A DIOPHANTINE EQUATION INVOLVING C-NOMIAL COEFFICIENTS

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Abstract. Let C_n be the nth Fibonacci number ($C_n = F_n$) or the nth Lucas number ($C_n = L_n$). For $1 \leq k \leq m$, let

$$\begin{bmatrix} m \\ k \end{bmatrix}_C = \frac{C_m C_{m-1} \cdots C_{m-k+1}}{C_1 \cdots C_k}$$

be the corresponding C-nomial coefficient. In this paper, we prove that the only solutions of the Diophantine equation

$$\begin{bmatrix} m \\ k \end{bmatrix}_C = m^a k^b,$$

in positive integers m, k, a, b with $a > 1$, are $(m, k, a, b) = (1, 1, a, b)$, $(5, 1, 1, b)$, $(12, 1, 2, b)$, and $(5, 3, 1, 1)$, for $C_n = F_n$ and $(m, k, a, b) = (1, 1, a, b)$ in the case $C_n = L_n$.

1. Introduction

Let $(C_n)_{n\geq 1}$ be a Lucas sequence given by

$$C_{n+2} = C_{n+1} + C_n, \quad \text{for } n \geq 1,$$

where the values C_0 and C_1 are previously fixed. For instance, if $C_0 = 0$ and $C_1 = 1$, then $C_n = F_n$ is the well-known *Fibonacci sequence*:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

Also, if $C_0 = 2$ and $C_1 = 1$, the sequence $C_n = L_n$ gives the *Lucas numbers*

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

According to the Binet's formula, for $n \geq 0$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = (1 + \sqrt{5})/2$ (the golden number) and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$.

It is well-known that the only solutions of $F_m = m$ are $m = 1$ and 5 and for $L_m = m$ one has $m = 1$. In fact, we have $C_m > m$, for all $m > 5$ (this can be proved by mathematical induction).

The *C-nomial coefficients* are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_C = \frac{C_m C_{m-1} \cdots C_{m-k+1}}{C_1 \cdots C_k},$$

for $1 \leq k \leq m$. For instance, if $C_n = F_n$, we have the well-known Fibonomial coefficients (sequence A001656 in OEIS[1] [7]). Some results on the spacing of these

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numbers can be found in [5]. We also refer the reader to [6] for several interesting identities involving this sequence.

Since C-nomial coefficients generalize the concept of the Fibonacci and Lucas numbers, as $\begin{bmatrix} m \\ 1 \end{bmatrix}_C = C_m$, it is worthwhile to find the solutions of the general equation

$$\begin{bmatrix} m \\ k \end{bmatrix}_C = m^a k^b. \tag{1.1}$$

The goal of this paper is to determine all the solutions of Diophantine equation (1.1) when $C_m = F_m,\ L_m$. Our main results are the following.

Theorem 1. *The only solutions of the Diophantine equation*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = m^a k^b$$

are $(m, k, a, b) = (1, 1, a, b),\ (5, 1, 1, b),\ (12, 1, 2, b)$, *and* $(5, 3, 1, 1)$.

Theorem 2. *The only solution of the Diophantine equation*

$$\begin{bmatrix} m \\ k \end{bmatrix}_L = m^a k^b$$

is $(m, k, a, b) = (1, 1, a, b)$.

We will prove these results in the next section.

2. Proofs of Theorems 1 and 2

Before the proofs of Theorems 1 and 2, we will recall some interesting and helpful properties of these sequences. Their proofs are well-known and can be found in any good text about sequences.

Lemma 1. *Let $(F_n)_{n\geq 0}$ be Fibonacci numbers and let $(L_n)_{n\geq 0}$ be Lucas numbers, then*

(i) $F_{2n} = F_n L_n$;
(ii) $L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n$;
(iii) *For all* $n \geq 3$,
$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \text{ and } \alpha^{n-1} \leq L_n \leq 2\alpha^n;$$
(iv) *If p is a prime number, then* $L_p \equiv 1 \pmod p$ *(see* [4, Theorem 7]*).*

Let C_n be Fibonacci or Lucas numbers. A *primitive divisor* p of the C_n is a prime factor of C_n which does not divide $5\prod_{1\leq j\leq n-1} C_j$. It is known that a primitive divisor p of C_n exists whenever $n \geq 13$ (see, for example, [3]). The above statement is usually referred to as the *Primitive Divisor Theorem* (see [1] and [2] for the most general version). It is also known that such a primitive divisor p satisfies $p \equiv \pm 1 \pmod n$. Now, we have the tools to study equation (1.1).

