Generalized Fibonacci numbers of the form $2^a + 3^b + 5^c$

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Abstract

For $k \geq 2$, the k-generalized Fibonacci sequence $(F_n^{(k)})_n$ is defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and such that each term afterwards is the sum of the k preceding terms. In this paper, we find all generalized Fibonacci numbers written in the form $2^a + 3^b + 5^c$. This work generalizes a recent Marques-Togbé result [19] concerning the case $k = 2$.

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [11] together with its very extensive annotated bibliography for additional references and history).

The problem of searching for Fibonacci numbers of a particular form has a very rich history, see for example [5] and references therein. In this same paper, Bugeaud et al [5, Theorem 1] showed that $0, 1, 8, 144$ are the only perfect powers in the Fibonacci sequence. Other related papers searched for Fibonacci numbers of the forms px^2+1, px^3+1 [22], k^2+k+2 [13], $p^a \pm p^b + 1$ [14], $p^a \pm p^b$ [15], $y^t \pm 1$ [6], $q^k y^t$ [7] and $2^a+3^b+5^c$ [19]. In particular, Marques and Togbé proved that the only solutions of the Diophantine equation

$$F_n = 2^a + 3^b + 5^c \tag{1}$$

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in integers n, a, b, c, with $0 \le a, b \le c$ are

$$(n, a, b, c) \in \{(4, 0, 0, 0),\ (6, 1, 0, 1)\}.$$

Let $k \ge 2$ and denote $F^{(k)} := (F_n^{(k)})_{n \ge -(k-2)}$, the *k-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)}, \tag{2}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k-step Fibonacci sequence*, *Fibonacci k-sequence*, or *k-bonacci sequence*. Clearly for $k = 2$, we obtain the classical Fibonacci numbers $(F_n)_n$, for $k = 3$, the *Tribonacci* numbers $(T_n)_n$, for $k = 4$, the *Tetranacci* numbers $(Q_n)_n$, etc.

Recently, these sequences have been the main subject of many works. We refer to [3] for results on the largest prime factor of $F_n^{(k)}$ and we refer to [1] for the solution of the problem of finding powers of two belonging to these sequences. In 2013, two conjectures concerning these numbers were proved. The first one, proved by Bravo and Luca [4] is related to *repdigits* (i.e., numbers with only one distinct digit in its decimal expansion) among k-Fibonacci numbers (proposed by Marques [18]) and the second one, a conjecture (proposed by Noe and Post [21]) about coincidences between terms of these sequences, proved independently by Bravo-Luca [2] and Marques [16] (see [17] for results on the spacing between terms of these sequences).

The aim of this paper is to find all generalized Fibonacci numbers which can be written in the form $2^a + 3^b + 5^c$, with $0 \le a, b \le c$. More precisely, our main result is the following

Theorem 1. *The solutions of the Diophantine equation*

$$F_n^{(k)} = 2^a + 3^b + 5^c \tag{3}$$

in integers n, k, a, b, c, *with* $0 \le a, b \le c$ *and* $n > k + 1$ *are*

$$\begin{aligned}(n, k, a, b, c) \in \{&(4, 2, 0, 0, 0), (6, 2, 1, 0, 1), (5, 3, 0, 0, 1), (15, 3, 1, 2, 5),\\ &(15, 3, 3, 1, 5)\}.\end{aligned}$$

We remark that the condition $n > k + 1$ is to avoid the infinitely many (uninteresting) solutions related to powers of two written in the form $2^a + 3^b + 5^c$ (since the first nonzero terms of $F^{(k)}$ are $1, 1, 2, \ldots, 2^{k-1}$) as for instance

$$F_7^{(k)} = 2^2 + 3^1 + 5^2,$$

for all $k \geq 6$.

