ell m} - 1}{10^{\ell} - 1}\right) \quad \text{and} \quad L_n = B \cdot \left(\frac{10^{\ell m} - 1}{10^{\ell} - 1}\right), \tag{1.3}$$

in positive integer numbers m, n and ℓ, with $m > 1$ and $1 \leq \ell \leq 10$, are $(m, n, \ell) = (2, 10, 1)$ and $(m, n, \ell) = (2, 5, 1)$ in the Fibonacci and Lucas case, respectively.

We organize this paper as follows. In Section 2, we will recall some useful properties such as a result of Matveev on linear forms in three logarithms and the reduction method of Baker-Davenport that we will use to prove Theorems 1 and 2. The third section is devoted to the proof of Theorem 1. In the last section, for each particular case (Fibonacci and Lucas), we first use Baker's method to obtain a bound for n, then we completely solve the problem by the means of Baker-Davenport reduction method. Therefore, we prove Theorem 2.

2. Auxiliary results

In this section, we recall some results that will be very useful for the proof of the above theorems. Let $G(x)$ be the characteristic polynomial of a linear recurrence G_n. One can factor $G(x)$ over the set of complex numbers as

$$G(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_\ell)^{m_\ell},$$

where $r_1, ..., r_\ell$ are distinct non-zero complex numbers (called the *roots* of the recurrence) and $m_1, ..., m_\ell$ are positive integers. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined polynomials $g_1, ..., g_\ell \in \mathbb{Q}(\{r_j\}_{j=0}^{\ell})[x]$, with $\deg g_j \le m_j - 1$, for $j = 1, ..., \ell$, such that

$$G_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n, \text{ for all } n. \tag{2.1}$$

For more details, one can refer to [4, Theorem C.1].

In the case of Fibonacci and Lucas sequence, the above formula is known as *Binet's formulas*:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = (1+\sqrt{5})/2$ (the golden number) and $\beta = (1-\sqrt{5})/2 = -1/\alpha$. Moreover, one can easily prove by induction that

$$\alpha^{n-2} \le F_n \le \alpha^{n-1}, \quad \alpha^{n-1} \le L_n \le 2\alpha^n,$$

for all $n \ge 1$.

The first lemma will be very useful in the proof of Theorem 1.

Lemma 1. *Let $(G_n)_{n\ge1}$ be a linear recurrence having a dominant root r_1 and an infinite subsequence of positive terms. Denote by $g_1(n)$ the dominant polynomial of $(G_n)_{n\ge1}$. Then $g_1(n)$ is a non-zero constant. Moreover, if $G_{2n+t} > 0$, for infinitely many integers n, with $t \in \{0,1\}$, then $g_1(n)r_1^t > 0$.*

Proof. We know that

$$G_n = g_1(n)r_1^n + \cdots + g_\ell(n)r_\ell^n,$$

where each r_j is a root of characteristic polynomial of G_n, with multiplicity m_j, and each $g_j(n)$ is a non-zero polynomial with degree $\le m_j - 1$. Suppose that r_1 is the dominant root, then we have immediately that $r_1 \ne r_j$, for all $j \ne i$. Thus $m_1 = 1$ and then the degree of dominant polynomial is at most $m_1 - 1 = 0$, so it is a constant, say g_1. Now, dividing G_n by r_1^n, we get

$$\frac{G_n}{r_1^n} = g_1 + \sum_{j=2}^{\ell} \frac{g_j(n)}{\kappa_j^n},$$

where $\kappa_j = r_1/r_j$. Since $|\kappa_j| > 1$, we have

$$\lim_{n\to\infty} \frac{g_j(n)}{\kappa_j^n} = 0, \text{ for all } 2 \le j \le \ell,$$

and so

$$\lim_{n\to\infty} \frac{G_n}{r_1^n} = g_1 \ne 0.$$

Now, if $G_{2n+t} > 0$ for infinitely many integers n, with $t \in \{0,1\}$, then

$$0 \le \lim_{n\to\infty} \sup \frac{G_{2n+t}}{r_1^{2n}} = g_1 r_1^t \ne 0.$$

Therefore $g_1 r_1^t > 0$ and the result follows by distinguishing the cases $t = 0$ and $t = 1$. □

In order to prove Theorems 1, 2, we will need to use a lower bound for a linear form in three logarithms *à la Baker* and such a bound was given by the following result of Matveev [3].