2.1. **Proof of Theorem 1: the Fibonacci case.** We consider equation (1.1) with $C_n = F_n$. Suppose that $m > \max\{24, k\}$. By the Primitive Divisor Theorem, there exists a primitive prime factor p for F_m. Since

$$F_m F_{m-1} \cdots F_{m-k+1} = m^a k^b F_1 \cdots F_k, \tag{2.1}$$

and p does not divide $\prod_{j=1}^{k} F_j$, then p divides $m^a k^b$. Therefore, p divides k, because $p \equiv \pm 1 \pmod m$. So it does not divide m. Moreover, the congruence $p \equiv \pm 1$

(mod m) implies that $p \geq m-1$. Thus, we conclude that $m-1 \leq p \leq k < m$ and then $k = p = m-1$ which implies that m is an even number. Now we can use item (i) of Lemma 1 to conclude that $F_m = F_{m/2}L_{m/2}$. Also equation (2.1) becomes

$$F_m = m^a(m-1)^b. \tag{2.2}$$

As $m > 24$, then $m/2 > 12$ and there exists a primitive prime factor q of $F_{m/2}$. Note that q divides F_m but does not divides m (because $q \equiv 1 \pmod{m/2}$). It follows that q divides $m-1 = p$ and hence $p = q$. This contradicts the fact that p does not divides $F_{m/2}$.

For the case $k = 1$, one can see that the solutions are $(1,1,a,b)$, $(5,1,1,b)$, and $(12,1,2,b)$. For the other cases, we need to determine an upper bound for the sum $a+b$. So we will use item (iii) in Lemma 1. Thus, we have

$$\left(\frac{F_m}{F_1}\right) < \alpha^{m-1} \text{ and } \left(\frac{F_{m-t}}{F_{t+1}}\right) < \alpha^{m-2t}, \text{ for } 1 \leq t \leq k-1.$$

Therefore, we obtain

$$\begin{bmatrix} m \\ k \end{bmatrix}_F \leq \alpha^{m-1+m-2+\cdots+m-2(k-1)} = \alpha^{m-1+(m-k)(k-1)}. \tag{2.3}$$

On the other hand, one can see that $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F = m^a k^b \geq k^{a+b}$. Combining this with inequality (2.3), we immediately get, for $2 \leq k < m \leq 24$,

$$a+b \leq \frac{(m-1)+(m-k)(k-1)}{2\log k} < 32.542,$$

as $\log\alpha < 1/2$ and the maximum occurs when $m = 24$ and $k = 9$. So for the remaining cases, it suffices to test the values in the obtained range. Therefore, we used a simple program in *Mathematica* [8]. It took a few minutes to show that the only zero of the difference $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_F - m^a k^b$ in the range $2 \leq k < m \leq 24,\ 2 \leq a \leq 32$, and $1 \leq b \leq 32-a$ is $(m,k,a,b) = (5,3,1,1)$. This completes the proof of Theorem 1.

2.2. **Proof of Theorem 2: the Lucas case.** In that case, equation (1.1) becomes

$$L_m L_{m-1} \cdots L_{m-k+1} = m^a k^b L_1 \cdots L_k. \tag{2.4}$$

Suppose that $m > \max\{12, k\}$, by using the Primitive Divisor Theorem, we get $p = k = m-1 > 3$. Thus we will only consider the solutions of

$$L_m = m^a(m-1)^b. \tag{2.5}$$

Here the parity of m is not useful, since there is no multiplicative identity for L_m. Actually, one has $L_{2n} = (5F_n^2 + L_n^2)/2$. Thus the method in the previous proof is not applicable. Instead, we explore the primality of p.

First, note that $b \geq 1$. Otherwise $L_m = m^a$ and thus any primitive divisor of L_m must divide m which contradicts the congruence $p \equiv \pm 1 \pmod{m}$. Therefore, as $p = m-1$, from equation (2.5), we deduce

$$L_{p+1} = (p+1)^a p^b \equiv 0 \pmod{p}.$$

By item (ii) of Lemma 1, one has $-5 = L_p^2 - L_{p-1}L_{p+1} \equiv L_p^2 \pmod{p}$. Combining this with item (iv) of Lemma 1, we see that p divides 6, which is impossible. Therefore, one must have $m \leq 12$.

Item (iii) of Lemma 1 leads to

$$k^{a+b} \leq \left(\frac{L_m}{L_1}\right) \cdots \left(\frac{L_{m-k+1}}{L_k}\right) \leq 2^k \alpha^{k(m-k+1)}.$$

This implies

$$a+b \leq \frac{k(\log 2 + (m-k+1)\log\alpha)}{\log k} < 15.9,$$

and the maximum occurs for $m = 11$ and $k = 2$. Again here we used a short program written in Mathematica [8] to show in a few seconds that the difference $\begin{bmatrix} m \\ k \end{bmatrix}_L - m^a k^b$ is not zero in the range $2 \leq k < m \leq 12,\ 2 \leq a \leq 15$ and $1 \leq b \leq 15-a$. This completes the proof of Theorem 2.

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