Let us give a brief overview of our strategy for proving Theorem 1. First, we use a Dresden formula [8, Formula (2)] to get an upper bound for a linear form in three logarithms related to equation (3). After, we use a lower bound due to Matveev to obtain an upper bound for n and c (and so for a and b) in terms of k. Very recently, Bravo and Luca solved the equation $F_n^{(k)} = 2^m$ and for that they used a nice argument combining some estimates together with the Mean Value Theorem (this can be seen in pages 77 and 78 of [1]). In our case, we use this Bravo and Luca approach to get an upper bound for a linear form in two logarithms (before using Bravo-Luca method, we solved some small cases, by using a reduction argument due to Dujella and Pethő). After, we use a result due to Laurent to get an absolute upper bound for the variables k, n and c. In the final section, we use some facts on continued fractions to deal with the case $c \leq k$. For the case $c \geq k$, we use again Dujella and Pethő lemma to make the final calculations feasible. The computations in the paper were performed using *Mathematica*®.

We remark some differences between our work and the one by Bravo and Luca in [1]. In their paper, the equation $F_n^{(k)} = 2^m$ was studied. By applying a key method, they get directly an upper bound for $|2^m - 2^{n-2}|$. In our case, the equation $F_n^{(k)} = 2^a + 3^b + 5^c$ needs a little more work, because we get an upper bound for $|2^{n-2} - 5^c|$. Then we use a lower bound for linear forms in two logarithms due to Laurent to get absolute upper bounds for the variables. Moreover, we also use two facts on convergents of continued fractions to solve quickly the case $c \leq k$. Our presentation is therefore organized in a similar way that the one in the papers [1, 2, 3, 4], since we think that those presentations are intuitively clear.

2. Auxiliary results

Before proceeding further, we shall recall some facts and tools which will be used after.

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1-2^{-k})$ and 2 (as can be seen in [24]). Also, in a recent paper, G. Dresden [8, Theorem 1] gave a simplified "Binet-like" formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{4}$$

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. Also, it was proved in [4, Lemma 1] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \text{ for all } n \geq 1, \tag{5}$$

where α is the dominant root of $\psi_k(x)$. Also, the contribution of the roots inside the unit circle in formula (4) is almost trivial. More precisely, it was proved in [8] that

$$|F_n^{(k)} - g(\alpha, k)\alpha^{n-1}| < \frac{1}{2}, \tag{6}$$

where we adopt throughout the notation $g(x, y) := (x-1)/(2+(y+1)(x-2))$.

Another tool to prove our theorem is a lower bound for a linear form logarithms *à la Baker* and such a bound was given by the following result of Matveev (see [20] or Theorem 9.4 in [5]).

Lemma 1. *Let* $\gamma_1, \ldots, \gamma_t$ *be real algebraic numbers and let* $b_1, \ldots, b_t$ *be non-zero rational integer numbers. Let* D *be the degree of the number field* $\mathbb{Q}(\gamma_1, \ldots, \gamma_t)$ *over* $\mathbb{Q}$ *and let* A_j *be a real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}, \text{ for } j = 1, \ldots, t.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, *then*

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

As usual, in the previous statement, the *logarithmic height* of an s-degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{s}(\log|a| + \sum_{j=1}^{s} \log\max\{1, |\gamma^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1\leq j\leq s}$ are the conjugates of γ (over $\mathbb{Q}$).

After finding an upper bound on n which is general too large, the next step is to reduce it. For that, our last ingredient is a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethő [9, Lemma 5 (a)]. For a real number x, we use $\| x \| = \min\{|x-n| : n \in \mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and let A, B be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon = \| \mu q \| - M \| \gamma q \|$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < A \cdot B^{-k}$$

in positive integers m, n and k with

$$m \leq M \text{ and } k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

See Lemma 5, a.) in [9].

Now, we are ready to deal with the proof Theorem 1.

3. The proof of Theorem 1

In light of Theorem 1.1 of [19], throughout this paper we shall assume $k \geq 3$.

3.1. Upper bounds for n and c in terms of k

In this section, we shall prove the following result

Lemma 3. *If (n, k, a, b, c) is an integer solution of Diophantine equation (3), with $0 \leq a, b \leq c$ and $n > k+1$, then*

$$c < 9 \cdot 10^{14} k^4 \log^3 k \quad \text{and} \quad n < 26.2 \cdot 10^{14} k^4 \log^3 k. \tag{7}$$

Proof. First, note that if $c = 0$ or 1, then $F_n^{(k)} \in \{3, 7, 8, 9, 11\}$ and so the only solutions satisfying the previous conditions are

$$(n, k, a, b, c) \in \{(4, 2, 0, 0, 0), (6, 2, 1, 0, 1), (5, 3, 0, 0, 1)\}.$$

Thus, we may suppose that $c \geq 2$.

