# FIXED POINTS OF THE ORDER OF APPEARANCE IN THE FIBONACCI SEQUENCE

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ABSTRACT. Let $F_n$ be the $n$th Fibonacci number. The order of appearance $z(n)$ of a natural number $n$ is defined as the smallest natural number $k$ such that $n$ divides $F_k$. In this paper, we prove that $z(n) = n$, if and only if $n = 5^k$ or $12 \cdot 5^k$, for some $k \geq 0$.

## 1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [5] together with its very extensive annotated bibliography for additional references and history).

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let $n$ be a positive integer number, the *order (or rank) of appearance* of $n$ in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer $k$, such that $n \mid F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For instance, $z(n) < \infty$ for all $n \geq 1$. The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [11, p. 300].

In recent papers, the author [6, 7, 8, 9] found explicit formulas for the order of appearance of integers related to Fibonacci numbers, such as $F_m \pm 1,\ L_n \pm 1, F_{mk}/F_k, \prod_{i=0}^{k} F_{n+i}$, $k = 1, 2, 3$ and $F_n^k$. In particular, it was proved that $z(F_n \pm 1) = \frac{n^2}{2} - 2$, for $4 \mid n$, $z(F_nF_{n+1}F_{n+2}) = \frac{n(n+1)(n+2)}{2}$, if $2 \mid n$, and $z(F_n^2) = nF_n$, if $n \equiv 3 \pmod 6$.

In this paper, we continue this program and search for fixed points of $z(n)$. Our main result is the following.

**Theorem 1.1.** *Let $n$ be a positive integer. Then $z(n) = n$ if and only if $n = 5^k$ or $12 \cdot 5^k$, for some $k \geq 0$.*

We remark that this assertion appears in the FORMULA section of the OEIS [13] sequence A001177 and it is due to Benoit Cloitre. However, according to Cloitre [1] his assertion is a conjecture. Thus, to the best of the author's knowledge, there is no proof for this fact in the literature.

We organize the paper as follows. In Section 2, we will recall some useful properties of Fibonacci numbers such as the d'Ocagne's identity and a result concerning the $p$-adic order of $F_n$. Section 3 will be devoted to the proof of Theorem 1.1. In the last section, we close the paper with a brief discussion about the equations $z(n) = n^k$ and $z(n) = an/b$.

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## 2. Auxiliary results

Before proceeding further, we prove some facts on Fibonacci numbers for the convenience of the reader.

**Lemma 2.1.** *We have*

(a) $n \mid m$ *if and only if* $F_n \mid F_m$.
(b) *(d'Ocagne's identity)* $(-1)^n F_{m-n} = F_m F_{n+1} - F_n F_{m+1}$.
(c) $F_{p-(\frac{5}{p})} \equiv 0 \pmod p$, *for all primes* $p$.

Here, as usual, $(\frac{a}{q})$ denotes the Legendre symbol of $a$ with respect to a prime $q > 2$.

This lemma can be proved by using the well-known Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, together with induction (for (a) and (b)) and the binomial theorem (for (c)). Note that (a) leads to the useful facts that $2 \mid F_n$ and $3 \mid F_n$ if and only if $3 \mid n$ and $4 \mid n$, respectively. We refer the reader to [4, 5, 12] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

**Lemma 2.2.** *If* $n \mid F_m$, *then* $z(n) \mid m$.

**Proof.** Write $m = z(n)q + r$, where $q$ and $r$ are integers, with $0 \le r < z(n)$. So, by Lemma 2.1 (b), we obtain

$$(-1)^{z(n)q} F_r = F_m F_{z(n)q+1} - F_{z(n)q} F_{m+1}.$$

Since $n$ divides both $F_m$ and $F_{z(n)q}$, then it also divides $F_r$ implying $r = 0$ (keep in mind the range of $r$). Thus $z(n) | m$. □

**Lemma 2.3.** *For all primes* $p \neq 5$, *we have that* $\gcd(z(p), p) = 1$.

**Proof.** By combining Lemma 2.1 (c) together with Lemma 2.2, we conclude that $z(p) \mid p - (\frac{5}{p})$. Thus, when $p \neq 5$, one has that $(\frac{5}{p}) = \pm 1$ and so $z(p) \mid p \pm 1$. This yields that $z(p) = p+1$ or $z(p) \le p-1$ and in any case $\gcd(z(p), p) = 1$. □

The $p$-adic order (or valuation) of $r$, $\nu_p(r)$, is the exponent of the highest power of a prime $p$ which divides $r$. Throughout the paper, we shall use the known facts that $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and that $a \mid b$ if and only if $\nu_p(a) \le \nu_p(b)$, for all primes $p$.

