THE FIBONACCI VERSION OF A VARIANT OF THE BROCARD-RAMANUJAN DIOPHANTINE EQUATION

DIEGO MARQUES

Abstract. Let F_n be the nth Fibonacci number. In this note, we prove that the Fibonacci version of a variant of the Brocard-Ramanujan Diophantine equation $n! + 1 = m^2$, that is, $F_n \cdots F_1 + 1 = F_m^t$, has at most finitely many solutions in positive integers n, m, t. Moreover, we prove that there is no solution when $1 \leq t \leq 10$.

1. Introduction

In 1876, Brocard [3] and independently Ramanujan [15],[16, p. 327], in 1913, posed the problem of finding all integral solutions of the Diophantine equation

$$n! + 1 = m^2 \tag{1.1}$$

which is then known as *Brocard-Ramanujan Diophantine equation*.

The only known solutions to (1.1) are $(n, m) \in \{(4, 5), (5, 11), (7, 71)\}$. In 1906, Gérardin [7] claimed that, if $m > 71$, then m must have at least 20 digits. Gupta [8] stated that calculations of $n!$ up to $n = 63$ gave no further solutions. Also, there are no solutions up to $n = 10^9$, see [1]. We also point out the existence of several variants for this equation, for instance, see [6] and the recent paper [9] for the equation

$$n! + A = m^2. \tag{1.2}$$

When $A = s^2$, $2 \leq s \leq 50$, Berndt and Galway [1] searched for solutions of (1.2) up to $n = 10^5$ and found either zero or one solution in each case. The largest n giving a solution was $11! + 18^2 = 6318^2$.

Another possible variant for equation (1.1) is the equation

$$n! + 1 = m^t, \text{ with } t \geq 2 \tag{1.3}$$

It is almost unnecessary to stress that the solubility of such equations are still open problems.

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$. The first few terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

Several authors become interested in Diophantine equations involving Fibonacci numbers. For instance, we mention here that the problem of showing that the only perfect powers in the Fibonacci sequence are $0, 1, 8$ and 144 was solved by Bugeaud et al [4] and its generalization can be found in [12], as well as the more recent [10, 13, 14].

Date: January 18, 2012.
2000 *Mathematics Subject Classification.* Primary 11Dxx, Secondary 11B39.
Key words and phrases. Diophantine equation, Fibonacci, Brocard-Ramanujan.

Also, a number of authors have considered, in varying degrees of generality, Fibonacci numbers appearing in additive Diophantine problems. For instance, the equation $F_n + 1 = y^2$ and more generally $F_n \pm 1 = y^\ell$ with integer y and $\ell \geq 2$ have been solved in [17] and [5], respectively. In a very recent paper, we prove that $F_1 \cdots F_n + 1 = F_m^2$ has no solution in positive integers m, n, see [13, Theorem 1.1].

In this note, we consider the solubility of the Fibonacci version of equation (1.3) when we replace m, n with their respective Fibonacci numbers and we use the usual notation $n_F! = F_n \cdots F_1$. Our main result is the following

Theorem 1. *Let $t \geq 1$ be an integer. If n, m are solutions of the Diophantine equation*

$$F_1 \cdots F_n + 1 = F_m^t, \tag{1.4}$$

then $n < 4t + 7$ and consequently $m < 8t^2 + 27t + 25$.

As application, we generalize the case $n = 1$ of Theorem 1.1 in [13] by solving the equation (1.4) for $1 \leq t \leq 10$.

Theorem 2. *The Diophantine equation* (1.4) *has no solution in positive integers n, m, t, with $1 \leq t \leq 10$.*

We point out that Luca and Shorey [11] proved, in particular, that if t is any fixed rational number which is not a perfect power of a different rational number, then the equation

$$F_1 \cdots F_n + t = y^m$$

has only finitely many integer solutions $n,\ y,\ m \geq 2$. However this does not apply to (1.4) since $t = 1$ is a perfect power.

2. Proof of the Theorems

2.1. **Auxiliary results.** Before proceeding further, some considerations will be needed for the convenience of the reader.

We recall that the problem of the existence of infinitely many prime numbers in the Fibonacci sequence remains open, however several results on the prime factors of a Fibonacci number are known. For instance, a *primitive divisor* p of F_n is a prime factor of F_n which does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor p of F_n exists whenever $n \geq 13$. The above statement is usually referred to the *Primitive Divisor Theorem* (see [2] for the most general version).

We cannot go very far in the lore of Fibonacci numbers without encountering the sequence of *Lucas numbers* $(L_n)_{n\geq 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

By the Binet's formulae, we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \text{ for all } n \geq 1,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2 = (-\alpha)^{-1}$. The first identity allows us to show that

$$\alpha^{n-2} < F_n < \alpha^{n-1}, \tag{2.1}$$

holds for all $n \geq 1$.