Now, we use Eq. (3) together with (4) and (6) to obtain

$$g(\alpha, k)\alpha^{n-1} - 5^c = 2^a + 3^b - E_k(n) > 0, \tag{8}$$

where $E_k(n) := \sum_{i=2}^{k} g(\alpha_i, k)\alpha_i^{n-1}$. Thus

$$\left| \frac{g(\alpha, k)\alpha^{n-1}}{5^c} - 1 \right| < \frac{1}{(1.35)^c}, \tag{9}$$

where we used that $|E_k(n)| < 1/2$ together with $2^c + 3^c + 1/2 < (3.7)^c$, for $c \geq 2$.

In order to use Lemma 1, we take $t := 3$,

$$\gamma_1 := g(\alpha, k),\ \ \gamma_2 := 5,\ \ \gamma_3 := \alpha$$

and

$$b_1 := 1,\ \ b_2 := -c,\ \ b_3 := n - 1.$$

For this choice, we have $D = [\mathbb{Q}(\alpha) : \mathbb{Q}] = k$. In [1, p. 73], an estimate for $h(g(\alpha, k))$ was given. More precisely, it was proved that

$$h(\gamma_1) = h(g(\alpha, k)) < \log(4k + 4).$$

Note that $h(\gamma_2) = \log 5$ and $h(\gamma_3) < 0.7/k$. It is a simple matter to deduce from inequality $2(1 - 2^{-k}) < \alpha < 2$ that $1/4 < g(\alpha, k) < 1$. Thus, we can take $A_1 := k \log(4k + 4), A_2 := k \log 5$ and $A_3 := 0.7$.

Note that $\max\{|b_1|, |b_2|, |b_3|\} = \max\{c, n - 1\}$. By using the inequalities in (5), we get $\alpha^{n-1} \geq F_n^{(k)} > 5^c$ and $\alpha^{n-2} \leq F_n^{(k)} < 2 \cdot 5^c$. Therefore

$$n - 1 > 2.3c \text{ and } n - 1 < 2.3 + 2.9c, \tag{10}$$

where we used that $7/4 < \alpha < 2$ (since $k \geq 3$). Thus, we can choose $B := 2.9c + 2.3$. Since $g(\alpha, k)\alpha^{n-1}5^{-c} > 1$ (by (8)), we are in position to apply Lemma 1. This lemma together with a straightforward calculation gives

$$\left| \frac{g(\alpha, k)\alpha^{n-1}}{5^c} - 1 \right| > \exp(-3.8 \cdot 10^{12} k^4 \log c \log^2 k), \tag{11}$$

where we used that $1+\log k < 2\log k$, for $k \geq 3$, $1+\log(2.9c+2.3) < 4.5\log c$, for $c \geq 2$, and $\log(4k+4) < 2.6\log k$, for $k \geq 3$.

By combining (9) and (11), we obtain

$$\frac{c}{\log c} < 1.3 \cdot 10^{13} k^4 \log^2 k.$$

Since the function $x/\log x$ is increasing for $x > e$, it is a simple matter to prove that

$$\frac{x}{\log x} < A \text{ implies that } x < 2A\log A. \tag{12}$$

A proof for that can be found in [1, p. 74].