We remark that the $p$-adic order of Fibonacci numbers was completely characterized, see [3, 10]. For instance, from the main results of Lengyel [10], we extract the following result.

**Lemma 2.4.** *For* $n \ge 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, *and if* $p$ *is prime* $\neq 2$ *or* $5$, *then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)} \end{cases}$$

A proof for this result can be found in [10].

## 3. Proof of the Theorem 1.1

For the "if" part, we have two cases to consider:

**Case 1:** $n = 5^k$.

Since $\nu_5(F_{5^k}) = \nu_5(5^k) = k$ (Lemma 2.4), we have that $5^k \mid F_{5^k}$ and so $z(5^k) \mid 5^k$. Now, suppose that $5^k \mid F_j$, then $k \leq \nu_5(F_j) = \nu_5(j)$ leading to $j = 5^k m$, for some positive integer. Thus $j \geq 5^k$ and then $5^k \leq z(5^k) \mid 5^k$. This yields that $z(5^k) = 5^k$.

**Case 2:** $n = 12 \cdot 5^k$.

We claim that $12 \cdot 5^k \mid F_{12 \cdot 5^k}$. In fact, note that $\nu_5(12 \cdot 5^k) = k = \nu_5(F_{12 \cdot 5^k})$ and it suffices to consider the $p$-adic order for $p = 2$ and 3. For these cases, we have

$$\nu_p(F_{12 \cdot 5^k}) = \nu_p(12 \cdot 5^k) + 1 + \delta_p > \nu_p(12 \cdot 5^k),$$

where $\delta_2 = 1$ and $\delta_3 = 0$. Thus the assertion is proved and therefore, by Lemma 2.2, $z(12 \cdot 5^k) \mid 12 \cdot 5^k$.

Suppose that $12 \cdot 5^k \mid F_j$, then $\nu_5(12 \cdot 5^k) \leq \nu_5(F_j) = \nu_5(j)$. Thus $k \leq \nu_5(j)$ yielding that $5^k \mid j$. Also, $1 = \nu_3(12 \cdot 5^k) \leq \nu_3(F_j)$, but this implies that $3 \mid F_j$, that is, $4 \mid j$. Finally, from $\nu_2(12 \cdot 5^k) \leq \nu_2(F_j)$, we infer that $2 \mid F_j$, that is, $3 \mid j$. Summarizing $12 \cdot 5^k = 3 \cdot 4 \cdot 5^k \mid j$. In conclusion, $j \geq 12 \cdot 5^k$ and hence $z(12 \cdot 5^k) = 12 \cdot 5^k$. This completes the proof of the first part.

For the "only if" part, we need the following key lemma:

**Lemma 3.1.** *If $n \mid F_n$ and $n > 1$, then the smallest prime factor of $n$ is 2 or 5. In particular, $5 \mid n$ or $12 \mid n$.*

**Proof.** Let $q$ be the smallest prime factor of $n$. Suppose, towards a contradiction, that $q > 5$. Thus $q \mid n \mid F_n$ and then $z(q) \mid n$ (by Lemma 2.2). Since $q \neq 5$, then $z(q) \leq q - 1$ or $z(q) = q + 1$. By the minimality of $q$ (as a prime divisor of $n$), we have that $z(q) = q + 1$ must be a prime, so $q = 2$ which is a contradiction. Therefore, we deduce that $q = 2, 3$ or 5. If $3 = q \mid n \mid F_n$, then 2 divides $n$ leading to an absurd (since 3 is the minimal prime factor of $n$). Hence $q = 2$ or 5. Note that, when $q = 2$, then $2 \mid n \mid F_n$ yielding $3 \mid n \mid F_n$ and so $4 \mid n$. In conclusion, $12 \mid n$. $\square$

We refer the reader to [2, 14] (and references therein) for results on terms of linear recurrence sequences divisible by their indexes.

We now turn back to our main proof.

Suppose that $z(n) = n$ and $n > 1$. In particular, $n \mid F_n$ and the previous result allows us to write $n = 12^a \cdot 5^k m$, with $a + k \geq 1$ and $\gcd(5 \cdot 12, m) = 1$. So, it suffices to prove that $a \in \{0, 1\}$ and $m = 1$.

In a few words, our approach is the following. Suppose, by contradiction, that either $a \geq 2$ or $m > 1$. In these cases, we shall exhibit an integer $j \in \{1, ..., n-1\}$, such that $n \mid F_j$. However, this contradicts $z(n) = n$.