Lemma 2. *Let $\alpha_1, \alpha_2, \alpha_3$ be non-zero algebraic numbers and let b_1, b_2, b_3 be non-zero integer rational numbers. Define*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$. Put

$$\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

Let A_1, A_2, A_3 be real numbers which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j = 1, 2, 3.$$

Assume that

$$B' \geq \max\{1, \max\{|b_j| A_j / A_1; 1 \leq j \leq 3\}\}.$$

Define also

$$C_1 = \frac{5 \cdot 16^5}{6\chi} \cdot e^3 (7 + 2\chi)(20.2 + \log(3^{5.5} D^2 \log(eD))).$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB' \log(eD)).$$

As usual, in the above statement, the *logarithmic height* of an s-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{s}(\log |a| + \sum_{j=1}^{s} \log \max\{1, |\alpha^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$), $(\alpha^{(j)})_{1 \leq j \leq s}$ are the conjugates of α and, as usual, the absolute value of the complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

After finding an upper bound on n which is general too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethő [1]. For a real number x, we use $\parallel x \parallel = \min\{|x - n| : n \in \mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 3. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of γ such that $q > 6M$ and let $\epsilon = \parallel \mu q \parallel - M \parallel \gamma q \parallel$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m, n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

See Lemma 5, a.) in [1]. Now, we are ready to deal with the proofs of our results.

3. The proof of Theorem 1

Equations (1.2) and (2.1) give

$$G_n = g_1(n) r_1^n + \cdots + g_s(n) r_s^n = B \cdot \left(\frac{g^{\ell m} - 1}{g^{\ell} - 1} \right), \tag{3.1}$$

where $g_1(n), ..., g_s(n)$ are polynomials with degree at most $k - 1$. Without loss of generality, we may suppose that $|r_1| > |r_t| := \max_{2 \leq j \leq s}\{|r_j|\}$. So r_1 is the dominant root.

If G_n assumes only finitely many positive numbers, then there exists n_0, such that $G_n \leq 0$ for all $n \geq n_0$. Thus, in our main equation (1.2) we must have $n \leq n_0$ (since its right-hand side is positive). Therefore, we combine the expressions of G_n in (3.1) and (4.2), to get

$$B(g^{\ell m}-1)/(g^{\ell}-1) = g_1(n)r_1^n + ... + g_s(n)r_s^n.$$

By applying the absolute value, the triangular inequality and since $B \geq 1$, we have

$$(g^{\ell m}-1)/(g^{\ell}-1) \leq |g_1(n)||r_1|^n + ... + |g_s(n)||r_s|^n$$

Since r_1 is the dominant root, then

$$(g^{\ell m}-1)/(g^{\ell}-1) \leq (|g_1(n)| + ... + |g_s(n)|)|r_1|^n.$$

Now, $|r_1| > 1$ and $n \leq n_0$, hence $|r_1|^n \leq |r_1|^{n_0}$ yielding

$$(g^{\ell m}-1)/(g^{\ell}-1) \leq K|r_1|^{n_0},$$

where $K = \max_{1\leq n\leq n_0}\{\sum_{j=1}^{s} |g_j(n)|\}$. Thus,

$$g^{\ell m} \leq K|r_1|^{n_0}(g-1) + 1,$$

and by applying the log function, we finally conclude that

$$m \leq \log(K|r_1|^{n_0}(g-1)+1)/\ell \log g =: M.$$

Thus, $n \leq n_0$ and $m \leq M$ and hence $n, m < \max\{n_0, M\} = C$ and the theorem is proved in this case.

So, we may suppose that G_n assumes infinitely many positive numbers. By Lemma 1, the dominant polynomial $g_1(n)$ is a constant, say g_1. Thus

$$\left| r_1^n - \left(\frac{B}{g^{\ell}-1}\right)\frac{g^{\ell m}}{g_1}\right| \leq |r_t|^n \cdot \sum_{j=2}^{s}\left|\frac{g_j(n)}{g_1}\right| + 1.$$

For all sufficiently large n, say $n \geq n_1$, we have

$$\left| r_1^n - \left(\frac{B}{g^{\ell}-1}\right)\frac{g^{\ell m}}{g_1}\right| < |r_t|^n(s-1)n^k$$

and so

$$\left| 1 - \left(\frac{B}{g^{\ell}-1}\right)\frac{g^{\ell m}r_1^{-n}}{g_1}\right| < \kappa^{-n}(s-1)n^k,$$

where $\kappa = |r_1/r_t| > 1$. Observe that

$$\kappa^{-n}(s-1)n^k = \frac{1}{\kappa^{n/2}} \cdot \frac{(s-1)n^k}{\kappa^{n/2}} < \kappa^{-n/2},$$

for all sufficiently large n, say $n > n_2 \geq n_1$. Therefore

$$\left|1 - e^{\Lambda}\right| < \kappa^{-n/2}, \tag{3.2}$$

where

$$\Lambda = \log\left(\frac{B}{(g^{\ell}-1)g_1}\right) + \ell m \log g - n \log r_1. \tag{3.3}$$