Thus, by using (12) for $x := c$ and $A := 1.4 \cdot 10^{13} k^4 \log^2 k$, we have that

$$c < 2(1.4 \cdot 10^{13} k^4 \log^2 k) \log(1.4 \cdot 10^{13} k^4 \log^2 k).$$

Now, the inequality $\log(1.4) + 13\log 10 + 2\log\log k < 28\log k$, for $k \geq 3$, yields

$$c < 9 \cdot 10^{14} k^4 \log^3 k, \tag{13}$$

and we use estimates in (10) to get

$$n < 26.2 \cdot 10^{14} k^4 \log^3 k.$$

This finishes the proof of lemma. □

3.2. The small cases: $3 \leq k \leq 176$

In this section, we shall prove the following result

Lemma 4. *If* (n, k, a, b, c) *is an nonnegative integer solution of Diophantine equation (3), with* $3 \leq k \leq 176$*,* $\max\{a, b, 2\} \leq c$ *and* $n > k+1$*, then*

$$(n, k, a, b, c) \in \{(15, 3, 1, 2, 5), (15, 3, 3, 1, 5)\}.$$

Proof. By using (8) and (9), we have that

$$0 < (n-1)\log\alpha - c\log 5 + \log g(\alpha, k) < (1.35)^{-c}.$$

Dividing by $\log 5$, we obtain

$$0 < (n-1)\gamma_k - c + \mu_k < 0.63 \cdot (1.35)^{-c}, \tag{14}$$

where $\gamma_k = \log \alpha^{(k)} / \log 5$ and $\mu_k = \log g(\alpha^{(k)}, k)/\log 5$. Here, we added the superscript to α for emphasizing its dependence on k.

We claim that γ_k is irrational, for any integer $k \geq 2$. In fact, if $\gamma_k = p/q$, for some positive integers p and q, we have that $5^p = (\alpha^{(k)})^q$ and we can conjugate this relation by some automorphism of the Galois group of the splitting field of $\psi_k(x)$ over $\mathbb{Q}$ to get $5^p = |(\alpha_i^{(k)})^q| < 1$, for $i > 1$, which is an absurdity, since $p \geq 1$. Let $q_{m,k}$ be the denominator of the m-th convergent of the continued fraction of γ_k. Taking $M_k := 26.2 \cdot 10^{14} k^4 \log^3 k \leq M_{176} < 3.5 \cdot 10^{26}$, we use *Mathematica* [23] to get

$$\min_{3 \leq k \leq 176} q_{90,k} > 6.8 \cdot 10^{33} > 6M_{176}.$$

Also

$$\max_{3 \leq k \leq 176} q_{90,k} < 1.3 \cdot 10^{53}.$$

Define $\epsilon_k := \| \mu_k q_{90,k} \| - M_k \| \gamma_k q_{90,k} \|$, for $3 \leq k \leq 176$, we get (again using Mathematica)

$$\min_{3 \leq k \leq 176} \epsilon_k > 7.7 \cdot 10^{-12}.$$

Note that the conditions to apply Lemma 2 are fulfilled for $A = 0.63$ and $B = 1.35$, and hence there is no solution to inequality (14) (and then no solution to the Diophantine equation (3)) for n and c satisfying

$$n < M_k \text{ and } c \geq \frac{\log(Aq_{90,k}/\epsilon_k)}{\log B}.$$

Since $n < M_k$ (Lemma 3), then

$$c < \frac{\log(Aq_{90,k}/\epsilon_k)}{\log B} \leq \frac{\log(0.63 \cdot 1.3 \cdot 10^{53}/7.7 \cdot 10^{-12})}{\log(1.35)} = 491.25\ldots.$$

Therefore $3 \leq k \leq 176$ and $2 \leq c \leq 491$. Now, by applying the estimate in (10), we have $n \leq 1427$.

Now, we use the Mathematica routine

```
Catch[Do[{n, k, a, b, c};
If[F[n, k] == 2^a + 3^b + 5^c, Print[{n, k, a, b, c}]], {k, 3,
  176}, {n, k + 2, 1427}, {c, 0, 491}, {a, 0, c}, {b, 0, c}]]
```

where the defined command

```
F[n_, k_] :=
SeriesCoefficient[Series[x/(1-Sum[x^j, {j,1,k}]), {x,0,1400}],
  n]
```

finds the n-th k-bonacci number.

