On the Diophantine equation $3^{2n} - 2 \cdot 3^m + 1 = k^2$

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Abstract

In 1981, F. Beukers used a hyper-geometric method for proving that the well-known generalized Ramanujan-Nagell equation

$$x^2 + C = p^n, \; p \text{ prime},$$

has at most one solution in positive integers x and n, where C and p are previously fixed, with a few exceptions.

In this note, we give an elementary proof of this result when n is even as well as the complete solution of a such equation when C is a power of 2009. Moreover, we prove that the previous result is surprisingly connected with the title equation which allows us to find all solutions for that equation.

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1 Introduction

The Diophantine equation

$$x^2 + C = y^n, \quad x \geq 1, \quad y \geq 1, \quad n \geq 3 \tag{1}$$

has a rich history and it has attracted the attention of several mathematicians. Several papers have been written on this topic, specially for particular values of C. The first non-trivial result is due to Lebesgue [21] and date back to the 1850.

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He proved that the above equation has no solutions for $C = 1$. In 1965, Ko [18] proved that if $C = -1$, then the only solution is $(x, y, n) = (3, 2, 3)$. In 2004, Tengely [31] solved the above equation with $C = B^2$ and $B \in \{3, 4, ..., 501\}$. The case when $C = p^k$, where p is a prime number, was studied for $p = 2$, in [8, 19, 20] for $p = 3$ in [6, 7, 22], for $p = 5$ in [1, 2] and for $p = 7$ in [25]. Some advances on an arbitrary prime p appear in [5]. The equations $x^2 + C = y^n$ with $1 \leq C \leq 100$ were completely solved in [12]. Also, the solutions when x and y are coprime $C = 2^a \cdot 3^b$, $C = 2^a \cdot 5^b$ and $C = 5^a \cdot 13^b$ can be found in [23, 24, 3], respectively. The more recent progress on the subject concerns to cases $C = 5^a \cdot 11^b$, $C = 2^a \cdot 11^b$, $C = 2^a \cdot 3^b \cdot 11^c$ and can be found in [14, 15, 16].

Also, several authors become interested in the equation (1) when the variable y is replaced by a positive integer number. The equation

$$x^2 + C = t^n,$$

where C and t are given integers, is called the *generalized Ramanujan-Nagell equation*. For instance, there is quite an extensive literature concerning the equation

$$x^2 + C = p^n, \ p \text{ prime}, \tag{2}$$

beginning for the case $C = 7$ and $p = 2$, which was posed in a work of Ramanujan [28, 29], in 1913 and first solved by Nagell [27] in 1948. The case $C = 11$ and $p = 3$ was treated by Cohen [13] in 1976. Consult its very extensive annotated bibliography for additional references and history. As a final remark, we point out that, in 1960, Apéry [4] showed that equation (2), when $p \nmid C$, has at most two solutions.

Here, we are particularly interested in solving the Diophantine equation

$$3^{2n} - 2 \cdot 3^m + 1 = k^2 \tag{3}$$

We prove that the possible solutions for the above equation are related to the solubility of the generalized Ramanujan-Nagell equation for $t = 9$. Our first result is the following

Theorem 1. *Let C be a positive integer. Then the Diophantine equation*

$$x^2 + C = 3^{2n} \tag{4}$$

has at most one solution in positive integers x and n.

It is important to pay attention that Eq. (4) has solution only when $C \equiv 0, 2 \pmod 3$.

After, we shall combine two powerful techniques in number theory, namely, the Baker's theory on linear forms in logarithms and some tools from Diophantine approximation, due to Baker and Davenport to find a general method for solving the equation (4) for values of C previously fixed. As application of it, we derive the following

Theorem 2. *The Diophantine equation*

$$x^2 + 2009^t = 3^{2n} \tag{5}$$

has no solution in positive integers x, t and n.

Finally, we prove

Theorem 3. *The only solutions of the Diophantine equation*

$$3^{2n} - 2 \cdot 3^m + 1 = k^2$$

in positive integers m, n and k, are those related to $m = n$, i.e., $(n, m, k) = (n, n, 3^n - 1)$.

We point out that our method is quite general and can be applied by replacing 3 in the title equation by any odd prime number p.

