Some consequences of Schanuel's conjecture

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Abstract

We study the algebraic independence of two inductively defined sets. Under the hypothesis of Schanuel's Conjecture we prove that the exponential power tower E and its related logarithmic ladder L are linearly disjoint. This generalizes an exercise given by Serge Lang.

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1 Introduction

We study the algebraic independence of two inductively defined sets: the exponential power tower and its related logarithmic ladder. Under the hypothesis of Schanuel's Conjecture we prove that E and L are linearly disjoint. This work was suggested to us by Professor Michel Waldschmidt and generalizes an exercise given by Serge Lang [1, p. 31].

Specifically, let $E = \bigcup E_n$ where $E_n = \overline{E_{n-1}(\{e^x : x \in E_{n-1}\})}$ for each integer $n \geqslant 0$. Similarly let $L = \bigcup L_n$, with $L_n = \overline{L_{n-1}(\{y : e^y \in L_{n-1}\})}$. We take $E_0 = L_0 = \overline{\mathbb{Q}}$ as the ground field. Now our conditional result is the following.

Theorem *Assuming the validity of the Schanuel Conjecture the sets E and L are linearly disjoint.*

This implies several interesting consequences. Namely $E \cap L = \overline{\mathbb{Q}}$ while $\pi \notin E$ and $e \notin L$. Furthermore the elements of the power tower $e, e^e, e^{e^e}, \ldots$ are $L-$algebraically independent and in the logarithmic ladder $\pi, \ln \pi, \ln \ln \pi, \ldots$ are $E-$algebraically independent.

2 Proof of Theorem

Conjecture *(Schanuel) Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field*

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

is at least n.

Definition *Two field extensions K/k and L/k are* linearly disjoint *(respectively* free*) over k when all finite subsets of K linearly independent (respectively algebraically independent) over k are again over L.*

To prove the theorem first observe that $E_n = \overline{\mathbb{Q}(\exp(E_{n-1}))}$. We also have $L_n = \overline{\mathbb{Q}(\exp^{-1}(L_{n-1}))}$. In addition, if $x \in E_n$ the coefficients of the minimal polynomial of x over $\mathbb{Q}(\exp(E_{n-1}))$ must be contained in $\mathbb{Q}(\exp(A_{n-1}))$ for some finite set $A_{n-1} \subset E_{n-1}$.

Lemma *For all $x \in E_n$ there exists a finite set $A \subseteq E_{n-1}$ such that $A \cup \{x\}$ is algebraic over $\overline{\mathbb{Q}(\exp(A))}$. Similarly, for all $x \in L_n$ there exists a finite set $C \subseteq \mathbb{C}$ with $\exp(C) \subseteq L_{n-1}$ such that $\exp(C) \cup \{x\}$ is algebraic over $\mathbb{Q}(C)$.*

Proof Given the set of coefficients A_{n-1} it follows that A_{n-1} is algebraic over $\mathbb{Q}(\exp(A_{n-2}))$. A descending chain terminates with A_1 algebraic over $\mathbb{Q}(\exp(A_0))$ for some finite $A_0 \subseteq E_0 = \overline{\mathbb{Q}}$. Let $A = \bigcup_{m \le n-1}$. Since $A_m \subseteq E_{n-1}$ is algebraic over $\mathbb{Q}(\exp(A_{m-1}))$ and $x \in \overline{\mathbb{Q}(\exp(A))}$ it follows that A_m is algebraic over $\mathbb{Q}(\exp(A))$. As a result A is algebraic over $\mathbb{Q}(\exp(A))$. $\square$

We state the proof for the exponential case. The logarithmic follows similarly.

Proof of Theorem It suffices to prove E_m and L_n are linearly disjoint for arbitrary m and n. We therefore assume Schanuel's Conjecture and by induction also assume E_{m-1} and L_n are linearly disjoint over $\overline{\mathbb{Q}}$ but E_m and L_n are not linearly disjoint. If $\{l_1, ..., l_k\} \subseteq L_n$ are linearly independent over $\overline{\mathbb{Q}}$ and $\{e_1, \ldots, e_k\} \subseteq E_m$ define the linear combination $\sum_{i=1}^{k} l_i e_i = 0$ where at least one $e_i \neq 0$. Then the Lemma implies there exist a finite set $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}_{i=1}^{k}$ is algebraic over $\mathbb{Q}(\exp(A))$. In addition the Lemma shows there exists another finite set $C \subseteq L_n$ such that $\exp(C) \cup \{l_i\}_{i=1}^{k}$ is algebraic over $\mathbb{Q}(C)$.

If $B \subseteq A$ and $D \subseteq C$ are sets such that $\exp(B)$ and D are transcendence bases of $\mathbb{Q}(\exp(A))$ and $\mathbb{Q}(C)$ respectively, we claim that $B \cup D$ is linearly independent over $\mathbb{Q}$. By considering

$$\sum_{b \in B} p_b b = \sum_{d \in D} q_d d$$

with $p_b, q_d \in \mathbb{Z}$ the induction hypothesis implies $E_{m-1} \cap L_n = \overline{\mathbb{Q}}$. However, if $r = \sum_{d \in D} q_d d$ is an algebraic relation of D with coefficients in $\overline{\mathbb{Q}}$ it must be trivial because D is $\overline{\mathbb{Q}}$-algebraically independent. Therefore $r = 0 = q_d$ for all $d \in D$. By exponentiating $\sum_{b \in B} p_b b = 0$ we have the product

$$\prod_{b \in B} (\exp(b))^{p_b} = 1.$$

This algebraic relation of $\exp(B)$ with coefficients in $\overline{\mathbb{Q}}$ is also trivial because $\exp(B)$ is $\overline{\mathbb{Q}}$-algebraically independent. Thus $B \cup D$ is $\mathbb{Q}$-linearly independent.