The program return us $\{15,3,3,1,5\}, \{15,3,1,2,5\}$ in roughly 23 hours on 2.5 GHz Intel Core i5 4GB Mac OSX. This finishes the proof. □

3.3. An absolute upper bound

In this section, we shall prove the following result

Lemma 5. *If* (n, k, a, b, c) *is a solution in integers of Diophantine equation (3), with* $0 \leq a, b \leq c$ *and* $n > k + 1$. *Then*

$$k \leq 777194, \ c < 8.2 \cdot 10^{41} \text{ and } n < 2.4 \cdot 10^{42}. \tag{15}$$

Proof. By Lemma 3, we may consider $k \geq 177$. In this case, we have

$$n < 26.2 \cdot 10^{14} k^4 \log^3 k < 2^{k/2}. \tag{16}$$

Now, we use a key argument due to Bravo and Luca [1, p. 77-78]. However, we shall present it for the sake of completeness.

Setting $\lambda = 2 - \alpha$, we deduce that $0 < \lambda < 1/2^{k-1}$ (because $2(1 - 2^{-k}) < \alpha < 2$). So

$$\alpha^{n-1} = (2-\lambda)^{n-1} = 2^{n-1}\left(1 - \frac{\lambda}{2}\right)^{n-1} > 2^{n-1}(1 - (n-1)\lambda),$$

since that the inequality $(1-x)^n > 1 - 2nx$ holds for all $n \geq 1$ and $0 < x < 1$. Moreover, $(n-1)\lambda < 2^{k/2}/2^{k-1} = 2/2^{k/2}$ and hence

$$2^{n-1} - \frac{2^n}{2^{k/2}} < \alpha^{n-1} < 2^{n-1} + \frac{2^n}{2^{k/2}},$$

yielding

$$|\alpha^{n-1} - 2^{n-1}| < \frac{2^n}{2^{k/2}}. \tag{17}$$

Now, we define for $x > 2(1 - 2^{-k})$ the function $f(x) := g(x, k)$ which is differentiable in the interval $[\alpha, 2]$. So, by the Mean Value Theorem, there exists $\xi \in (\alpha, 2)$, such that $f(\alpha) - f(2) = f'(\xi)(\alpha - 2)$. Thus

$$|f(\alpha) - f(2)| < \frac{2k}{2^k}, \tag{18}$$

where we used the bounds $|\alpha - 2| < 1/2^{k-1}$ and $|f'(\xi)| < k$ (see p. 77 of [1]). For simplicity, we denote $\delta = \alpha^{n-1} - 2^{n-1}$ and $\eta = f(\alpha) - f(2) = f(\alpha) - 1/2$. After some calculations, we arrive at

$$2^{n-2} = f(\alpha)\alpha^{n-1} - 2^{n-1}\eta - \frac{\delta}{2} - \delta\eta.$$

Therefore

$$\begin{aligned} |2^{n-2} - 5^c| &\leq 2^a + 3^b + 1/2 + 2^{n-1}|\eta| + \left|\frac{\delta}{2}\right| + |\delta\eta| \\ &\leq 2^a + 3^b + 1/2 + \frac{2^n k}{2^k} + \frac{2^{n-1}}{2^{k/2}} + \frac{2^{n+1}k}{2^{3k/2}}, \end{aligned}$$

where we used (17) and (18). Since $n > k+1$, one has that $2^{n-2}/2^{k/2} \geq 2^{k/2} > 7/4$ (for $k \geq 3$) and we rewrite the above inequality as

$$|2^{n-2} - 5^c| < 2^a + 3^b + 1/2 + \left(\frac{4k}{2^{k/2}}\right)\frac{2^{n-2}}{2^{k/2}} + 2 \cdot \frac{2^{n-2}}{2^{k/2}} + \left(\frac{8k}{2^k}\right)\frac{2^{n-2}}{2^{k/2}}.$$

