Even perfect numbers among generalized Fibonacci sequences

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Abstract

A positive integer n is said to be a perfect number, if $\sigma(n) = 2n$, where $\sigma(N)$ is the sum of all positive divisors of N. In 2000, F. Luca proved that there is no perfect number in the Fibonacci sequence. For $k \geq 2$, the k-generalized Fibonacci sequence $(F_n^{(k)})_n$ is defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and such that each term afterwards is the sum of the k preceding terms. In this paper, we prove, among other things, that there is no even perfect numbers belonging to k-generalized Fibonacci sequences when $k \not\equiv 3 \pmod 4$.

Keywords: Generalized Fibonacci sequences, linear forms in logarithms, perfect numbers, Diophantine equations

2000 MSC: 11B39, 11J86

1. Introduction

Let $\sigma(N)$ be the sum of all positive divisors of N. A natural number n is called *perfect* if $\sigma(n) = 2n$. Perfect numbers have a very rich history (see [7, Chapter 1] and [20, Chapter 1]) and were already considered by Euclid, who proved that if the number $2^n - 1$ is a prime then its product by 2^{n-1} is perfect. Euler was the first to prove that Euclid's method gives all even perfect numbers:

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[1]Supported by FAP-DF, DPP/UnB and CNPq-Brazil

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Euclid-Euler Theorem. *An even integer n is perfect if and only if there exists a prime number p, such that $2^p - 1$ is also prime and $n = 2^{p-1}(2^p - 1)$.*

Although being studied since the ancient Greek, there are many open questions related to perfect numbers, as for instance:

- Are there infinitely many perfect numbers?
- Does exist an odd perfect number?

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [10] together with its very extensive annotated bibliography for additional references and history).

In 2000, F. Luca [12] proved that there is no perfect Fibonacci number. More generally, Luca and Huguet [13] and Phong [19] showed, independently, that there is no perfect number of the form F_{mk}/F_k. Both proofs used that an odd perfect number has the form $p^a x^2$, where p is a prime number with $p \equiv a \equiv 1 \pmod 4$. However, Phong used the, to the best of authors' knowledge, unproved fact that every prime factor of x is congruent to 3 modulo 4.

Let $k \geq 2$ and denote $F^{(k)} := (F_n^{(k)})_{n\geq -(k-2)}$, the *k-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)}, \tag{1}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Clearly for $k = 2$, we obtain the classical Fibonacci numbers $(F_n)_n$, for $k = 3$, the *Tribonacci* numbers $(T_n)_n$, for $k = 4$, the *Tetranacci* numbers $(Q_n)_n$, etc.

Recently, these sequences have been the main subject of many works. We refer to [3] for results on the largest prime factor of $F_n^{(k)}$ and we refer to [1] for the solution of the problem of finding powers of two belonging to these sequences. In 2013, two conjectures concerning these numbers were proved. The first one, proved by Bravo and Luca [4] is related to *repdigits* (i.e., numbers with only one distinct digit in its decimal expansion) among k-Fibonacci numbers (proposed by Marques [16]) and the second one, a conjecture (proposed by Noe and Post [18]) about coincidences between terms of these sequences, proved independently by Bravo-Luca [2] and Marques [14] (see [15] for results on the spacing between terms of these sequences).

The aim of this paper is to search for even perfect numbers belonging to generalized Fibonacci numbers. In other words, we shall study the Diophantine equation

$$F_n^{(k)} = 2^{p-1}(2^p - 1) \tag{2}$$

with p and $2^p - 1$ primes. More precisely, our main result is the following

Theorem 1. *The Diophantine equation (2) has no solution in positive integers n, k and p, where p and $2^p - 1$ are prime numbers, if at least one of the following conditions is satisfied:*

(i) $2 \leq k \leq 167$.

(ii) $n \neq 2p + 1$.

(iii) $n = 2p + 1$ *and* $p \geq k$.

(iv) $n = 2p + 1$, $p < k$ *and* $k \not\equiv 3 \pmod 4$.

(v) $n = 2p + 1$, $p < k, k \equiv 3 \pmod 4$ *and* $2p - k + 1 \neq 2^{k-p+1}$.

In particular, there is no even perfect number in $F^{(k)}$ *when* $k \not\equiv 3 \pmod 4$.

Our proof combines lower bounds for linear forms in three logarithms, a variation of a Dujella and Pethő reduction lemma and a fruitful method developed by Bravo and Luca concerning approximation of some convenient number (related to the dominant root of the characteristic polynomial of $F_n^{(k)}$) by a power of 2. In order to finish the proof (solve the particular case when $n = 2p+1$), it is still necessary to use a helpful trick related to a simple closed form to $F_n^{(k)}$ when $n \leq 2k + 2$.

