ON A VARIANT OF A QUESTION PROPOSED BY K. MAHLER CONCERNING LIOUVILLE NUMBERS

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Abstract. In this note, we shall prove the existence of an uncountable subset of Liouville numbers (which we call the set of *ultra-Liouville numbers*) for which there exists uncountably many transcendental analytic functions mapping the subset into itself.

1. Introduction

A real number ξ is called a *Liouville number*, if there exists a rational sequence $(p_k/q_k)_{k\geq 1}$, with $q_k > 1$, such that

$$0 < \left|\xi - \frac{p_k}{q_k}\right| < q_k^{-k}, \text{ for } k = 1, 2, \ldots.$$

The set of the Liouville numbers is denoted by $\mathbb{L}$.

The name arises because Liouville [4] in 1844 showed that all Liouville numbers are transcendental, establishing thus the first explicit examples of transcendental numbers. The number $\ell := \sum_{n\geq 1} 10^{-n!}$, the so-called *Liouville constant*, is a standard example of a Liouville number. In 1962, Erdős [3] proved that every real number can be written as the sum and (if it is non zero) the product of two Liouville numbers, as a consequence of the fact that $\mathbb{L}$ is a rather large set in a topological sense: it is a dense G_δ set.

In his pioneering book, Maillet [6, Chapitre III] discusses some arithmetic properties of Liouville numbers. One of them is that, given a rational function f, with rational coefficients, if ξ is a Liouville number, then so is $f(\xi)$. We observe that the converse of this result is not valid in general, e.g., taking $f(x) = x^2$, then $\zeta := \sqrt{(3+\ell)/4}$ is not a Liouville number [1, Theorem 7.4], but $f(\zeta)$ is. Also the rational coefficients cannot be taken algebraic (with at least one of them non-rational). For instance, $\ell\sqrt{3/2}$ is not a Liouville number, see [6, Théorème I_3]. In fact, $\ell\sqrt{3/2}$ is a U_2-number (for the definition of a U_2-number and this result, see [2]).

An *algebraic function* is a function $f(x)$ which satisfies $P(x, f(x)) = 0$, where $P(x, y)$ is a polynomial with complex coefficients. For instance, functions that can be constructed using only a finite number of elementary operations are examples of algebraic functions. A function which is not algebraic is, by definition, a *transcendental function*. Common examples are the trigonometric functions, the exponential function, and their inverses.

In 1984, in one of his last papers, K. Mahler [5] stated several questions for which, according to him, 'perhaps further research might lead to interesting results'. His

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first question is related to Liouville numbers. In particular, this question asks the following:

Question. *Are there transcendental entire functions $f(z)$ such that if ξ is any Liouville number, then so is $f(\xi)$?*

He also said that: 'The difficulty of this problem lies of course in the fact that the set of all Liouville numbers is non-enumerable'.

The study of similar problems has attracted the attention of several mathematicians. Let A and B be subsets of $\mathbb{C}$ with $A \subset B$ and let $\Sigma_A(B)$ be the set of all transcendental analytic functions $f : B \to B$ such that $f(A) \subseteq A$. In 1886, Weierstrass proved that the set $\Sigma_{\mathbb{Q}}(\mathbb{R})$ has the power of continuum. Moreover, he asserted that $\Sigma_{\overline{\mathbb{Q}}}(\mathbb{C}) \neq \emptyset$. In 1896, Stäckel [7] confirmed the Weierstrass assertion by proving that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there is a transcendental entire function f such that $f(\Sigma) \subseteq T$. In particular, if A is a countable dense subset of $\mathbb{C}$, then $\Sigma_A(\mathbb{C})$ is uncountable. Consult the very extensive annotated bibliography of [8] for additional references and history. Note that the Mahler question can be rephrased as: is $\Sigma_{\mathbb{L}}(\mathbb{C}) \neq \emptyset$?

Set, inductively, $\exp^{[n]}(x) = \exp(\exp^{[n-1]}(x))$ and $\exp^{[0]}(x) = x$. Now, let us define the following class of numbers:

Definition. *A real number ξ is called an* ultra-Liouville number*, if for every positive integer k, there exist infinitely many rational numbers p/q, with $q > 1$, such that*

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{\exp^{[k]}(q)}.$$

The set of the ultra-Liouville numbers will be denoted by $\mathbb{L}_{\text{ultra}}$.