Since the inequalities $4k < 8k < 2^{k/2} < 2^k$ hold for all $k > 13$, then

$$|2^{n-2} - 5^c| < 2^a + 3^b + \frac{4.5 \cdot 2^{n-2}}{2^{k/2}}. \tag{19}$$

After some manipulations, we arrive at

$$|1 - 5^c \cdot 2^{-(n-2)}| < \frac{7}{(1.3)^\ell}, \tag{20}$$

where we used that $n > 2.3c + 1$ and $\ell := \min\{k, c\}$. One can rewrite the above inequality as

$$|e^{\Lambda^*} - 1| < \frac{7}{(1.3)^\ell},$$

where $\Lambda^* = c\log 5 - (n-2)\log 2$. Clearly, $\Lambda^* \neq 0$. If $\Lambda^* > 0$, then $\Lambda^* < e^{\Lambda^*} - 1 < 7/(1.3)^\ell$. In the case of $\Lambda^* < 0$, it holds that $|1 - e^{\Lambda^*}| = 1 - e^{-|\Lambda^*|}$ yielding $e^{|\Lambda^*|} < 1/(1 - 7/(1.3)^\ell)$. Therefore $|\Lambda^*| < e^{|\Lambda^*|} - 1 < 50/(1.3)^\ell$, where we used that $1 - 7/(1.3)^\ell > 0.14$, for all $c \geq 8$ (in the case of $2 \leq c \leq 7$, the equation is easily solved as in the beginning of subsection 3.1). Thus, in any case, we have $|\Lambda^*| < 50/(1.3)^\ell$ and by applying the log function, we obtain

$$\log|\Lambda^*| < \log 50 - \ell\log(1.3). \tag{21}$$

Now, we will determine a lower bound for $\log|\Lambda^*|$. We remark that the bounds available for linear forms in two logarithms are substantially better than those available for linear forms in three logarithms. Here we choose to use a result due to Laurent [12, Corollary 2] with $m = 24$ and $C_2 = 18.8$ (we can use that because 5 and 2 are multiplicatively independent). First let us introduce some notations. Let α_1, α_2 be real algebraic numbers, with $|\alpha_j| \geq 1$, b_1, b_2 be positive integer numbers and

$$\Gamma = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let A_j be real numbers such that

$$\log A_j \geq \max\{h(\alpha_j), |\log \alpha_j|/D, 1/D\},\ j \in \{1, 2\},$$

where D is the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$ over $\mathbb{Q}$. Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

Laurent's result asserts that if α_1, α_2 are multiplicatively independent, then

$$\log|\Gamma| \geq -18.8 \cdot D^4 \left(\max\{\log b' + 0.38, m/D, 1\}\right)^2 \cdot \log A_1 \log A_2.$$

We then take

$$b_1 = n - 2,\ b_2 = c,\ \alpha_1 = 2,\ \alpha_2 = 5.$$

So, $D = 1$ and we can choose $\log A_1 = 1$ and $\log A_2 = \log 5$. We then get

$$b' = \frac{n-2}{\log 5} + c < 2.9c + 0.9,$$

where we used that $n - 2 < 2.9c + 1.3$.

Thus, by Corollary 2 of [12] we get

$$\log|\Lambda^*| \geq -21 \cdot \left(\max\{\log(2.9c + 0.9) + 0.38, 24\}\right)^2. \tag{22}$$

Now, we combine the estimates (21) and (22) to obtain

$$\ell < 81 \cdot \left(\max\{\log(2.9c + 0.9) + 0.38, 24\}\right)^2 \tag{23}$$

which yields

$$k \leq 777194,\ c < 8.2 \cdot 10^{41} \text{ and } n < 2.4 \cdot 10^{42}$$

as desired. Here we used (13) when $\ell = k$. □

3.4. Finishing the proof

We shall split the proof in two cases:

3.4.1. If $\ell = c$

For this case, we already get by Lemma 4 that $c \leq 777194$ and so $n \leq 2253865$ a serious improvement. Now, we shall improve these estimates to solve completely this case.