2. Auxiliary results

Before proceeding further, we shall recall some facts and tools which will be used after.

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and 2 (as can be seen in [22, Lemma 3.6]). Also, in a recent paper, Dresden and Du [8, Theorem 1] gave a simplified "Binet-like" formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{3}$$

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. Also, it was proved in [4, Lemma 1] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \text{ for all } n \geq 1, \tag{4}$$

where α is the dominant root of $\psi_k(x)$. Also, the contribution of the roots inside the unit circle in formula (3) is almost trivial. More precisely, it was proved in [8] that

$$|F_n^{(k)} - g(\alpha, k)\alpha^{n-1}| < \frac{1}{2}, \tag{5}$$

where we adopt throughout the notation $g(x, y) := (x-1)/(2+(y+1)(x-2))$.

Another tool to prove our theorem is a lower bound for a linear form logarithms *à la Baker* and such a bound was given by the following result of Matveev (see [17] or Theorem 9.4 in [5]).

Lemma 1. *Let $\gamma_1, \ldots, \gamma_t$ be real algebraic numbers and let $b_1, \ldots, b_t$ be non-zero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_t)$ over $\mathbb{Q}$ and let A_j be a real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}, \text{ for } j = 1, \ldots, t.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

As usual, in the previous statement, the *logarithmic height* of an s-degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{s}(\log |a| + \sum_{j=1}^{s} \log \max\{1, |\gamma^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1 \leq j \leq s}$ are the conjugates of γ (over $\mathbb{Q}$).

After finding an upper bound on n which is too large for practical purposes, the next step is to reduce it. For that, our last ingredient can be found in [4, Lemma 4] and it is a variant of the famous Dujella and Pethő [9, Lemma 5 (a)] reduction lemma. For a real number x, we use $\parallel x \parallel = \min\{|x-n| : n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and let A, B be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon = \parallel \mu q \parallel - M \parallel \gamma q \parallel$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < A \cdot B^{-k}$$

in positive integers m, n and k with

$$m \leq M \ \textit{and}\ k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Now, we are ready to deal with the proof of the theorem.

3. Some key lemmas

In light of the main result of [12], throughout this paper we shall assume $k \geq 3$. Also, p will denote always (unless otherwise stated) a prime number with $2^p - 1$ also prime.

3.1. Upper bounds for n and c in terms of k

Since $F_n^{(k)}$ is a power of 2, for all $1 \leq n \leq k+1$, we may suppose that $n > k+1$.

In this section, we shall prove the following result

Lemma 3. *If (n, k, p) is an integer solution of Diophantine equation (2) with $n > k+1$, then*

$$n < 1.7 \cdot 10^{14} k^4 \log^3 k \quad \textit{and} \quad p < 1.2 \cdot 10^{14} k^4 \log^3 k. \tag{6}$$

Proof. By using Eq. (2), we obtain

$$g(\alpha, k)\alpha^{n-1} - 2^{2p-1} = -2^{p-1} - E_k(n) < 0, \tag{7}$$

where $E_k(n) := \sum_{i=2}^{k} g(\alpha_i, k)\alpha_i^{n-1}$. Thus

$$\left|\frac{g(\alpha,k)\alpha^{n-1}}{2^{2p-1}} - 1\right| < \frac{1}{2^{p-1}}, \tag{8}$$

where we used that $|E_k(n)| < 1/2$ (which follows from (3) and (5)).

In order to use Lemma 1, we take $t := 3$,

$$\gamma_1 := g(\alpha, k),\ \ \gamma_2 := 2,\ \ \gamma_3 := \alpha$$

and

$$b_1 := 1,\ \ b_2 := -2p+1,\ \ b_3 := n-1.$$

For this choice, we have $D = [\mathbb{Q}(\alpha) : \mathbb{Q}] = k$. In [1, p. 73], an estimate for $h(g(\alpha, k))$ was given. More precisely, it was proved that

$$h(\gamma_1) = h(g(\alpha, k)) < \log(4k+4).$$

Note that $h(\gamma_2) = \log 2$ and $h(\gamma_3) < 0.7/k$. It is a simple matter to deduce from inequality $2(1-2^{-k}) < \alpha < 2$ that $1/4 < g(\alpha, k) < 1$. Thus, we can take $A_1 := k\log(4k+4), A_2 := k\log 2$ and $A_3 := 0.7$.