It follows from the definition that $\mathbb{L}_{\text{ultra}} \subseteq \mathbb{L}$ is also a dense G_δ set (in particular it is uncountable) which means that $\mathbb{L}_{\text{ultra}}$ is a large set in a topological sense. In particular, every real number can be written as the sum and (if it is not zero) the product of two ultra-Liouville numbers, as in [3]. However, from a metrical point of view, both sets $\mathbb{L}$ and $\mathbb{L}_{\text{ultra}}$ are very small: they have Hausdorff dimension zero.

The aim of this paper is to investigate a problem related to Mahler's question concerning $\mathbb{L}_{\text{ultra}}$. More precisely, our main result is the following

Theorem 1. *The set $\Sigma_{\mathbb{L}_{\text{ultra}}}(\mathbb{C})$ is uncountable.*

In order to prove that, we shall prove a stronger result about the behavior of some functions in $\Sigma_{\mathbb{Q}}(\mathbb{C})$. For a rational number z, we denote by $\text{den}(z)$ its denominator. We prove that

Theorem 2. *There exist uncountably many functions $f \in \Sigma_{\mathbb{Q}}(\mathbb{C})$ with $1/2 < f'(x) < 3/2, \forall x \in \mathbb{R}$, such that*

$$(*) \qquad \text{den}(f(p/q)) < q^{8q^2},$$

for all $p/q \in \mathbb{Q}$, with $q > 1$. In particular, $\text{den}(f(p/q)) < e^{e^q} - 1$, *if $q \geq 7$.*

2. The proofs

2.1. **Proof that Theorem 2 implies Theorem 1.** Given an ultra-Liouville number ξ and a positive integer k, there exist infinitely many rational numbers p/q with

$q \geq 7$ and such that
$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{\exp^{[k+2]}(q)}.$$
Let f be a function as in Theorem 2. By the Mean Value Theorem, we obtain
$$\left|f(\xi) - f\left(\frac{p}{q}\right)\right| \leq \frac{3}{2}\left|\xi - \frac{p}{q}\right| < \frac{3}{2\exp^{[k+2]}(q)}.$$
We know that $f(p/q) = a/b$, with $b < e^{e^q} - 1$. Then $\frac{3}{2}\exp^{[k]}(b) < \exp^{[k+2]}(q)$ and hence
$$\left|f(\xi) - \frac{a}{b}\right| = \left|f(\xi) - f\left(\frac{p}{q}\right)\right| < \frac{1}{\exp^{[k]}(b)}.$$
This implies that $f(\xi)$ is an ultra-Liouville number as desired. □

2.2. **Proof of Theorem 2.** Before starting the proof, we shall state three useful facts (which can be easily proved)

- For any distinct $y, b \in [-1, 1]$, we have $|\sin(y - b)| > |y - b|/3$. (Indeed, the function $\sin(x)/x$ is decreasing for $x \in (0, \pi]$, and $\sin(2)/2 > 1/3$.)
- For any distinct $x, y \in \mathbb{Q} \cap [0, 1/2]$, with $\mathrm{den}(x), \mathrm{den}(y) \leq n$, we have
$$|\cos(2\pi x) - \cos(2\pi y)| \geq \frac{4}{n^3}.$$
(Indeed, we can assume $x < y$; we can also assume $y \leq 1/4$: if $1/4 \leq x \leq 1/2$, we use that $|\cos(2\pi x) - \cos(2\pi y)| = |\cos(2\pi(1/2 - x)) - \cos(2\pi(1/2 - y))|$, and, if $x < 1/4 < y$ we use that $|\cos(2\pi x) - \cos(2\pi y)| > |\cos(2\pi x) - \cos(2\pi \cdot 1/4)| > 1 - 4x \geq 1/n \geq 4/n^3$, since $\mathrm{den}(y) \geq 2$; now we have two cases: if $x = 0$ then $\cos(2\pi x) - \cos(2\pi y) = 1 - \cos(2\pi y) = 2\sin^2(\pi y) \geq 8/n^2 \geq 4/n^3$; and, if $0 < x < y$ then $x \geq 1/n$ and, by the mean value theorem, $|\cos(2\pi x) - \cos(2\pi y)| \geq 2\pi \sin(2\pi\xi)(2\pi y - 2\pi x) \geq 8\pi x(y - x) \geq 8\pi(y - x)/n \geq 8\pi/n^3 > 4/n^3$.)
- For every $\epsilon \in (0, 2]$, any interval of length $> \epsilon$ contains at least two rational numbers with denominator $\leq \lceil 2/\epsilon \rceil$. (Indeed, if $m = \lceil 2/\epsilon \rceil$ and (a, b) is the interior of the interval, we have $b - a > \epsilon \geq 2/m$, and so, for $k = \lfloor ma \rfloor + 1$, we have $ma < k \leq ma + 1$, and so $ma < k < k + 1 \leq ma + 2 < ma + m(b - a) = mb$, which implies $a < k/m < (k + 1)/m < b$.)