2 The Diophantine equation $x^2 + C = 3^{2n}$

2.1 The proof of Theorem 1

It is important to get noticed that Beukers [10, 11] proved that the equation (2) (and consequently Eq. (4)) has at most one solution except when $(p, C) = (3, 2)$ or $(4t^2 + 1, 3t^2 + 1)$, for a positive t. In all these exceptional cases, the pair $(x, n) = (1, 1)$ is a direct solution and so Theorem 1 is according to Beukers result. He used refined techniques on hyper-geometric methods for proving these results.

Here we will present an elementary demonstration of the Theorem 1 which was discovered by Professor F. A. Germano who has communicated us his nice proof by e-mail.

Proof. Suppose that x, y, m, n are positive integer numbers such that $x^2 + C = 3^{2m}$ and $y^2 + C = 3^{2n}$. We shall show that $m = n$ and consequently $x = y$. First of all, we note that

$$(3^m + x)(3^m - x) = C = (3^n + y)(3^n - y)$$

Without losing any generality, we can suppose $\gcd(C, 3) = \gcd(x, 3) = 1$. In fact, we have $x = 3^u a$, $C = 3^v b$, where $a, b \in \mathbb{N}$, $3 \nmid ab$, u and v are nonnegative integer numbers. Hence

$$x^2 + C = 3^{2u}a^2 + 3^v b = 3^{2m}$$

Of course, $2m \geq \max\{2u, v\}$. Set $\ell = \min\{2u, v\}$, we have $\ell \leq 2u$, $v \leq 2m$ and $3^\ell(3^{2u-\ell}a^2 + 3^{v-\ell}b) = 3^{2m}$. We then conclude that either $2u = \ell = v$ or $3 \nmid (3^{2u-\ell}a^2 + 3^{v-\ell}b)$. In the first case, we have

$$a^2 + b = 3^{2(m-u)}, \tag{6}$$

with $m - u > 0$ and whence it is enough to prove the theorem for the equation (6). In the second case, we infer that $1 = 3^{2u-t}a^2 + 3^{v-t}b > 1$ which is an absurd.

We have then $C = (3^m - x)(3^m + x) = r(2 \cdot 3^m - r)$, where $0 < r = 3^m - x < 3^m$. Thus, if (x, m) is a solution of (4), we get an integer number $0 < r < 3^m$ such that $C = r(2 \cdot 3^m - r)$ and $3 \nmid r$. Therefore, for another solution (y, n) of (4), there exists $0 < s = 3^n - y < 3^n$ such that $h = s(2 \cdot 3^n - s)$ and $3 \nmid s$.

We claim that $m = n$. Towards a contradiction, we may suppose $n > m$ (the other case can be handled in much the same way). This implies that $C = s(2 \cdot 3^n - s) = r(2 \cdot 3^m - r)$ and then $0 < s < r < 3^m$. Therefore, r and s have the same parity, since $s^2 \equiv r^2 \pmod 2$. By considerations modulo 3^m, it is easy to deduce that $s^2 \equiv r^2 \pmod{3^m}$ and so $3^m | (r - s)(r + s)$. Recall that the numbers $r - s$ and $r + s$ can not be both multiples of 3 (otherwise $3|r$ and $3|s$). It follows that $r \equiv \pm s \pmod{3^m}$ which yields

$$r \pm s \in \{..., -3^{m+1}, -2 \cdot 3^m, -3^m, 0, 3^m, 2 \cdot 3^m, 3^{m+1}, ...\} = 3^m \mathbb{Z}.$$

Since $0 < s < r < 3^m$, we get $0 < r \pm s < 2 \cdot 3^m$ and therefore $r \pm s = 3^m$, but this is an absurd because $r \pm s$ is even (keep in mind that r and s have the same parity). Thus $m = n$ as desired. □

2.2 The proof of Theorem 2

2.2.1 Auxiliary results

Before proceeding further, we recall some results which will be very useful in what follows.

The main idea for proving the Theorem 2 is to use bounds *à la Baker* for a suitable linear form in three logarithms and then to deduce an upper bound on t. From the main result of Matveev [26], we extract the following result.