Case 1. If $\Lambda^* > 0$, then we can use the estimate in (21) to obtain

$$0 < \frac{\log 5}{\log 2} - \frac{n-2}{c} < \frac{50}{c(1.3)^c \log 2}.$$

If $c \leq 39$, then $n \leq 116$ and so $k < n - 1 \leq 115$ and these cases were treated in Lemma 4. Thus, assume that $c \geq 40$. Then $(1.3)^c > 902c$ and we get

$$\left| \frac{\log 5}{\log 2} - \frac{n-2}{c} \right| < \frac{50}{902c^2 \log 2}. \tag{24}$$

By a criterion of Legendre, the previous inequality implies that $(n-2)/c$ is a convergent of the continued fraction of $\log 5/\log 2 = [a_1; a_2, a_3, \ldots] = [2; 3, 9, 2, 2, 4, 6, 2, 1, 1, \ldots] = \lim_{s\to\infty} p_s/q_s$. Thus, $(n-2)/c = p_t/q_t$ for some $t > 0$. Since $\gcd(p_t, q_t) = 1$, then $q_t \mid c$ yielding

$$c \geq q_t \geq 1838395 > 777194 \geq c,$$

if $t \geq 13$. Therefore $t \leq 12$. On the other hand, a well-known fact on continued fractions gives

$$\left| \frac{\log 5}{\log 2} - \frac{n-2}{c} \right| > \frac{1}{(a_{t+1}+2)q_t^2}.$$

But $\max_{1\leq s\leq 12}\{a_{s+1}\} = a_3 = 9$ and so

$$\left| \frac{\log 5}{\log 2} - \frac{n-2}{c} \right| > \frac{1}{11c^2}. \tag{25}$$

Combining (24) and (25) we reach the absurdity that $550 > 902 \log 2 = 625.219\ldots$. This completes the proof in this case.

Case 2. If $\Lambda^* < 0$, then we can use that $0 < -\Lambda^* = |\Lambda^*|$ together with estimate (21) to get

$$0 < \frac{\log 2}{\log 5} - \frac{c}{n-2} < \frac{36.4}{(n-2)(1.08)^{n-2}},$$

where we used that $c > (n - 3.3)/2.9 > 0.3n - 1.2$. If $n \leq 152$, then $k < n - 1 \leq 151$ and these cases were treated in Lemma 4. Assume that $n \geq 153$. Then $(1.08)^{n-2} > 737(n - 2)$ and we obtain

$$\left|\frac{\log 2}{\log 5} - \frac{c}{n-2}\right| < \frac{36.4}{737(n-2)^2}. \tag{26}$$

Again, by the criterion of Legendre, we infer that $c/(n-2)$ is a convergent of the continued fraction of $\log 2/\log 5 = [b_1; b_2, b_3, \ldots] = [0; 2, 3, 9, 2, 2, \ldots] = \lim_{s\to\infty} m_s/n_s$. Thus, $c/(n-2) = m_\nu/n_\nu$ for some $\nu > 0$. Since $\gcd(m_\nu, n_\nu) = 1$, then $n_\nu \mid n - 2$. Therefore

$$n - 2 \geq n_\nu \geq 4268621 > 2253863 \geq n - 2,$$

if $\nu \geq 14$. Therefore $\nu \leq 13$. On the other hand, we have

$$\left|\frac{\log 2}{\log 5} - \frac{c}{n-2}\right| > \frac{1}{(b_{\nu+1} + 2)n_\nu^2}.$$

But $\max_{1\leq s\leq 13}\{b_{s+1}\} = b_{14} = 18$ and so

$$\left|\frac{\log 2}{\log 5} - \frac{c}{n-2}\right| > \frac{1}{20(n-2)^2}. \tag{27}$$

Combining (26) and (27) we reach the absurdity that $36.4 \cdot 20 = 728 > 737$. This completes the proof in this case. □

3.4.2. If $\ell = k$

In this case, we use all machinery and definitions given in Subsection 6 with

$$\epsilon_k := \| \mu_k q_{1000,k} \| - M_k \| \gamma_k q_{1000,k} \|, \text{ for } 177 \leq k \leq 777194,$$

to get

$$\min_{177\leq k\leq 777194} q_{1000,k} > 6 \cdot 10^{41} > 6M_{777194},$$

$$\max_{177\leq k\leq 777194} q_{1000,k} < 5.3 \cdot 10^{21311}$$

and

$$\min_{177\leq k\leq 777194} \epsilon_k > 1.5 \cdot 10^{-20315}.$$

Now, we apply Lemma 2 to obtain that $c \leq 319382$ and then $n \leq 926210$. To finish, we use the Mathematica routine given previously adding the condition $k < c$. It was needed roughly 18 days (on 2.5 GHz Intel Core i5 4GB Mac OSX) to the program return us that there is solution in this case. □

Remark 1. *We point out that we split the use of Dujella-Pethő reduction lemma in two parts in order to make our presentation more natural to the reader, since at the first point we only needed* $k \geq 177$ *to obtain the inequality (16).*

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References

[1] J. J. Bravo, F. Luca, Powers of two in generalized Fibonacci sequences, *Rev. Colombiana Mat.* **46** (2012), 67–79.