We may note that the Fibonacci and Lucas sequences can be extrapolated backwards using $F_n = F_{n+2} - F_{n+1}$ and $L_n = L_{n+2} - L_{n+1}$. Thus, for example,

$F_{-1} = 1, F_{-2} = -1$, and so on. Since that the Binet's formulae remain valid for Fibonacci and Lucas numbers with negative indices, one can deduce the following result (which we shall prove for the sake of completeness)

Lemma 1. *For any integers a, b, we have*

$$F_a L_b = F_{a+b} + (-1)^b F_{a-b}.$$

Proof. The identity $\alpha = (-\beta)^{-1}$ leads to

$$F_a L_b = \frac{\alpha^a - \beta^a}{\alpha - \beta}(\alpha^b + \beta^b) = F_{a+b} + \frac{\alpha^a \beta^b - \beta^a \alpha^b}{\alpha - \beta} = F_{a+b} + (-1)^b F_{a-b}.$$

□

Lemma 1 gives immediately the following factorizations for $F_n \pm 1$, depending on the class of n modulo 4:

$$\begin{array}{lll} F_{4\ell} + 1 = F_{2\ell-1} L_{2\ell+1} & ; & F_{4\ell} - 1 = F_{2\ell+1} L_{2\ell-1} \\ F_{4\ell+1} + 1 = F_{2\ell+1} L_{2\ell} & ; & F_{4\ell+1} - 1 = F_{2\ell} L_{2\ell+1} \\ F_{4\ell+2} + 1 = F_{2\ell+2} L_{2\ell} & ; & F_{4\ell+2} - 1 = F_{2\ell} L_{2\ell+2} \\ F_{4\ell+3} + 1 = F_{2\ell+1} L_{2\ell+2} & ; & F_{4\ell+3} - 1 = F_{2\ell+2} L_{2\ell+1} \end{array} \tag{2.2}$$

Now, we are ready to deal with the proof of theorems.

2.2. **The proof of Theorem 1.** The equation (1.4) can be rewritten as $F_1 \cdots F_n = (F_m - 1) \sum_{j=0}^{t-1} F_j$. By the relations in (2.2), we have that $F_m - 1 = F_a L_b$, where $2a \in \{m-2, m-1, m+1\}$. Therefore, our equation becomes

$$F_1 \cdots F_n = F_a L_b (F_m^{t-1} + \cdots + 1) \tag{2.3}$$

Now, the estimates in (2.1) and the identity (1.4) yield

$$\alpha^{t(m-1)} > F_m^t > F_1 \cdots F_n > \alpha^{n(n-3)/2},$$

and so $m > \frac{n(n-3)}{2t} + 1$. We claim that $n < 4t + 7$. Towards a contradiction, suppose that $n \geq 4t + 7$. It is a simple matter to prove that the inequality

$$4t + 3 \leq \frac{3 + 4t + \sqrt{16t^2 + 32t + 9}}{2} < 4t + 4$$

holds for all $t \geq 1$, yielding that $\lfloor (3 + 4t + \sqrt{16t^2 + 32t + 9})/2 \rfloor + 4 = 4t + 7$. It follows that $4t + 7$ is greater than the largest root of the polynomial $x^2 - (3 + 4t)x - 2t$, then $n^2 - (3 + 4t)n - 2t > 0$ or equivalently $n(n-3)/2t > 2n + 1$. In view, of the lower bound on m, we get $m > 2n + 2$ and thus $a \geq (m-2)/2 > n$. If $m > 26$, then $a \geq (m-2)/2 > (26-2)/2 = 12$, the Primitive Divisor Theorem implies in the existence of a primitive divisor p of F_a, but this contradicts the identity (2.3), because $a > n$. In the case of $m \leq 26$, we get $n(n-3) < 50t$ which implies that $n < 2t + 9 < 4t + 7$. This completes the proof.

□

2.3. **The proof of Theorem 2.** The possible solutions of equation (1.4), with $1 \le t \le 10$, occurs when $n < 4 \cdot 10 + 7 = 47$ (since $4t + 7$ is an increasing function in t). We then use Mathematica to print all the values of $F_1 \cdots F_n + 1$ in the range $1 \le n \le 46$. By looking at the sequence of perfect powers (sequence A001597 in OEIS [18]), we convince ourselves that there is no any such power in that list. In order to facilitate this task, we point out that $F_1 \cdots F_n + 1$ ends in 0...01 with at least $\lfloor n/5 \rfloor - 1$ zeros.

$\square$

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, BRASÍLIA, DF, BRAZIL

E-mail address: diego@mat.unb.br