Now, we claim that $\Lambda \neq 0$. Suppose that $\Lambda = 0$. Thus, $Bg^{\ell m}/(g^{\ell}-1) = g_1 r_1^n$ and then the identity (3.1) leads to an absurdity as $\sum_{j=2}^{s} g_j(n)r_j^n = -B/(g^{\ell}-1)$, for all $n \in \mathbb{N}$. Hence $\Lambda \neq 0$ as desired.

Note that the signs of g_1 and r_1 affect in Λ being real or not. In order to solve this problem, we shall split our proof according to the positivity of G_n.

Case 1. If $G_{2n} < 0$ for all sufficiently large n, then we repeat the above argument in order to find a constant C such that if

$$G_{2n} = B \cdot \left(\frac{g^{\ell m} - 1}{g^{\ell} - 1}\right)$$

then $m, 2n < C$.

Thus, we shall need only to consider the odd indexes (since the even ones are already bounded). Thus, we replace n by $2n+1$ in equation (4.2) to get

$$G_{2n+1} = B \cdot \left(\frac{g^{\ell m} - 1}{g^{\ell} - 1}\right)$$

Note that $G_{2n+1} > 0$ for infinitely many n (since $G_n > 0$ for infinitely many integers n). Therefore, by Lemma 1, we have $r_1 g_1 > 0$. Thus, the form in (3.3) becomes

$$\Lambda_0 = \log\left(\frac{B}{(g^{\ell} - 1) g_1 r_1}\right) + \ell m \log g - n \log r_1^2.$$

which is now a real number. Since $r_1 g_1$ and r_1^2 are positive numbers.

Case 2. If $G_{2n} > 0$ for infinitely many n (by Lemma 1, we get $g_1 > 0$), the we have two subcases :

Case 2.1. If $G_{2n+1} > 0$ for infinitely many n, then again Lemma 1 yields $r_1 g_1 > 0$ and so $r_1 > 0$, since $g_1 > 0$. So, our linear form in (3.3) is a real number.

Case 2.2. If $G_{2n+1} < 0$ for all sufficiently large n. In this case, we proceed as before to get a bound C, such that if

$$G_{2n+1} = B \cdot \left(\frac{g^{\ell m} - 1}{g^{\ell} - 1}\right)$$

then $m, 2n+1 < C$. So, we must bound only the even indexes, by considering the equation

$$G_{2n} = B \cdot \left(\frac{g^{\ell m} - 1}{g^{\ell} - 1}\right)$$

Thus, the form in (3.3) becomes

$$\Lambda_1 = \log\left(\frac{B}{(g^{\ell} - 1) g_1}\right) + \ell m \log g - 2n \log r_1,$$

which is also a real number.

Summarizing, in any case, we can consider the real linear form

$$\Lambda_t = \log\left(\frac{B}{(g^{\ell} - 1) g_1 r_1^t}\right) + \ell m \log g - n \log r_1^2, \quad t \in \{0, 1\}.$$

If $\Lambda_t > 0$, then $\Lambda_t < e^{\Lambda_t} - 1 < \kappa^{-n/2}$. In the case of $\Lambda_t < 0$, we get

$$1 - e^{-|\Lambda_t|} = |e^{\Lambda_t} - 1| < \kappa^{-n/2}.$$

Therefore,

$$|\Lambda_t| < e^{|\Lambda_t|} - 1 < \frac{\kappa^{-n/2}}{1 - \kappa^{-n/2}} < \kappa^{-n/2+1},$$

for all sufficiently large $n > n_3 \geq n_2$. Hence, in any case $|\Lambda_t| < \kappa^{-n/2+1}$ or equivalently

$$-\log|\Lambda_t| > \left(\frac{n}{2} - 1\right) \log \kappa. \tag{3.4}$$

To apply Lemma 2, we take

$$\alpha_1 = \frac{B}{(g^\ell - 1)g_1 r_1^t}, \quad \alpha_2 = g, \quad \alpha_3 = r_1^2, \quad b_1 = 1, \quad b_2 = \ell m, \quad b_3 = -n.$$

