# ON THE ARITHMETIC NATURE OF HYPERTRANSCENDENTAL FUNCTIONS AT COMPLEX POINTS

DIEGO MARQUES

Abstract. Most of well-known transcendental functions usually take a transcendental value at an algebraic point belonging to its domain, the algebraic exceptions forming the so-called exceptional set. For instance, the exceptional set of the function $e^{z-\sqrt{2}}$ is the set $\{\sqrt{2}\}$, which follows from the Hermite-Lindemann theorem. In this paper, we shall use interpolation formulae, to prove that any subset of $\overline{\mathbb{Q}}$ is the exceptional set of uncountable many hypertranscendental entire functions with order of growth as small as we wish. Moreover these functions are algebraically independent over $\mathbb{C}$.

## 1. Introduction: a little survey on transcendental numbers

We say that a complex number $\alpha$ is *algebraic* if there exists a nonzero polynomial $P \in \mathbb{Q}[x]$ with $P(\alpha) = 0$. If no such polynomial exists, $\alpha$ is *transcendental.* The set of algebraic numbers forms a field denoted by $\overline{\mathbb{Q}}$.

Euler was probably the first person to define transcendental numbers in the modern sense (see [4]). But transcendental number theory began in 1844 with Liouville's proof [10] that if an algebraic number $\alpha$ has degree $n > 1$, then there exists a constant $C > 0$ such that $|\alpha - p/q| > Cq^{-n}$, for all $p/q \in \mathbb{Q} \setminus \{0\}$. Using this result, Liouville gave the first explicit examples of transcendental numbers, e.g., the "Liouville number" $\sum_{n\geq 0} 10^{-n!}$. There are several classical theorems on transcendental numbers, Let us state three of them for making this text self-contained.

In 1872 Hermite [7] proved that $e$ is transcendental, and in 1884 Lindemann [9] extended Hermite's method to prove that $\pi$ is also transcendental. In fact, Lindemann proved a more general result.

**Theorem 1** (Hermite-Lindemann)**.** *The number $e^{\alpha}$ is transcendental for any nonzero algebraic number $\alpha$.*

As a consequence, the numbers $e^{\sqrt{2}}$ and $e^{\,i}$ are transcendental ($i = \sqrt{-1}$), as are $\log 2$ and $\pi$, since $e^{\log 2} = 2$ and $e^{\pi i} = -1$ are algebraic.

At the 1900 International Congress of Mathematicians in Paris, as the seventh in his famous list of 23 problems, Hilbert gave a big push to transcendental number theory with his question of the arithmetic nature of the power $\alpha^{\beta}$ of two algebraic numbers $\alpha$ and $\beta$. In 1934, Gelfond and Schneider, independently, completely solved the problem (see [1, p. 9]).

**Theorem 2** (Gelfond-Schneider)**.** *Assume $\alpha$ and $\beta$ are algebraic numbers, with $\alpha \neq 0$ or $1$, and $\beta$ irrational. Then $\alpha^{\beta}$ is transcendental.*

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In particular, $2^{\sqrt{2}}, (-1)^{\sqrt{2}}$, and $e^{\pi} = i^{-2i}$ are all transcendental (we refer the reader to [18, 13, 6] for recent results on the arithmetic nature of $x^y$, with both $x$ and $y$ are transcendental). Since the sum of transcendental numbers can be algebraic (e.g., $e+(-e)$), one may ask about the nature of the sum of transcendental numbers as in the Hermite-Lindemann theorem. For istance, is $e + e^{\sqrt{2}}$ transcendental? This natural question leads to a beautiful generalization of the Hermite-Lindemann theorem due to Lindemann and Weierstrass.

**Theorem 3** (Lindemann-Weierstrass)**.** *Let $\alpha_1, ...\alpha_n$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then $e^{\alpha_1}, ..., e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

An *algebraic function* is a function $f(x)$ which satisfies $P(x, f(x)) = 0$, where $P(x, y)$ is a polynomial with complex coefficients. For instance, functions that can be constructed using only a finite number of elementary operations are examples of algebraic functions. A function which is not algebraic is, by definition, a *transcendental function*, as example the trigonometric functions, the exponential function, and their inverses. A interesting question is to study the arithmetic nature of a function at algebraic points. For instance, it is a simple matter to show that an *entire* function, namely a function which is analytic in $\mathbb{C}$, is a transcendental function if and only if it is not a polynomial. Thus, one may interesting to think only in the case of transcendental functions.