By Schanuel's Conjecture $\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(B, D, \exp(B), \exp(D)) \geq |B| + |D|$. On the other hand

$$\begin{aligned}
\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(B, D, \exp(B), \exp(D)) &= \text{trdeg}_{\mathbb{Q}} \mathbb{Q}(B, C, \exp(A), \exp(D)) \\
&= \text{trdeg}_{\mathbb{Q}} \mathbb{Q}(C, \exp(A)) \\
&= \text{trdeg}_{\mathbb{Q}} \mathbb{Q}(D, \exp(B)) \\
&\leq |B| + |D|.
\end{aligned}$$

We conclude $\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(D, \exp(B)) = |B| + |D|$.

Both $\mathbb{Q}(\exp(B))$ and $\mathbb{Q}(D)$ as well as their algebraic closures $\overline{\mathbb{Q}(\exp(B))}$ and $\overline{\mathbb{Q}(D)}$ are $\overline{\mathbb{Q}}$-free. Now $\mathbb{Q}(\exp(B))$ and $\mathbb{Q}(D)$ are linearly disjoint over $\overline{\mathbb{Q}}$ (see [2, Theorem 4.12, p. 367]). So the coefficients $\{l_i\}$ are $\mathbb{Q}(C)$-algebraic while the $\{e_i\}$ are $\mathbb{Q}(\exp(A))$-algebraic. Because of this $\{l_i\} \subseteq \overline{\mathbb{Q}(D)}$ and $\{e_i\} \subseteq \overline{\mathbb{Q}(\exp(B))}$ render our previously constructed nontrivial linear relation $\sum l_i e_i = 0$, which is a contradiction. □

Now from on E and L are the sets constructed in Introduction. Assuming Schanuel's conjecture to be true, we have the following results

Corollary 1 *The constant $\pi \notin E$ and the constant $e \notin L$.*

Proof Follows imediately by Theorem.

□

Corollary 2 *The numbers $\pi, \ln\pi, \ln\ln\pi, \ldots$ are E-algebraically independent.*

Proof Let us write $\ln_{[k]}\pi$ for the k^{th}-iterated logarithm of π. Observe that $\pi, \ln\pi, \ln\ln\pi, \ldots \in L$. By the theorem E and L are free, so it is enough to prove that $i\pi, \log\pi, \log\log\pi, \ldots$ are algebraically independent over $\mathbb{Q}$, for that we use Schanuel's conjecture again. Without loss of generality, we may assume the statement true for $i\pi, \ln\pi, \ln\ln\pi, \ldots, \ln_{[n-1]}\pi$ (by induction). Now define the linear combination

$$i\pi q + \sum_{k=1}^{n-1} q_k \ln_{[k]}\pi = 0$$

with $q, q_k \in \mathbb{Z}$. Exponentiation gives

$$(-1)^q \prod_{k=1}^{n} \left(\ln_{[k-1]}\pi\right)^{q_k} = 1$$

$$\prod_{k=0}^{n-1} \left(\ln_{[k]}\pi\right)^{q_{k+1}} = (-1)^q.$$

Because the assumption is that $i\pi, \ln\pi, \ln\ln\pi, \ldots, \ln_{[n-1]}\pi$ are $\mathbb{Q}$-algebraically independent this last algebraic relation must be trivial. Therefore the set $A = \{i\pi, \ln\pi, \ln\ln\pi, \ldots, \ln_{[n]}\pi\}$ is $\mathbb{Q}$-linearly independent, hence Schanuel's Conjecture implies the transcendence degree of $\mathbb{Q}(A, \exp(A))$ should be at least $n+1$. The conclusion follows because $\exp(A)$ is algebraic over $\mathbb{Q}(A)$ and this implies $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(i\pi, \ln\pi, \ln\ln\pi, \ldots, \ln_{[n]}\pi) \geq n+1$. □

Corollary 3 *The numbers $e, e^e, e^{e^e}, \ldots$ are L-algebraically independent.*

Proof Set $\exp^{[n]}(1) = \exp(\exp^{[n-1]}(1))$ and $\exp^{[0]}(1) = 1$. Then assuming $\{\exp^{[k]}(1)\}_{k=1}^n$ are $\mathbb{Q}$-algebraically independent the set

$$A = \{1, e, e^e, \ldots, \exp^{[n]}(1)\} = \{\exp^{[k]}(1)\}_{k=0}^n$$

is $\mathbb{Q}$-linearly independent. Schanuel's conjecture implies

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\exp(A)) = \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(A, \exp(A)) \geq n+1.$$

Now $\exp(A) = \{\exp^{[k]}(1)\}_{k=1}^{n+1}$ are $\mathbb{Q}$-algebraically independent. The induction is complete. $\square$

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