Then, we can choose

$$D = k, \quad A_1 = kh_1 + 0.16, \quad A_2 = k \log g, \quad A_3 = kh_3 + 0.16,$$

where k is the degree of the number field $\mathbb{Q}(g_1, r_1^2)$ over $\mathbb{Q}$ and h_1 and h_3 are the logarithmic height of α_1 and α_3, respectively. Moreover, we have

$$B' = \max\left\{1, \frac{\ell m k \log g}{kh_1 + 0.16}, \frac{n(kh_3 + 0.16)}{kh_1 + 0.16}\right\}.$$

Since $\chi = 1$, we obtain

(3.5)
$$-\log|\Lambda| < 1.6 \cdot 10^8 (21 + \log(3^{5.5} k^2 \log(ek))) k^5 (\max\{h_1, \log g, h_3\})^2 \log(4.1 k B' \log(ek)).$$

Combining estimates (3.4) and (3.5), we get a constant $C > 0$ which depends only on g, ℓ and the parameters of G_n, such that $m, n < C$.

□

4. The proof of Theorem 2

Note that the dominant root for Fibonacci and Lucas sequences is $\alpha = (1+\sqrt{5})/2$. The dominant polynomials are $g(n) = 1/\sqrt{5}$ and $g(n) = 1$ in the Fibonacci and Lucas cases, respectively. Since α^n is irrational for all non-zero integer number n, we can apply Theorem 1 to conclude that the Diophantine equations in (1.3) have only finitely many solutions. The goal of this section is to improve our estimates and therefore to completely solve these equations. First, we prove Theorem 2 in the Fibonacci case. The proof of the Lucas case will be handled in a similar way. This will be done later.

4.1. **The Fibonacci case.**

4.1.1. *Finding a bound on n.* We assume that $n > 47$. By Binet's formula and equation (1.3), we have

$$\alpha^n - \beta^n = \left(\frac{\sqrt{5}B}{10^\ell - 1}\right)(10^{m\ell} - 1);$$

that is

$$\alpha^n - \left(\frac{\sqrt{5}B}{10^\ell - 1}\right)10^{m\ell} = \beta^n - \frac{B\sqrt{5}}{10^\ell - 1}; \tag{4.1}$$

and hence

$$\left|\alpha^n - \left(\frac{\sqrt{5}B}{10^\ell - 1}\right)10^{m\ell}\right| \le \alpha^{-47} + \sqrt{5} < 2.4. \tag{4.2}$$

Define $\Lambda_F = \log(\sqrt{5}B/(10^\ell - 1)) - n\log\alpha + m\ell\log 10$. Then (4.2) becomes

$$|e^{\Lambda_F} - 1| < \frac{2.4}{\alpha^n} < \alpha^{-n+2}. \tag{4.3}$$

We also claim that $\Lambda_F > 0$. In fact, from equation (4.1), we deduce that

$$1 - e^{\Lambda_F} = \frac{1}{\alpha^n}\left(\beta^n - \frac{B\sqrt{5}}{10^\ell - 1}\right) \leq \frac{1}{\alpha^n}\left(\alpha^{-47} - \frac{\sqrt{5}}{10^{10} - 1}\right) < 0;$$

so $\Lambda_F > 0$. Thus $\Lambda_F < e^{\Lambda_F} - 1 < \alpha^{-n+2}$ (see (4.3)). Therefore

$$\log|\Lambda_F| < -(n-2)\log\alpha. \tag{4.4}$$

Now, we will apply Lemma 2, but first we must be sure that $\Lambda_F \neq 0$. Indeed, if $(\sqrt{5}B/(10^\ell - 1))10^{m\ell}\alpha^{-n} = 1$ then $\alpha^{2n} \in \mathbb{Q}$, which is an absurd. So $\Lambda_F \neq 0$. To apply Lemma 2, we take

$$\alpha_1 = \sqrt{5}B/(10^\ell - 1),\ \alpha_2 = \alpha,\ \alpha_3 = 10,\ b_1 = 1,\ b_2 = -n,\ b_3 = m\ell.$$

Observe that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\sqrt{5})$ and then $D = 2$. The conjugates of α_1, α_2 and α_3 are $\alpha_1' = -\alpha_1,\ \alpha_2' = \beta,\ \alpha_3' = \alpha_3$, respectively. Surely, α_2 and α_3 are algebraic integers, while the minimal polynomial of α_1 is

$$(X - \alpha_1)(X - \alpha_1') = X^2 - \frac{5B^2}{(10^\ell - 1)^2}.$$

Thus, the minimal polynomial of α_1 is a divisor of $(10^\ell - 1)^2X^2 - 5B^2$. Therefore,