At the end of XIXth century, after the proof by Hermite and Lindemann of the transcendence of $e^{\alpha}$ for all nonzero algebraic $\alpha$, a question arose:

$(*)$ Does a transcendental analytic function usually takes transcendental values at algebraic points?

In the example of the exponential function $e^z$, the word "usually" stands for avoiding the exception $z = 0$. After the Hermite-Lindemann theorem, it was expected that by evaluating a transcendental function $f$ at an algebraic point of its domain, we would find a transcendental number, but exceptions can take place. All these exceptions (i.e., algebraic numbers at which the function assumes algebraic values) form the so-called *exceptional set*, denoted by $S_f$. This set plays an important role in transcendental number theory (see, e.g., [23] and references therein).

In 1886, Weierstrass found a positive answer for the question $(*)$, when he gave an example of a transcendental entire function which takes rational values at all rational points. Later, Stäckel [20] proved that for each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$, there is a transcendental entire function $f$ such that $f(\Sigma) \subseteq T$. Another construction due to Stäckel [21] produces an entire function $f$ whose derivatives $f^{(t)}$, for $t = 0, 1, 2, \dots$, all map $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}$ and so $S_{f^{(t)}} = \overline{\mathbb{Q}}$. Two years later, G. Faber refined this result by showing the existence of a transcendental entire function such that $f^{(t)}(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}(i)$, for all $t \geq 0$. A more elegant discussion on this subject can be found in [11] and [23].

In this paper, we were able to generalize these two Stäckel's and the Faber's result. Before of state our main theorem, we need a couple of definitions: a set of functions $f_1, ..., f_m$ is said to be *algebraically independent over* a field $K$, if there is no nonzero polynomial $P$, with coefficients in $K$, such that $P(f_1(z), ..., f_m(z))$ is the zero function. Otherwise, they are called *algebraically dependent over* $K$. In 1949, Morduhai-Boltovskoi introduce the term *hypotranscendental function* to $f$ by saying that there exists $n \geq 0$ such that the functions $z, f(z), ..., f^{(n)}(z)$ are algebraically dependent over $\mathbb{Q}$. Otherwise, the function is called *hypertranscendental*, or *transcendentally transcendental*, see [15].

**Definition 1.** *(Order) Let $f$ be an entire function and $R > 0$, the order of growth of $f$ is defined to be*

$$\lim_{R\to\infty} \sup \frac{\log\log|f|_R}{\log R}, \text{ where } |f|_R = \sup_{|z|=R} |f(z)|.$$

By definition, it follows that a function $f$ that satisfies $|f|_R \leq e^{R^\rho}$ for some $\rho > 0$ and for all $R$ sufficiently large has order $\leq \rho$. Surprisingly, there exists a straight relation between the order of a function and its integer values, G. Chudnovsky [3] proved that if $f$ has order $\rho$, then the set

$$\{z \in \mathbb{C} \ : \ f^{(t)}(z) \in \mathbb{Z} \text{ for all } t \geq 0\}$$

has cardinality at most $\rho$. For more see [3, Chapter 9].

Let us state our main result

**Theorem 4.** *Let $A$ be a countable subset of $\mathbb{C}$ and let $\rho$ be a positive real number. For any integer $s \geq 0$ and any $\alpha \in A$, let $E_{\alpha,s}$ be a dense subset in $\mathbb{C}$. Then there exists a set $\mathcal{F}$ of entire functions with the following properties:*

(a) *For any $f \in \mathcal{F}$, any $\alpha \in A$ and any integer $s \geq 0$, $f^{(s)}(\alpha) \in E_{\alpha,s}$;*
(b) *Any function $f \in \mathcal{F}$ has order at most $\rho$;*

*If $\mathcal{F}^{(s)}$ denotes the set of $s$-th derivates of functions in $\mathcal{F}$, that is, $\mathcal{F}^{(s)} = \{f^{(s)} : f \in \mathcal{F}\}$, then*

(c) *For any integer $m \geq 1$, any distinct functions $f_1, \ldots, f_m \in \bigcup_{s\geq 0} \mathcal{F}^{(s)}$ and any non zero polynomial $P \in \mathbb{C}[X_0, X_1, \ldots, X_m]$, the entire function $P(z, f_1(z), \ldots, f_m(z))$ is not the zero function;*
(d) *The set $\mathcal{F}$ has the power of continuum.*

Note that the property (c) ensures that the functions in $\mathcal{F}$ are hypertranscendental.

