On repdigits as product of consecutive Fibonacci numbers

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Abstract

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. In 2000, F. Luca proved that $F_{10} = 55$ is the largest repdigit (i.e., a number with only one distinct digit in its decimal expansion) in the Fibonacci sequence. In this note, we show that if $F_n \cdots F_{n+(k-1)}$ is a repdigit, with at least two digits, then $(k, n) = (1, 10)$.

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties. In 1963, the Fibonacci Association was created to provide an opportunity to share ideas about these intriguing numbers and their applications. We remark that, in 2003, Bugeaud et al [2] proved that the only perfect powers in the Fibonacci sequence are $0, 1, 8$ and 144 (see [6] for the Fibonomial version). In 2005, Luca and Shorey [5] showed, among other things, that a non-zero product of two or more consecutive Fibonacci numbers is never a perfect power except for the trivial case $F_1 \cdot F_2 = 1$.

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Recall that a positive integer is called a *repdigit* if it has only one distinct digit in its decimal expansion. In particular, such a number has the form $a(10^m - 1)/9$, for some $m \geq 1$ and $1 \leq a \leq 9$. The problem of finding all perfect powers among repdigits was posed by Obláth [8] and completely solved, in 1999, by Bugeaud and Mignotte [1]. One can refer to [3] and its extensive annotated bibliography for additional references, history and related results.

In 2000, F. Luca [4], using elementary techniques, proved that $F_{10} = 55$ is the largest repdigit in the Fibonacci sequence. In a very recent paper, the authors [7] used bounds for linear forms in logarithms *à la Baker*, in order to prove that there is no Fibonacci number of the form $B \cdots B$ (concatenation of B, m times), for $m > 1$ and $B \in \mathbb{N}$ with at most 10 digits.

In this note, we follow the same ideas by using elementary tools for searching repdigits as product of consecutive Fibonacci numbers. More precisely, our main result is the following

Theorem 1. *The only solution of the Diophantine equation*

$$F_n \cdots F_{n+(k-1)} = a\left(\frac{10^m - 1}{9}\right), \tag{1}$$

in positive integers n, k, m, a, *with* $1 \leq a \leq 9$ *and* $m > 1$ *is* $(n, k, m, a) = (10, 1, 2, 5)$.

We point out that all relations which will appear in the proof of the above result can be easily proved by elementary ways (mathematical induction, the Fibonacci recurrence pattern, congruence properties etc). So, we will leave them as exercises to the reader.

2. The proofs

2.1. Proof of Theorem 1

First, we claim that $k \leq 4$. Indeed, we suppose the contrary, i.e. there exist at least 5 consecutive numbers among $n, ..., n + (k-1)$. Thus, $3|(n+i)$ and $5|(n+j)$, for some $i, j \in \{0, ..., k-1\}$. This implies that $2|F_{n+i}$ and $5|F_{n+j}$ leading to an absurdity as $10|F_n \cdots F_{n+(k-1)} = a(10^m - 1)/9$ and hence $k \in \{1, 2, 3, 4\}$. If $k = 1$, Luca's result [4, Theorem 1] ensures that $(n, m, a) = (10, 2, 5)$. Hence, we must to prove that Eq. (1) has no solution for $k \in \{2, 3, 4\}$.

Note that $a(10^2-1)/9 = a\cdot 11$ and $a(10^3-1)/9 = a\cdot 3\cdot 37$ are not products of at least two Fibonacci numbers, for $1 \leq a \leq 9$. So, from now on, we can assume that $m \geq 4$.

a	1	2	3	4	5	6	7	8	9	
$a\cdot(\frac{10^m-1}{9})$	7	14	5	12	3	10	1	8	15	(mod 16)

Table 1: Residues classes modulo 16, for $m \geq 4$.

Case $k=4$. The sequence $(F_nF_{n+1}F_{n+2}F_{n+3})_{n\geq 1}$ has period 12 modulo 16. In fact,

$$F_nF_{n+1}F_{n+2}F_{n+3} \equiv 6, 14, 0, 8, 8, 0, 14, 6, 0, 0, 0, 0 \pmod{16}.$$

So, by Table 1, it suffices to consider $a = 2$ and 8. Since 4 divides one of the numbers $n, n+1, n+2, n+3$, then

$$3 = F_4 | F_nF_{n+1}F_{n+2}F_{n+3} = a\left(\frac{10^m-1}{9}\right)$$

and so $3|(10^m-1)/9$. Thus we deduce that $3|m$ (in what follows, we will use this fact on several occasions).

For $a = 2$ and 8, one has $n \equiv 2, 7 \pmod{12}$ and $n \equiv 4, 5 \pmod{12}$, respectively. Therefore $F_nF_{n+1}F_{n+2}F_{n+3} \equiv 0, 1 \pmod{5}$. Thus, Eq. (1) is not valid, since $2\cdot\left(\frac{10^m-1}{9}\right) \equiv 2 \pmod{5}$ and $8\cdot\left(\frac{10^m-1}{9}\right) \equiv 3 \pmod{5}$, for $m \geq 2$. We conclude that the assumption $k = 4$ is impossible.

Case $k=3$. The period of $(F_nF_{n+1}F_{n+2})_{n\geq 1}$ modulo 16 is 12. Actually, we have

$$F_nF_{n+1}F_{n+2} \equiv 2, 6, 14, 8, 8, 8, 2, 6, 14, 0, 0, 0 \pmod{16}$$

Again, by looking at Table 1, we deduce that $a = 2$ or 8.