Note that $\max\{|b_1|, |b_2|, |b_3|\} = \max\{2p-1, n-1\}$. By using (4) and (2), we get $2^{n-1} \geq F_n^{(k)} = 2^{p-1}(2^p-1) > 2^{3p/2-1}$ (where we used that $2^p-1 > 2^{p/2}$, for $p \geq 2$) and $\alpha^{n-2} \leq F_n^{(k)} = 2^{p-1}(2^p-1) = 2^{2p-1} - 2^{p-1} < 2^{2p-1}$. Therefore

$$n > 3p/2 \text{ and } 2p-1 > 0.8n - 1.7, \tag{9}$$

where we used that $7/4 < \alpha < 2$ (since $k \geq 3$). Thus, we can choose $B := 2n-3$. Since $g(\alpha,k)\alpha^{n-1}2^{-2p+1} < 1$ (by (7)), we are in position to apply Lemma 1. This lemma together with a straightforward calculation gives

$$\left|\frac{g(\alpha,k)\alpha^{n-1}}{2^{2p-1}} - 1\right| > \exp(-3.7\cdot 10^{11}k^4\log^2 k(1+\log(2n-3))), \tag{10}$$

where we used that $1+\log k < 2\log k$ and $\log(4k+4) < 2.6\log k$, for $k \geq 3$.

By combining (8) and (10), we obtain

$$\frac{n}{\log n} < 2.7\cdot 10^{12}k^4\log^2 k,$$

where we used that $1+\log(2n-3) < 2\log n$, for $n \geq 2$. Since the function $x/\log x$ is increasing for $x > e$, it is a simple matter to prove that

$$\frac{x}{\log x} < A \ \text{ implies that } \ x < 2A\log A. \tag{11}$$

A proof for that can be found in [1, p. 74].

Thus, by using (11) for $x := n$ and $A := 2.7\cdot 10^{12}k^4\log^2 k$, we have that

$$n < 1.7\cdot 10^{14}k^4\log^3 k \tag{12}$$

and we use estimates in (9) to get

$$p < 1.2\cdot 10^{14}k^4\log^3 k.$$

This finishes the proof of lemma. □

3.2. The small cases: $3 \leq k \leq 167$

In this section, we shall prove the following result

Lemma 4. *There is no solution to Diophantine equation (2), with* $n > k+1$ *and* $3 \leq k \leq 167$.

Proof. By using Mathematica, Equation (2) has no solutions for $p \leq 23$, so we can assume $p \geq 23$. By (7) and (8) one has

$$0 < (2p-1)\log 2 - (n-1)\log\alpha + \log(1/g(\alpha,k)) < 1.001\cdot 2^{-p+1}.$$

Dividing by $\log\alpha$, we obtain

$$0 < (2p-1)\gamma_k - (n-1) + \mu_k < 3.6\cdot 2^{-p}, \tag{13}$$

where $\gamma_k = \log 2/\log\alpha^{(k)}$ and $\mu_k = \log(1/g(\alpha^{(k)},k))/\log\alpha^{(k)}$. Here, we added the superscript to α for emphasizing its dependence on k.

We claim that γ_k is irrational, for any integer $k \geq 2$. In fact, if $\gamma_k = p/q$, for some positive integers p and q, we have that $2^q = (\alpha^{(k)})^p$ and we can conjugate this relation by some automorphism of the Galois group of the splitting field of $\psi_k(x)$ over $\mathbb{Q}$ to get $2^q = |(\alpha_i^{(k)})^q| < 1$, for $i > 1$, which is an absurdity, since $q \geq 1$. Let $q_{m,k}$ be the denominator of the m-th convergent of the continued fraction of γ_k. Taking $M_k := 2.41\cdot 10^{14}k^4\log^3 k \leq M_{167} < 2.6\cdot 10^{25}$, we use *Mathematica* [21] to get

$$\min_{3\leq k\leq 167} q_{90,k} > 7\cdot 10^{38} > 6M_{167}.$$

Also

$$\max_{3\le k\le 167} q_{90,k} < 4.4\cdot 10^{102}.$$

Define $\epsilon_k := \| \mu_k q_{90,k} \| - M_k \| \gamma_k q_{90,k} \|$, for $3 \le k \le 167$, we get (again using Mathematica)

$$\min_{3\le k\le 167} \epsilon_k = 0.0000571469\ldots.$$

Note that the conditions to apply Lemma 2 are fulfilled for $A = 3.6$ and $B = 2$, and hence there is no solution to inequality (13) (and then no solution to the Diophantine equation (2)) for p satisfying

$$2p - 1 < M_k \text{ and } p \ge \frac{\log(Aq_{90,k}/\epsilon_k)}{\log B}.$$

Since $2p - 1 < M_k$ (Lemma 3), then

$$p < \frac{\log(Aq_{90,k}/\epsilon_k)}{\log B} \le \frac{\log(3.6\cdot 4.4\cdot 10^{102}/0.000057146)}{\log 2} = 356.917\ldots.$$