Lemma 1. *Let $\alpha_1, \alpha_2, \alpha_3$ be real algebraic numbers and let b_1, b_2, b_3 be nonzero integer rational numbers. Define*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ and let A_1, A_2, A_3 be real numbers which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j = 1, 2, 3.$$

Assume that

$$B \geq \max\{1, \max\{|b_j| A_j / A_1; 1 \leq j \leq 3\}\}.$$

Define also

$$C_1 = 6750000 \cdot e^4 (20.2 + \log(3^{5.5} D^2 \log(eD))).$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)).$$

As usual, in the previous statement, the *logarithmic height* of an s-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{s}(\log |a| + \sum_{j=1}^{s} \log \max\{1, |\alpha^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1\leq j\leq s}$ are the conjugates of α.

After finding an upper bound on t which is general too large, the next step is to reduce it. For this purpose, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethö [17]. For a real number x, we use $\parallel x \parallel = \min\{|x-n| : n \in \mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and let $\epsilon = \parallel \mu q \parallel - M \parallel \gamma q \parallel$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < A \cdot B^{-m}$$

in positive integers m, n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

See Lemma 5, a.) in [17].

Now, we are ready to deal with the proof of our result.

2.2.2 The proof

Finding a bound on k

First, note that t in the equation (5) must be odd, say $2k+1$, because $x^2 \equiv 0, 1 \pmod 3$ and $2009 \equiv -1 \pmod 3$. So, equation (5) can be rewritten in the form

$$2009^{2k+1} = (3^n - x)(3^n + x) \tag{7}$$

Since $3 \nmid x$ (because $3 \nmid 2009$), we get

$$\{3^n - x, 3^n + x\} = \pm\{1, 2009^{2k+1}\}$$

Hence, we may suppose that $3^n - x = 1$ and $3^n + x = 2009^{2k+1}$. Thus

$$2 \cdot 3^n - 2009^{2k+1} = 1 \tag{8}$$

We point out that the above equation has no solution when $n = 2k+1$. This fact is an immediate consequence of a result due to Bennett [9]: for any positive integer a, the equation

$$(a+1)x^n - ay^n = 1, \text{ in integers } x \geq 1,\ y \geq 1,\ n \geq 3,$$

has no solution other than given by $x = y = 1$.

For the remaining cases ($n \neq 2k+1$), we shall use bounds for linear forms in three logarithms of algebraic numbers (for more details on transcendental methods to Diophantine equations we refer the reader to [30]).

First, on dividing Eq. (8) through by 2009^{2k+1}, we get

$$2 \cdot 3^n \cdot 2009^{-(2k+1)} - 1 = 2009^{-(2k+1)}$$

Let $\Lambda = (2k+1)\log(1/2009) - n\log(1/3) + \log 2$, then the previous equality becomes $e^{\Lambda} - 1 = 2009^{-(2k+1)} > 0$ and so $\Lambda > 0$. Therefore $\Lambda < e^{\Lambda} - 1 = 2009^{-(2k+1)}$ which yields

$$\log \Lambda < -(2k+1)\log 2009 \tag{9}$$

Now, we will apply Lemma 1. Take

$$\alpha_1 = 1/2009,\ \alpha_2 = 1/3,\ \alpha_3 = 2,\ b_1 = 2k+1,\ b_2 = -n,\ b_3 = 1.$$

Observe that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}$ and then $D = 1$. Surely, we can take $A_1 = \log 2009$, $A_2 = \log 3$ and $A_3 = \log 2$.

Note that

$$\max\{1, \max\{|b_j|A_j/A_1; 1 \le j \le 3\}\} = \max\{2k+1, n\log 3/\log 2009\},$$

and then it suffices to choose $B = 2k+1$ as

$$2 \cdot 3^n = 2009^{2k+1} + 1 < 2 \cdot 2009^{2k+1} \text{ and then } n\log 3 < (2k+1)\log 2009.$$

Since, for $D = 1$, it holds that $C_1 < 9.7 \cdot 10^9$, Lemma 1 yields

$$\log|\Lambda| > -56.2 \cdot 10^9 \log(4.08(2k+1)). \tag{10}$$

Combining the estimates (9) and (10), we get

$$56.2 \cdot 10^9 \log(4.08(2k+1)) > (2k+1)\log 2009,$$

and this inequality implies $k < 2 \cdot 10^{11}$ (for the sake of preciseness $k < 101389315227$).