$$h(\alpha_1) < \frac{1}{2}(2\log(10^\ell - 1) + 2\log\sqrt{5}) < 23.84.$$

Also, $h(\alpha_2) = \log\alpha/2 < 0.25$ and $h(\alpha_3) = \log 10 < 2.31$. We take $A_1 = 47.68,\ A_2 = 0.5$ and $A_3 = 4.62$. Since $n > 47$, we have

$$\max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\} = \max\{n/12, 5m\ell/48\},$$

and then it suffices to choose $B' = 5n/48$ as $n > m\ell$. Since $C_1 < 4.45 \cdot 10^9$, Lemma 2 yields

$$\log|\Lambda_F| > -1.97 \cdot 10^{12}\log(1.439n). \tag{4.5}$$

Combining the estimates (4.4) and (4.5), we get

$$1.97 \cdot 10^{12}\log(1.439n) > (n-2)\log\alpha,$$

and this inequality implies $n < 1.4 \cdot 10^{14}$.

Now, let us determine some estimates for m in terms of n that will useful later. Equation (1.3) yields

$$m\ell = \left\lfloor \frac{\log F_n}{\log 10} \right\rfloor + 1.$$

Hence

$$(n-2)\frac{\log\alpha}{\log 10} < m\ell \leq (n-1)\frac{\log\alpha}{\log 10} + 1. \tag{4.6}$$

Thus, we deduce from the estimate on n that $m < 3 \cdot 10^{13}$.

4.1.2. *Reducing the bound.* We know that $0 < \Lambda_F < \alpha^{-n+2}$. Since $m \geq 2$ and $\alpha^c = 10$, where $c = \log 10/\log\alpha$. So we have

$$\alpha^{n-2} \geq \alpha^{cm\ell-6} > (\alpha^c)^m 10^{-6} = 10^{m-6}.$$

Therefore

$$0 < m\ell\log\alpha_3 - n\log\alpha_2 + \log\alpha_1 < 10^{-m+6}.$$

On dividing through by $\log\alpha_2$, we get

$$0 < m\ell\gamma - n + \mu < 3\cdot 10^6\cdot 10^{-m}, \tag{4.7}$$

with $\gamma = \log\alpha_3/\log\alpha_2$ and $\mu = \log\alpha_1/\log\alpha_2$.

Surely γ is an irrational number[1] (because α and 10 are multiplicatively independent). So, let us denote p_n/q_n be the nth convergent of its continued fraction.

In order to reduce our bound on m (which is too large!), we will use Lemma 3. For that, take $M = 3\cdot 10^{13}$, we have that

$$\frac{p_{34}}{q_{34}} = \frac{9146274886090674}{1911458405521733},$$

then $q_{34} \geq 1911458405521733 > 1.9\cdot 10^{15} > 6M$. Moreover, we get

$$M \parallel q_{34}\gamma \parallel = 0.00736166... < 0.0075,$$

and the minimal value of $\parallel q_{34}\mu \parallel$ is at least 0.008. Hence

$$\epsilon = \parallel \mu q \parallel - M \parallel \gamma q \parallel > 0.008 - 0.0075 = 0.0005.$$

Thus all the hypotheses of the Lemma 3 are satisfied and we take $A = 3\cdot 10^6$ and $B = 10$. It follows from Lemma 3 that there is no solution of the inequality in (4.7) (and then for the Diophantine equation (1.3)) in the range

$$\left[\left\lfloor\frac{\log(Aq_{34}/\epsilon)}{\log B}\right\rfloor + 1, M\right] = [26, 3\cdot 10^{13}].$$

Therefore $m \leq 26$ and then inequality (4.6) tells us that $n < 1246$. To finish, we use Mathematica to print the values of all the Fibonacci numbers in the range $47 < n < 1246$ and we see that there are no Fibonacci numbers as desired in the theorem. This completes the proof.

4.2. **The Lucas case.** From the Binet's formula $L_n = \alpha^n + \beta^n$, we can take

$$\Lambda_L = \log(B/(10^\ell - 1)) - n\log\alpha + m\ell\log 10.$$

Since $B/(10^\ell - 1) < \sqrt{5}B/(10^\ell - 1)$, we get the same estimates as in (4.3) and (4.4), so the possible solutions appear when $n < 1.4\cdot 10^{14}$. Therefore $m < 3\cdot 10^{13}$. Then the Baker-Davenport reduction method can be applied to prove that actually $n < 1245$. Finally, we use again Mathematica to complete the proof of Theorem 2.

[1]Actually, this number is transcendental by Gelfond-Schneider theorem: if α and β are algebraic numbers, with $\alpha \neq 0$ or 1, and β irrational, then α^β is transcendental.

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