Consider the following enumeration of $\mathbb{Q} \cap [0, 1/2]$:
$$\{x_1, x_2, \ldots\} = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{1}{6}, \ldots\},$$
where we consider only irreducible fractions ordered in the following way: $x_1 = 0/1$; for every $k \geq 1$, if $x_k = p/q$ with $2p < q - 2$ then $x_{k+1} = r/q$ where r is the minimum with $p < r \leq q/2$ and $\gcd(r, q) = 1$, and if $2p \geq q - 2$ then $x_{k+1} = 1/(q + 1)$. The set $A = \mathbb{Q} \cap [0, 1/2]$ has the properties that $\cos(2\pi x) \neq \cos(2\pi y)$ for every $x \neq y$ in A, and that for every $z \in \mathbb{Q}$ there is (exactly one) $x \in A$ with $\cos(2\pi x) = \cos(2\pi z)$.

One can see that $\mathrm{den}(x_n) \geq \sqrt{n}$, for all $n \geq 1$: indeed, the number of positive integers n for which the denominator of x_n is equal to k is at most k for every $k \geq 1$, so the maximum positive integer n for which the denominator of x_n is at most k is at most $1 + 2 + \cdots + k = k(k + 1)/2 \leq k^2$.

Define $B_n = \{y_1, y_2, \ldots, y_n\}$ with $y_k := \cos(2\pi x_k)$ and define f by
$$f(x) = x + g(\cos(2\pi x)),$$

where $g(y) = \sum_{n=1}^{\infty} c_n g_n(y)$ and $g_n(y) = \prod_{b\in B_n} \sin(y-b)$. Note that $f(x+1) = f(x)+1$ and so it is enough to consider $\mathbb{Q}\cap[0,1)$ in order to characterize f on $\mathbb{Q}$. Notice also that, in order to show that $f(x)\in\mathbb{Q}$ for every $x\in\mathbb{Q}$, it is enough to prove this for $x\in A$. Indeed, given $z\in\mathbb{Q}$, take $x\in A$ with $\cos(2\pi x)=\cos(2\pi z)$. Then we have $f(z)-z = g(\cos(2\pi z)) = g(\cos(2\pi x)) = f(x)-x$, and so, if $f(x)\in\mathbb{Q}$, then $f(z) = f(x)+z-x\in\mathbb{Q}$; in particular, if $z\in\mathbb{Z}$ then $f(z)=z$, since $f(0)=0$.

Now, we shall choose inductively the constants c_n so that f will satisfy the desired conditions in Theorem 2. The first requirements are $c_n = 0$ for $1\le n\le 5$ and $|c_n| < 1/n^n$ for every positive integer n. On the other hand, for all y belonging to the open ball $B(0,R)$ one has that

$$|g_n(y)| < \prod_{b\in B_n} e^{|y-b|} \le e^{n(R+1)},$$

where we used the fact that $b\in[-1,1]$. Thus, since $|c_n|<1/n^n$, we get $|c_n g_n(y)| \le (e^{R+1}/n)^n$ from which g (and so f) is an entire function, since the series $g(y) = \sum_{n=1}^{\infty} c_n g_n(y)$, which defines g, converges uniformly in any of these balls. Moreover, for $x\in\mathbb{R}$, we have $|g_n'(x)|\le n$, and so $f'(x) = 1-2\pi\sin(2\pi x)\sum_{n=1}^{\infty} c_n g_n'(\cos(2\pi x)) \in (1/2,3/2)$, since $\sum_{n=6}^{\infty} n/n^n < 1/4\pi$.