Reducing the bound

Since $0 < \Lambda < 2009^{-2k-1}$, we have that

$$0 < (2k+1)\log\alpha_1 - n\log\alpha_2 + \log\alpha_3 < 2009^{-2k}.$$

On dividing through by $\log\alpha_2$, we get

$$0 < (2k+1)\gamma - n + \mu < 2009^{-2k}, \tag{11}$$

with $\gamma = \log\alpha_1/\log\alpha_2$ and $\mu = \log\alpha_3/\log\alpha_2$.

Surely γ is an irrational number[a] (because 2009 and 3 are multiplicatively independent). So, let us denote p_ℓ/q_ℓ be the ℓth convergent of its continued fraction.

In order to reduce our bound on k (which is too large!), we will use the Lemma 2.

For that, take $M = 2 \cdot 10^{11}$. Since

$$\frac{p_{27}}{q_{27}} = \frac{24782374449400}{3579857528251},$$

[a]Actually, this number is transcendental by Gelfond-Schneider theorem: if α and β are algebraic numbers, with $\alpha \neq 0$ or 1, and β irrational, then α^{β} is transcendental.

then $q_{27} \geq 3579857528251 > 1.2 \cdot 10^{12} = 6M$. Moreover, a straight calculation gives

$$M \parallel q_{27}\gamma \parallel = 0.02760... < 0.02,$$

and

$$\parallel q_{27}\mu \parallel = 0.33016... > 0.34$$

Hence

$$\epsilon = \parallel \mu q_{27} \parallel - M \parallel \gamma q_{27} \parallel > 0.34 - 0.02 = 0.32$$

Thus all the hypotheses of the Lemma 2 are satisfied with $A = 1$ and $B = 2009^2$. It follows from that lemma that there is no solution of the Diophantine equation (7) in the range

$$\left[\left\lfloor \frac{\log(Aq_{27}/\epsilon)}{\log B} \right\rfloor + 1, M\right] = [115, 2 \cdot 10^{11}]$$

For the remaining possibilities (that is $k < 115$), we define a function $\mathcal{T} : \mathbb{N} \to \mathbb{R}$ given by

$$\mathcal{T}(s) := \frac{\log\left(\frac{2009^{2s+1}+1}{2}\right)}{\log 3}$$

Thus in view of the relation in (8), if the equation (7) has solution for a certain k, then $\mathcal{T}(k)$ must be an integer number. To finish, we use Mathematica to print all the values of this function, for $1 \leq k \leq 114$. This task took less than one second on a 1.86 GHz Pentium Core Duo. Finally, we convince ourselves that $\mathcal{T}(k)$ is never an integer number in the obtained range. This completes the proof.

$\square$

3 The proof of Theorem 3

Note that if $m = n$, then $3^{2n} - 2 \cdot 3^n + 1 = (3^n - 1)^2$. If k is positive, then $(n, m, k) = (n, n, 3^n - 1)$ is solution for (3) for all positive integer n. Our goal is to prove that this one is the only possibility.

For that, in order to facilitate our work, we shall denote $\delta_{m,n} = 3^{2n} - 2 \cdot 3^m + 1$ and let m, n, k be positive integer numbers such that $\delta_{m,n} = k^2$. First, take $p = 3^n + k$ and $q = 3^n - k$. So, we have $p > q \geq 1$, $p + q = 2 \cdot 3^n$ and $pq = 2 \cdot 3^m - 1$. Now, if $x = 3^m - 1$ and $y = 3^n - q = k$, we get

$$x = 3^m - 1 = pq - 3^m \text{ and } y = 3^n - q = p - 3^n = k$$

yielding

$$(3^m + x)(3^m - x) = pq = (3^n + y)(3^n - y)$$

Thus (x, n) and (y, m) are solutions of the equation (4) with $C = pq$. Hence we apply the Theorem 1 to get $m = n$ and this completes our proof. $\square$

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