One basic problem in the theory of transcendental numbers is to determine $S_f$, or at least to find properties of this set. It is almost unnecessary to stress that this is not an easy problem. The question on the possible exceptional sets was partially solved in 1965, when K. Mahler [12] proved that if $A$ is *closed relative to* $\overline{\mathbb{Q}}$, that is if $\alpha \in A$ then all its algebraic conjugates lie also in $A$, then it is the exceptional set of some transcendental function. Since the exceptional sets of a function and its derivative can be different, in this work we consider a more general definition (including multiplicity): Let $f$ be an entire function. We define *the exceptional set with multiplicity of $f$* to be the set of pairs $(\alpha, t) \in \overline{\mathbb{Q}} \times (\mathbb{N} \cup \{0\})$, such that $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$. We denote it by $M_f$.

In this paper we solved completely the problem of the possible exceptional sets with multiplicity of a hypertranscendental function.

**Theorem 5.** *If $A \times N \subseteq \overline{\mathbb{Q}} \times \mathbb{N}_0$, then there is an uncountable set $\mathcal{F}_{A,N}$, of hypertranscendental entire functions such that*

$$M_f = A \times N, \tag{1.1}$$

*for all $f \in \mathcal{F}_{A,N}$. Moreover the set*

$$\{f^{(t)}(\alpha) \ : \ (\alpha, t) \notin A \times N \text{ and } f \in \mathcal{F}_{A,N}\} \tag{1.2}$$

*is algebraically independent over* $\mathbb{C}$.

## 2. Preliminary results

One piece of notation: throughout the paper we write $L(P)$ for the sum of absolute values of coefficients of a polynomial $P$, well-known as the *length of* $P$, $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$ and $[a, b] = \{a, a+1, ..., b\}$, where $a < b$ are integers.

Before upsetting the reader with a plenty of technical lemmas, we start with a brief overview of our strategy to prove the Theorem 1. We hope this one becomes the next lemmas a little more natural. In the Theorem 1, we wish to find functions with certain prescribed properties. Well, a such function will be taken as $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$, where the polynomials $P_n$'s will be appropriately chosen. First of all, we need ensure that $f$ is an entire function and which has a precribed growth order, for that the $a_k$'s will be chosen as a center of a ball with radius depending on $P_k$ and of the required order. Secondly, the sequence $(P_n)_{n\geq 0}$ will be explicited and it must be a key property, namely, for a certain sequence $(s_n)_{n\geq 0}$ (to be explicited and depending on an enumeration of $\overline{\mathbb{Q}} = \{\alpha_1, ...\}$) the set of the indexes for which $P_k^{(s_n)}(\alpha_n) \neq 0$ is bounded. This ensures that $f^{(s)}(\alpha)$ is actually a finite sum. After, since $f^{(s)}(\alpha)$ is a finite sum, we can proceed by induction for finding infinite possibilities for each $a_k$, which can be chosen in a infinite set, namely the intersection of a ball with a dense set. Finally, the possibility of choosing $a_k$ in an infinite set together with the property of the $P_k^{(s)}$'s guarantee the uncountablity of these possible functions.

Now, let us to the work.

**Lemma 1.** *Let* $P(z) \in \mathbb{C}[z]$ *be a polynomial and* $d \geq \deg(P)$ *(in the case of* $P \equiv 0$*,* $d$ *can be taken as any non-negative real number), then*

$$|P(z)| \leq L(P) \max\{1, |z|\}^d, \text{ for all } z \in \mathbb{C}. \tag{2.1}$$

*Proof.* Write

$$P(z) = a_0 + a_1 z + \cdots + a_{\deg(P)} z^{\deg(P)}.$$

The triangular inequality yields

$$|P(z)| \leq |a_0| + |a_1||z| + \cdots + |a_{\deg(P)}||z^{\deg(P)}|$$

Since $|z|^k \leq \max\{1, |z|\}^{\deg(P)}$, for all $k \in [0, \deg(P)]$ and $z \in \mathbb{C}$, we get

$$\begin{aligned} |P(z)| &\leq (|a_0| + \cdots + |a_{\deg(P)}|) \max\{1, |z|\}^{\deg(P)} \\ &\leq L(P) \max\{1, |z|\}^d \end{aligned}$$

□

**Lemma 2** (Analicity)**.** *Let* $(P_n(z))_{n\geq 0} \in \mathbb{C}[z]$ *be a sequence of nonzero polynomials, and let* $\rho$ *be a positive real number. Set* $m_0 = 1$ *and by recurrence* $m_k = \max\{\lceil \frac{\deg(P_k)}{\rho} \rceil, m_{k-1} + 1\}$ *for* $k \geq 1$*. If the sequence* $(a_n)_{n\geq 0} \in \mathbb{C}$ *satisfies*

$$|a_k| \leq \frac{1}{L(P_k) m_k!} \tag{2.2}$$

*for all $k \geq 0$, then the series $\sum a_n P_n(z)$ converges absolutely and uniformly on any compact sets, particularly this gives an entire function, moreover its sum $f(z)$ has order at most $\rho$.*

