

# ALGEBRAIC NUMBERS OF THE FORM $P(T)^{Q(T)}$ , WITH $T$ TRANSCENDENTAL

DIEGO MARQUES

ABSTRACT. Let  $P(x), Q(x) \in \mathbb{Q}[x]$  be non-constant polynomials. We show the existence of algebraic numbers which can be written of the form  $P(T)^{Q(T)}$ , for some  $T$  transcendental.

## 1. INTRODUCTION

When I was a high-school student, I liked writing rational numbers as “combination” of irrational ones, for instance

$$2 = \sqrt{\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}} = \sqrt{3}^{\frac{\log 4}{\log 3}} = e^{\log 2}$$

In particular, the last equality above shows us one way of writing the algebraic number 2 as power of two transcendental numbers. In 1934, the mathematicians A. O. Gelfond [2] and T. Schneider [3], proved the following well-known result: if  $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$  and  $\beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ , then  $\alpha^\beta$  is a transcendental number. This result, named as Gelfond-Schneider theorem, classifies completely the arithmetic nature of the numbers of the form  $A_1^{A_2}$ , for  $A_1, A_2 \in \overline{\mathbb{Q}}$ . Returning to our subject, but now using Gelfond-Schneider theorem, we also can easily write 2 as  $T^T$ , for some  $T$  transcendental. Actually, all prime number and all algebraic number  $A \geq e^{-1/e}$ , satisfying  $A^n \notin \mathbb{Q}$  for all  $n \geq 1$ , can be written in this form, for a more general result see [4, Proposition 1]. Using again the Gelfond-Schneider theorem and Galois Theory, we show that for all non-constant polynomials  $P(x), Q(x) \in \mathbb{Q}[x]$ , there are infinitely many algebraic numbers which can be written in the particular “complicated” form  $P(T)^{Q(T)}$ , for some  $T$  transcendental.

## 2. MAIN RESULT

**Proposition.** *Fix non-constant polynomials  $P(x), Q(x) \in \mathbb{Q}[x]$ . Then the set of algebraic numbers of the form  $P(T)^{Q(T)}$ , with  $T$  transcendental, is dense in some connected subset either of  $\mathbb{R}$  or  $\mathbb{C}$ .*

As we said in Section 1, all algebraic numbers  $A \geq e^{-1/e}$  satisfying  $A^n \notin \mathbb{Q}$  for all  $n \geq 1$ , can be written in the form  $T^T$ , for some  $T \notin \overline{\mathbb{Q}}$ . An example of such  $A$  is  $1 + \sqrt{2}$ . But that is only one case of our Proposition, namely when  $P(x) = Q(x) = x$ . So for proving our result we need a stronger condition satisfied for an algebraic number  $A$  and that is exactly what our next result asserts

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**Lemma.** *Let  $Q(x)$  be a polynomial in  $\mathbb{Q}[x]$  and set  $\mathcal{F} = \{Q(x) - d : d \in \mathbb{Q}\}$ . Then there exists  $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$ , such that*

$$(2.1) \quad \alpha^n \notin \mathbb{Q}(\mathcal{R}_{\mathcal{F}}), \text{ for all } n \geq 1,$$

where  $\mathcal{R}_{\mathcal{F}}$  denotes the set  $\{x \in \mathbb{C} : f(x) = 0 \text{ for some } f \in \mathcal{F}\}$ .