**Case 1:** If $a \geq 2$.

In this case, we shall prove that $12^a \cdot 5^k m \mid F_{12^{a-1} \cdot 5^k m}$ arriving at a contradiction. In fact, since $\nu_p(12^a \cdot 5^k m) \leq \nu_p(F_{12^a \cdot 5^k m})$, for all primes $p$, then $\nu_p(12^a \cdot 5^k m) \leq \nu_p(F_{12^{a-1} \cdot 5^k m})$, for all primes $p \neq 2$ and 3. Now, we prove that the previous inequality also holds for $p = 2$ and 3. For $p = 2$, since $12 \mid 12^{a-1}$, we get

$$\nu_2(F_{12^{a-1} \cdot 5^k m}) = \nu_2(12^{a-1} \cdot 5^k m) + 2 = 2a = \nu_2(12^a \cdot 5^k m).$$

For $p = 3$, since $4 = z(3) \mid 12^{a-1}$,

$$\nu_3(F_{12^{a-1}\cdot 5^k m}) = \nu_3(12^a \cdot 5^k m) + 1 = a + 1 > a = \nu_3(12^a \cdot 5^k m).$$

Thus $a = 0$ or $1$ as desired.

**Case 2:** If $m > 1$.

Suppose that $q$ is a prime factor of $m$ and write $m = qt$, for some positive integer $t$. Since $\gcd(60, m) = 1$, then $q \neq 2, 3, 5$. Thus, the hypothesis $z(n) = n$ implies that $z(12^a \cdot 5^k qt) = 12^a \cdot 5^k qt$. We claim that the last equality is not true. More precisely, we shall prove that $12^a \cdot 5^k qt \mid F_{12^a \cdot 5^k t}$. Since $\nu_p(12^a \cdot 5^k qt) \leq \nu_p(12^a \cdot 5^k qt)$, for all primes $p$, we infer that $\nu_p(12^a \cdot 5^k qt) \leq \nu_p(F_{12^a \cdot 5^k qt})$, for all primes $p \neq q$. Thus, we must only treat the case $p = q$. Note that $12^a \cdot 5^k qt \mid F_{12^a \cdot 5^k qt}$ yields $q \mid F_{12^a \cdot 5^k qt}$ and so $z(q) \mid 12^a \cdot 5^k qt$. However, Lemma 2.3 gives that $q$ and $z(q)$ are coprime leading to $z(q) \mid 12^a \cdot 5^k t$. Hence, we get

$$\nu_q(F_{12^a \cdot 5^k t}) = \nu_q(12^a \cdot 5^k t) + \nu_q(F_{z(q)}) \geq \nu_q(t) + 1 = \nu_q(12^a \cdot 5^k qt).$$

In conclusion, we get an absurdity as $12^a \cdot 5^k qt = z(12^a \cdot 5^k qt) \mid 12^a \cdot 5^k t$. This completes the proof. $\square$

## 4. Further results and comments

It seems tempting to search for solutions to the equation $z(n) = n^k$, for some integers $k \geq 2$ and $n > 1$. However, this equation has no solution. This fact is an immediate consequence of the below result (which is a slight improvement of the proof of Theorem in page 52 of [15]).

**Proposition 4.1.** *If $n > 2$, then $z(n) < (n-1)^2 + 1$. In particular, $z(n) = n^k$, for some $k \geq 2$ if and only if $n = 1$.*

**Proof.** First, we consider the sequence $\mathcal{S} = \{(F_k, F_{k+1}) \pmod{n}\}_{k \in \mathbb{N}}$. Note that there are at most $(n-1)^2$ distinct pairs in $\mathcal{S}$, such that $n$ does not divide $F_k$ or $F_{k+1}$. Thus if we take $(n-1)^2 + 1$ terms of $\mathcal{S}$ we have that either $n$ divides some $F_k$ (and we are done) or there exist $m > s > 0$ such that $F_{m+1} \equiv F_{s+1} \pmod{n}$ and $F_m \equiv F_s \pmod{n}$. In the last case, by subtracting the previous congruences and using the recurrence pattern of $(F_n)_n$, we get $F_{m-1} \equiv F_{s-1} \pmod{n}$. Repeating this procedure $s$ times, we obtain $F_{m-s} \equiv F_2 - F_1 \equiv 0 \pmod{n}$. Since $m - s > 0$, we get the desired $n \mid F_{m-s}$. $\square$

We remark that the sequence A023172 of OEIS is dedicated to numbers $n$ dividing $F_n$. A few terms of this sequence are

$$1, 5, 12, 24, 25, 36, 48, 60, 72, 96, 108, 120, 125, 144, 168, 180, 192, \ldots$$

As one can see, there are many terms in this list which are not of the form $5^k$ or $12 \cdot 5^k$ (the fixed points of $z(n)$). So, a natural question arises: for which $n$ and $k > 1$, we have that $n^k \mid F_n$? The answer lies in the below result.