First, we suppose that $a = 2$. Thus, one has $n \equiv 3, 9 \pmod{12}$. If $n \equiv 3 \pmod{12}$, then $F_nF_{n+1}F_{n+2} \equiv 25, 29, 22, 18, 30 \pmod{31}$. Since $3|m$ then $4|(n+1)$ and we get

$$2\left(\frac{10^m-1}{9}\right) \equiv 5, 14, 24, 11, 0 \pmod{31}.$$

Thus Eq. (1) is not true in this case. In the case of $n \equiv 9 \pmod{12}$, we have $4 \nmid (n+j)$, for $j \in \{0, 1, 2\}$. Thus $3 \nmid m$ and we split the proof in two subcases:

- $m \equiv 1 \pmod 3$: In this case, $2(10^m - 1)/9 \equiv 14 \pmod{32}$, but on the other hand $F_n F_{n+1} F_{n+2} \equiv 30 \pmod{32}$;
- $m \equiv 2 \pmod 3$: Then $2(10^m - 1)/9 \equiv 4, 1 \pmod 7$, while $F_n F_{n+1} F_{n+2} \equiv 2, 5 \pmod 7$.

So, we have no solutions in the case $a = 2$.

Second, we take $a = 8$. One has $n \equiv 4, 5, 6 \pmod{12}$. In the case of $n \equiv 4 \pmod{12}$, we have $F_n F_{n+1} F_{n+2} \equiv 0, 1, 4 \pmod 5$. Since $4|n$, then $3|m$ yields $8(10^m - 1)/9 \equiv 3 \pmod 5$. When $n \equiv 6 \pmod{12}$, we obtain $F_n F_{n+1} F_{n+2} \equiv 0, 6, 9 \pmod{15}$. Again $3|m$, because $4|(n+2)$ and so $8(10^m - 1)/9 \equiv 3 \pmod{15}$. Therefore, a possible solution may appear for $n \equiv 5 \pmod{12}$. In this case, $3 \nmid m$, so we have the following two cases:

- $m \equiv 1 \pmod 3$ implies $8(10^m - 1)/9 \equiv 15, 4, 5, 17, 9, 8 \pmod{19}$. On the other hand, $F_n F_{n+1} F_{n+2} \equiv 0, 12, 7 \pmod{19}$;
- $m \equiv 2 \pmod 3$ yields $8(10^m - 1)/9 \equiv 7, 10 \pmod{13}$, while

$$F_n F_{n+1} F_{n+2} \equiv 9, 2, 0, 11, 4, 0, 0 \pmod{13}.$$

Thus, we also have no solution for $k = 3$.

Case $k = 2$. Since

$$F_n F_{n+1} \equiv 1, 2, 6, 15, 8, 8, 1, 10, 14, 15, 0, 0 \pmod{16},$$

we need to consider $a = 2, 6, 7, 8$, and 9. For $a = 6$, we have $n \equiv 8 \pmod{12}$ and then $F_n F_{n+1} \equiv 0, 2, 4 \pmod 5$, while $6(10^m - 1)/9 \equiv 1 \pmod 5$. When $a = 9$, one has $n \equiv 10 \pmod{12}$ and therefore Eq. (1) becomes $F_n F_{n+1} = 10^m - 1 \equiv 0 \pmod 9$. However, $F_n F_{n+1} \equiv 8 \pmod 9$, for $n \equiv 10 \pmod{12}$. In the case of $a = 7$, one gets $n \equiv 1, 7 \pmod{12}$ (and then $4 \nmid n$). On the other hand, Eq. (1) implies that $7|F_n$ or $7|F_{n+1}$ and thus $n \equiv 0 \pmod 8$ or $n \equiv -1 \pmod 8$. Therefore, $n \equiv 7 \pmod{12}$ and $n \equiv -1 \pmod 8$. We then get $n \equiv 7 \pmod{24}$ leading to $F_n F_{n+1} \equiv 0, 1, 3 \pmod 5$, but $7(10^m - 1)/9 \equiv 2 \pmod 5$. For $a = 2$, one has $n \equiv 9 \pmod{12}$ and so $4 \nmid (n + j)$, for $j \in \{0, 1\}$. Thus $3 \nmid m$ and then $2(10^m - 1)/9 \equiv 2 \pmod 5$, but $F_n F_{n+1} \equiv 0, 1, 3 \pmod 5$. For $a = 8$, we have $n \equiv 5, 6 \pmod{12}$. If $n \equiv 5 \pmod{12}$, similarly as in previous cases, we deduce that $3 \nmid m$.

- $m \equiv 1 \pmod 3$ implies $8(10^m - 1)/9 \equiv 5, 2, 8 \pmod 9$, however $F_n F_{n+1} \equiv 4 \pmod 9$;
- $m \equiv 2 \pmod 3$ yields $8(10^m - 1)/9 \equiv 2, 4 \pmod 7$, again Eq. (1) is not valid, since $F_n F_{n+1} \equiv 1, 5 \pmod 7$.

We finish by considering the case $n \equiv 6 \pmod{12}$. Again $3 \nmid m$ and so $8(10^m - 1)/9 \equiv 3 \pmod 5$, while $F_n F_{n+1} \equiv 0, 2, 4 \pmod 5$.

In conclusion, Eq. (1) has no solution for $k > 1$. □

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