*Proof.* Set  $\mathcal{F} = \{F_1, F_2, \dots\}$ , and for each  $n \geq 1$ , set  $K_n = \mathbb{Q}(\mathcal{R}_{F_1 \dots F_n})$  and  $[K_n : \mathbb{Q}] = t_n$ . Since  $K_n \subseteq K_{n+1}$ , then  $t_n | t_{n+1}$ , for all  $n \geq 1$ . Therefore, there are integers  $(m_n)_{n \geq 1}$  such that  $t_n = m_{n-1} \dots m_1 t_1$ . Note that  $K_{n+1} = K_n(\mathcal{R}_{F_{n+1}})$  and  $\deg F_{n+1} = \deg Q$ . It follows that  $[K_{n+1} : K_n] \leq (\deg Q)!$ . Because  $\mathbb{Q} \subseteq K_n \subseteq K_{n+1}$ , we also have that  $\frac{t_{n+1}}{t_n} \leq (\deg Q)!$  for all  $n \geq 1$ . On the other hand  $\frac{t_{n+1}}{t_n} = m_n$ , so the sequence  $(m_n)_{n \geq 1}$  is bounded. Thus, we ensure the existence of a prime number  $p > \max_{n \geq 1} \{m_n, t_1, 3\}$ . Hence  $p$  does not divide  $t_n$ , for  $n \geq 1$ . We pick a real number  $\alpha$  that is a root of the irreducible polynomial  $F(x) = x^p - 4x + 2$  and we claim that  $\alpha \notin \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$ . Indeed, if this is not the case, then there exists a number  $s \geq 1$ , such that  $\alpha \in \mathbb{Q}(\mathcal{R}_{F_1 \dots F_s}) = K_s$ . Since  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = p$ , we would have that  $p | t_s$ , however this is impossible. Moreover, given  $n \geq 1$ , we have the field inclusions  $\mathbb{Q} \subseteq \mathbb{Q}(\alpha^n) \subseteq \mathbb{Q}(\alpha)$ . So  $[\mathbb{Q}(\alpha^n) : \mathbb{Q}] = 1$  or  $p$ , but  $\alpha^n$  can not be written as radicals over  $\mathbb{Q}$ , since that  $F(x)$  is not solvable by radicals over  $\mathbb{Q}$ , see [1, p. 189]. Hence  $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^n)$  and then such  $\alpha$  satisfies the condition (2.1).  $\square$

Without referring the Lemma, we have the following special remarks:

**Remark 1.** *If  $\deg Q(x) = 1$ , then  $\mathbb{Q}(\mathcal{R}_{\mathcal{F}}) = \mathbb{Q}$ . Therefore  $\alpha = 1 + \sqrt{2}$  satisfies our desired condition (2.1).*

**Remark 2.** *More generally, if  $\deg Q \leq 4$ , then we take  $\alpha$  one of the real roots of the polynomial  $F(x) = x^5 - 4x + 2$ . I assert that this  $\alpha$  satisfies (2.1). In fact, note that all element of the field  $\mathbb{Q}(\mathcal{R}_{\mathcal{F}})$  is solvable by radicals (over  $\mathbb{Q}$ ), on the other hand the Galois group of  $F(x) = 0$  over  $\mathbb{Q}$  is isomorphic to  $S_5$  (the symmetric group), see [1, p. 189]. Hence if  $\alpha^n \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$ , it would be expressed as radicals over  $\mathbb{Q}$ , but this can not happen.*

Now we are able to prove our main result

*Proof.* (Proposition) Let us suppose that  $P$  assumes a positive value. In this case, we have  $0 < P(x) \neq 1$  for some interval  $(a, b) \subseteq \mathbb{R}$ . Therefore, the function  $f : (a, b) \rightarrow \mathbb{R}$ , given by  $f(x) := P(x)^{Q(x)}$  is well defined. Since  $f$  is a non-constant and continue function, then  $f((a, b))$  is a non-degenerate interval, say  $(c, d)$ . Now, take  $\alpha$  as in the Lemma. Note that the set  $\{\alpha Q : Q \in \mathbb{Q} \setminus \{0\}\}$  is dense in  $(c, d)$ . For a such  $\alpha Q \in (c, d)$ , we have

$$(2.2) \quad \alpha Q = P(T)^{Q(T)}$$

for some  $T \in (a, b)$ . We must prove that  $T$  is a transcendental number. Supposing the contrary, then  $P(T)$  and  $Q(T)$  are algebraic numbers. Since  $P(T) \notin \{0, 1\}$ , then by Gelfond-Schneider theorem, we infer that  $Q(T) = \frac{r}{s} \in \mathbb{Q}$ ,  $s > 0$ . It follows that  $T \in \mathcal{R}_{Q(x) - \frac{r}{s}} \subseteq \mathcal{R}_{\mathcal{F}}$ , so  $P(T)^r \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$ . By (2.2),  $(\alpha Q)^s = P(T)^r$ , hence  $\alpha^s \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$ , but that contradicts the Lemma.

For the case that  $P(x) \leq 0$  for all  $x \in \mathbb{R}$ , we can consider a subinterval  $(a, b) \subseteq \mathbb{R}$  such that  $\mathcal{R}_P \cap (a, b) = \emptyset$ , therefore the proof follows the same idea. But in this case the image of  $(a, b)$  under  $f$  is a connected subset of  $\mathbb{C}$  and our basic dense subset (in  $\mathbb{C}$ ) is the set  $\{\alpha Q : Q \in \mathbb{Q}(i) \setminus \{0\}\}$ .  $\square$

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, BRASÍLIA,  
DF, BRAZIL

*E-mail address:* `diego@mat.unb.br`