**Proposition 4.2.** *If $n^k \mid F_n$ for positive integers $n > 1$ and $k > 1$, then $(n, k) = (12, 2)$.*

**Proof.** It suffices to consider the case $k = 2$ (because $12^k \mid F_{12} = 144$, only for $k = 1, 2$). Since $n > 1$, then Lemma 3.1 implies that $5 \mid n$ or $12 \mid n$. Note that $n^2 \mid F_n$ yields $2\nu_5(n) \leq \nu_5(F_n) = \nu_5(n)$ (by Lemma 2.4). Thus $5 \nmid n$ and then $n = 12^a m$, for some positive integers

$n$ and $a$, with $\gcd(m,12)=1$. So, we shall prove that $a=m=1$. Note that $12^{2a}m^2 \mid F_{12^a m}$ and in particular, we have

$$4a=\nu_2(12^{2a}m^2)\le \nu_2(F_{12^a m})=\nu_2(12^a m)+2=2a+2$$

and so $a=1$. Also, suppose that, on the contrary, $m>1$, then we can write $m=p_1^{a_1}\cdots p_k^{a_k}$, where $a_i$'s are positive integers and $p_i$'s are prime numbers, with $5<p_1<p_2<\cdots<p_k$. Since $p_1 \mid F_{12p_1^{a_1}\cdots p_k^{a_k}}$, then $z(p_1)\mid 12p_1^{a_1}\cdots p_k^{a_k}$. However, either $z(p_1)<p_1$ or $z(p_1)=p_1+1$. When $z(p_1)<p_1$, then $\gcd(z(p_1),p_i)=1$, for all $i=1,...,k$. Thus $z(p_1)\mid 12$ implying that $p_1=2$ or 3 which is a contradiction. Suppose now that $z(p_1)=p_1+1$, then $p_1\in\{7,23,43,67,...\}$. It is a simple matter to infer that $p_3>p_1+1$ which yields $\gcd(p_1+1,p_i)=1$, for $i=1,3,...,k$. Also $p_1+1$ and $p_2$ are coprime. In fact, on the contrary, $p_2\mid p_1+1$ and therefore $p_1=2$ and $p_2=3$, contradiction. In conclusion, $p_1+1\mid 12$ which can not happen for $p_1\in\{7,23,...\}$. Hence $m=1$ and the proof is complete. □

Another kind of related problem which could be considered is about the equation $z(n)=an/b$ for some positive integers $a$ and $b$. For example, we have

**Proposition 4.3.** *Let $a$ and $b$ be positive integers. If $\nu_5(a)\ne\nu_5(b)$, then the equation $z(n)=an/b$ has no solution in positive integers $n$.*

**Proof.** If $\nu_5(b)>\nu_5(a)$, then $\nu_5(an/b)=\nu_5(a)-\nu_5(b)+\nu_5(n)<\nu_5(n)$ and so $n\nmid F_{an/b}$ yielding that $z(n)\ne an/b$. When $\nu_5(b)<\nu_5(a)$, then

$$\nu_5(F_{an/5b})=\nu_5(an/5b)=\nu_5(a)-1-\nu_5(b)+\nu_5(n)\ge\nu_5(n).$$

Therefore $n\mid F_{an/5b}$ and thus $z(n)\mid an/5b<an/b$. □

However, there are several related equations with infinitely many solutions. For instance, we can take the same approach as in the proof of Theorem 1.1 to prove the following result:

**Proposition 4.4.** *We have that*

(a) *$z(n)=n/2$ if and only if $n=24\cdot 5^k$, for $k\ge 0$;*
(b) *$z(n)=2n$ if and only if $n=6\cdot 5^k$, for $k\ge 0$;*
(c) *$z(n)=n/3$ if and only if $n=4\cdot 3^k\cdot 5^\ell$, for $k\ge 2$ and $\ell\ge 0$;*
(d) *$z(n)=2n/3$ if and only if $n=2\cdot 3^k$ or $n=10\cdot 3^k$, for $k\ge 2$;*
(e) *$z(n)=3n/4$ if and only if $n=2^k\cdot 5^\ell$, for $k\ge 3$ and $\ell\ge 0$;*
(f) *$z(n)=6n/7$ if and only if $n=4\cdot 5^k\cdot 7^\ell$, for $k\ge 0$ and $\ell\ge 1$.*

We finish by pointing that the general equation $z(n)=m^t$, for $t>1$, was not solved completely (here we treated only the case $n=m$). However, Theorem 1.1 allows us to conclude that for any fixed $t>1$, this equation has infinitely many solutions, with $n\ne m$, namely the pairs $(n,m)=(5^{kt},5^k)$, for all $k\ge 1$.

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