Suppose that $c_1,\dots,c_{n-1}$ have been chosen such that $f(x_1),\dots,f(x_n)$ have the desired property (notice that the choice of $c_1,\dots,c_{n-1}$ determines the values of $f(x_1),\dots,f(x_n)$, independently of the values of $c_k, k\ge n$; in particular, since $c_k = 0$ for $1\le k\le 5$, we have $f(x_n) = x_n$ for $1\le n\le 6$). Now, we shall choose c_n for which $f(x_{n+1})$ satisfies the requirements.

Let $t\le n$ be positive integers with $n\ge 5$. Then $\mathrm{den}(x_{n+1}), \mathrm{den}(x_t)\le n$ (indeed, $\mathrm{den}(x_6) = 5$ and $\mathrm{den}(x_{n+1})-\mathrm{den}(x_n)\le 1, \forall n\ge 1$). Since $\cos(2\pi x_{n+1})\ne\cos(2\pi x_t)$, then $|y_{n+1}-y_t|\ge 4/n^3$. Therefore

$$|\sin(y_{n+1}-y_t)| > \frac{|y_{n+1}-y_t|}{3} > \frac{4}{3n^3} > \frac{1}{n^3}$$

yielding $|g_n(y_{n+1})| > n^{-3n}$. Thus $c_n g_n(y_{n+1})$ runs through an interval of length larger than $2/n^{4n}$. Now, we may choose (in at least two ways) c_n such that $g(y_{n+1})$ is a rational number with denominator at most n^{4n}.

Given $z\in\mathbb{Q}$, let $q = \mathrm{den}(z)$; if $q=1$ then $z\in\mathbb{Z}$ and so $f(z)=z$ and thus $\mathrm{den}(f(z)) = 1\le q^{8q^2}$. Otherwise, $q>1$, and there is a positive integer k with $\cos(2\pi x_k) = \cos(2\pi z)$, so x_k and z have the same denominator; indeed, in this case, we have $z-x_k\in\mathbb{Z}$ or $z+x_k\in\mathbb{Z}$. Thus $\mathrm{den}(f(z)-z) = \mathrm{den}(g(\cos(2\pi z)) = \mathrm{den}(g(\cos(2\pi x_k)) = \mathrm{den}(g(y_k)) \le (k-1)^{4(k-1)} < k^{4(k-1)}$. Since $q = \mathrm{den}(z) = \mathrm{den}(x_k)\ge\sqrt{k}$, we get $\mathrm{den}(f(z)-z)\le k^{4(k-1)}\le (q^2)^{4(q^2-1)} = q^{8(q^2-1)}$. Then we have

$$\mathrm{den}(f(z)) \le \mathrm{den}(z)\,\mathrm{den}(f(z)-z) = q\,\mathrm{den}(f(z)-z) \le q\cdot q^{8(q^2-1)} \le q^{8q^2}$$

as desired.

The proof that we can choose f to be transcendental follows because there is a binary tree of different possibilities for f. (If we have choosen $c_1, c_2,\dots,c_{n-1}$, different choices of c_n give different values of $f(y_{n+1})$, which does not depend on the values of c_k for $k>n$, and so different functions f.) Thus, we have constructed uncountably many possible functions, and the algebraic entire functions taking $\mathbb{Q}$ into itself must be polynomials belonging to $\mathbb{Q}[z]$, which is a countable subset.

In fact, we can prove that all functions constructed above are transcendental, unless $c_n = 0, \forall n \in \mathbb{N}$: if such a function f is not transcendental, then f would be polynomial, since it is an entire function. However, the property $f(x+1) = f(x)+1$ would imply $f(x) = x + c$, for some $c > 0$. Then $g(\sin(2\pi x))$ is a constant, but this leads to a contradiction, since $g(y_1) = 0$ and $g(y_{k+1}) = c_k \prod_{b\in B_k} \sin(y_{k+1} - b) \neq 0$, where k is minimal such that $c_k \neq 0$. □

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