[2] J. J. Bravo, F. Luca, Coincidences in generalized Fibonacci sequences, *J. Number Theory.* **133** (2013), 2121–2137.

[3] J. J. Bravo, F. Luca, On the largest prime factor of the k-Fibonacci numbers. *Int. J. Number Theory* **9** (2013), 1351-1366.

[4] J. J. Bravo, F. Luca, On a conjecture about repdigits in k-generalized Fibonacci sequences, *Publ. Math. Debrecen* **82** Fasc. 3-4 (2013).

[5] Y. Bugeaud, M. Mignotte, S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas powers, *Annals of Math.* **163** (2006), 969-1018.

[6] Y. Bugeaud, M. Mignotte, F. Luca, S. Siksek, Fibonacci numbers at most one away from a perfect power, *Elem. Math.* **63** (2008), 65–75.

[7] Y. Bugeaud, M. Mignotte, S. Siksek, Sur les nombres de Fibonacci de la forme $q^k y^t$. *C. R. Math. Acad. Sci. Paris* **339** (2004), no. 5, 327–330.

[8] G. P. Dresden, *A simplified Binet formula for k-generalized Fibonacci numbers*, Preprint, arXiv:0905.0304v2 (2011). Accessed 31 December 2011.

[9] A. Dujella, A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.* (2) **49** (1998), 291–306.

[10] F. Heppner, Über Primzahlen der Form $n2^n+1$ bzw. $p2^p+1$, *Monatsh. Math.* **85** (1978), 99-103.

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[12] M. Laurent, Linear forms in two logarithms and interpolation determinants II, *Acta Arith.* **133.4** (2008), 325–348.

[13] F. Luca, Fibonacci numbers of the form k^2+k+2. *Applications of Fibonacci numbers,* Vol. 8 (Rochester, NY, 1998), 241–249, Kluwer Acad. Publ., Dordrecht, 1999.

[14] F. Luca, L. Szalay, Fibonacci numbers of the form $p^a \pm p^b + 1$. *Fibonacci Quart.* **45** (2007), no. 2, 98-103.

[15] F. Luca, P. Stănică, Fibonacci numbers of the form $p^a \pm p^b$. Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications. *Congr. Numer.* **194** (2009), 177-183.

[16] D. Marques, The proof of a conjecture concerning the intersection of k-generalized Fibonacci sequences, *Bull. Brazilian Math. Soc.* **44** (3) (2013), 455-468.

[17] D. Marques, On the spacing between terms of generalized Fibonacci sequences, To appear in *Coll. Math.*

[18] D. Marques, On k-generalized Fibonacci numbers with only one distinct digit, To appear in *Util. Math.*

[19] D. Marques, A. Togbé, Fibonacci numbers of the form $2^a + 3^b + 5^c$, *Proc. Japan Acad. Ser. A Math. Sci*, **89** (2013), 47–50.

[20] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180. English translation in *Izv. Math.* **64** (2000), 1217–1269.

[21] T. D. Noe and J. V. Post, Primes in Fibonacci n-step and Lucas n-step sequences, *J. Integer Seq.*, **8** (2005), Article 05.4.4.

[22] N. Robbins, Fibonacci numbers of the forms $pX^2 \pm 1$, $pX^3 \pm 1$, where p is prime. *Applications of Fibonacci numbers* (San Jose, CA, 1986), 77–88, Kluwer Acad. Publ., Dordrecht, 1988.

[23] Wolfram Research, Inc., Mathematica, Version `7.0`, Champaign, IL (2008).

[24] A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (1998), 129–145.