Yet Another "Direct" Proof of the Uncountability of the Transcendental Numbers

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The most known proof of uncountability of the transcendental numbers is based on proving that $\mathbb{A}$ is countable and concluding that $\mathbb{R}\backslash\mathbb{A}$ is uncountable since $\mathbb{R}$ is. Very recently, J. Gaspar [1] gave a nice "direct" proof that the set of transcendental numbers is uncountable. In this context, the word *direct* means a proof which does not follow the previous steps. However, we point out that his proof is based on the transcendence of π which is, to the best of the author's knowledge, proved by an *indirect* argument. In this note, in the *spirit* of Gaspar, we present a "direct" proof of the following stronger result.

Theorem. *There exist uncountable many algebraically independent real numbers. So the set of the transcendental real numbers is uncountable.*

Proof. Let $\mathcal{B}$ be a transcendence basis (which exists by Zorn's lemma) of the field extension $\mathbb{R}/\mathbb{Q}$. If $\mathcal{B}$ were countable, then $\mathbb{Q}(\mathcal{B})$ would be countable (because its elements are of the form $P(\vec{b})/Q(\vec{c})$ with $n \in \mathbb{N}, P, Q \in \mathbb{Q}[X_1, \ldots, X_n]$ and $\vec{b}, \vec{c} \in \mathcal{B}^n$), so $\mathbb{R}$ would be countable (because its elements are roots of polynomials in $\mathbb{Q}(\mathcal{B})[X]\backslash\{0\}$ and there would be only countable roots), which is false. The elements of $\mathcal{B}$ are uncountable many algebraically independent real numbers. ■

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References

[1] J. Gaspar, Direct proof of the uncountability of the transcendental numbers, *The American Mathematical Monthly* **121** (1) (2014), p. 80.

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