Therefore $p \le 356$. By applying the estimate in (9), we have $n \le 890$. Since $2^p - 1$ is a prime number, then

$$p \in \{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127\}.$$

Now, we can use Mathematica to conclude that Eq. (2) has no solution for each of these 12 cases. In particular, we may suppose that $p \ge 521$. $\square$

4. The proof of Theorem 1

Note that the item (i) follows directly from Lemma 4.

4.1. The case $n \neq 2p + 1$

If $n \neq 2p + 1$. By Lemma 4, we may consider $k \ge 168$. In this case, we have, by Lemma 3,

$$n < 1.7\cdot 10^{14} k^4 \log^3 k < 2^{k/2}. \tag{14}$$

Now, we use a key argument due to Bravo and Luca [1, p. 77-78]. By following their same steps, we arrive at

$$|2^{n-2} - 2^{2p-1}| < 2^p + \frac{5\cdot 2^{n-2}}{2^{k/2}}, \tag{15}$$

or equivalently

$$|1 - 2^{2p-n+1}| < 2^{p-n+2} + \frac{5}{2^{k/2}}. \tag{16}$$

By using (9) and after some manipulations, we obtain

$$|1 - 2^{2p-n+1}| < \frac{4}{2^{n/3}} + \frac{5}{2^{k/2}} < \frac{9}{(\sqrt[3]{2})^k}, \tag{17}$$

where we used that $n > k$. Since $n \neq 2p+1$, then either $2p \geq n$ or $n \geq 2p+2$. In any case, we have $|1-2^{2p-n+1}| > 1/2$ and then $(\sqrt[3]{2})^k < 18$ yielding $k \leq 13$ which is a contradiction. This finishes the proof of this case.

□

4.2. The case $n = 2p + 1$

The proof of the case $n = 2p+1$ splits in two parts and in both of them we shall use the helpful fact that

$$F_{k+i}^{(k)} = 2^{k+i-2} - i2^{i-3},$$

for all $2 \leq i \leq k+2$ (see [6, Theorem 2.2]).

Thus, we shall prove that there is no solution for

$$F_{2p+1}^{(k)} = 2^{p-1}(2^p - 1) \tag{18}$$

in the following two cases:

Case 1. $p \geq k$. Indeed, in this case, we claim that

$$F_{2p+1}^{(k)} < 2^{p-1}(2^p - 1),$$

for all $p \geq k$ (it is not necessary to suppose primality of p). Let us proceed by induction on p. The base step $p = k$ follows because

$$\begin{aligned} 2^{k-1}(2^k - 1) - F_{2k+1}^{(k)} &= 2^{k-1}(2^k - 1) - 2^{2k-1} + (k+1)2^{k-2} \\ &= (k+1)2^{k-2} - 2^{k-1} > 0, \end{aligned}$$

if $k \geq 2$. By induction hypothesis, suppose that $F_{2p+1}^{(k)} < 2^{p-1}(2^p - 1)$. Now, we use that $F_{n+1}^{(k)} \leq 2F_n^{(k)}$ to obtain

$$F_{2p+3}^{(k)} \leq 4F_{2p+1}^{(k)} < 2^p(2^{p+1} - 2) < 2^p(2^{p+1} - 1)$$

which finished the inductive process. □

Case 2. $p < k$. Since, we are supposing that $2p+1 > k+1$ (to avoid the powers of two), then $k/2 < p < k$. Thus $2 \le 2p+1-k \le k$ and we can used the closed form for $F_{k+i}^{(k)}$ to get

$$F_{2p+1}^{(k)} = 2^{2p-1} - (2p+1-k)2^{2p-k-2}.$$

Combining with (18) we obtain

$$2p+1-k = 2^{k-p+1}.$$

In particular, k is odd (on the contrary $k-p+1=0$ and $2p+1-k=1$). If $k \equiv 1 \pmod 4$, then the 2-adic valuation of $2p+1-k$ is 1. On the other hand, the above equality leads to $p=k$ which is a contradiction (since $p < k$).

□

Acknowledgement

Part of this work was made during a postdoctoral position of the first author in the Department of Mathematics at University of British Columbia. He thanks the UBC for excellent working conditions.

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