*Proof.* We define $(Q_n(z))_{n\geq 0} \in \mathbb{C}[z]$ and $(b_n)_{n\geq 0} \in \mathbb{C}$ as follows

$$Q_n(z) = \begin{cases} 0, & \text{if} \quad n \neq m_k \\ P_k(z), & \text{if} \quad n = m_k \end{cases}$$

and $b_{m_k} = a_k$ for $k \geq 0$. Since $1 = m_0 < m_1 < m_2 < \cdots$, we have that the $Q_n$'s and $b_n$'s are well defined and moreover $\sum_{n=0}^{\infty} a_n P_n(z) = \sum_{n=0}^{\infty} b_n Q_n(z)$. Below one can see the gaps of zeros in the sequence $(Q_n)_{n\geq 0}$

$$0, P_0(z), \underbrace{0, ..., 0}_{m_1-m_0-1}, P_1(z), \underbrace{0, ..., 0}_{m_2-m_1-1}, P_2(z), 0, ....$$

Also, if $Q_n$ is nonzero, then $n = m_k$ for some $k \geq 0$. Hence

$$\deg(Q_n) = \deg(Q_{m_k}) = \deg(P_k) \leq m_k\rho = n\rho.$$

Thus we get by Lemma 1

$$|Q_n(z)| \leq L(Q_n) \max\{1, |z|\}^{n\rho},$$

for all $n \geq 0$. Let $K \subseteq \mathbb{C}$ be a compact set, then $|z| \leq R$, for some $R > 0$ and any $z \in K$. Therefore, in $K$, we have

$$\begin{array}{rcl} |b_{m_k} Q_{m_k}(z)| & \leq & |b_{m_k}||L(Q_{m_k})| \max\{1, |z|\}^{m_k\rho} \\ & = & |a_k||L(P_k)| \max\{1, |z|\}^{m_k\rho} \\ & \leq & \frac{\max\{1,|z|\}^{m_k\rho}}{m_k!} \end{array}$$

We conclude that $|b_{m_k} Q_{m_k}(z)| \leq M_k$, for all $z \in K$, where $M_k = \frac{\max\{1,R\}^{m_k\rho}}{m_k!}$. On the other hand,

$$\sum_{k=0}^{\infty} M_k \leq \sum_{n=0}^{\infty} \frac{\max\{1, R\}^{n\rho}}{n!} = e^{\max\{1,R\}^{\rho}} \tag{2.3}$$

Therefore $f(z) = \sum_{k=0}^{\infty} b_{m_k} Q_{m_k}(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ is an entire function (Weierstrass $M$-test). From the inequality in (2.3), we deduce that $f$ has order at most $\rho$. □

Now, let us enumerate the set $A$ in Theorem 4 as $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. All integer number $n \geq 1$ can be written uniquely in form $n = \frac{m_n(m_n+1)}{2} + j_n$, for $m_n \geq 0$ and $1 \leq j_n \leq m_n + 1$, define $\gamma_n = \alpha_{m_n+2-j_n}$. Now, let us construct a sequence of polynomials as follows

$$P_0(z) = 1 \text{ and } P_n(z) = (z - \gamma_1) \cdots (z - \gamma_n) \text{ for } n \geq 1,$$

Here we list the first few polynomials:

$$\begin{aligned}
P_0(z) &= 1\\
P_1(z) &= (z-\alpha_1)\\
P_2(z) &= (z-\alpha_1)(z-\alpha_2)\\
P_3(z) &= (z-\alpha_1)^2(z-\alpha_2)\\
P_4(z) &= (z-\alpha_1)^2(z-\alpha_2)(z-\alpha_3)\\
P_5(z) &= (z-\alpha_1)^2(z-\alpha_2)^2(z-\alpha_3)\\
P_6(z) &= (z-\alpha_1)^3(z-\alpha_2)^2(z-\alpha_3)\\
P_7(z) &= (z-\alpha_1)^3(z-\alpha_2)^2(z-\alpha_3)(z-\alpha_4)\\
&\vdots
\end{aligned}$$

The pattern can be seen by following the arrows and picking up the corresponding term at each node in the figure 1:

FIGURE 1. Building the $P_n$'s

With the same notation, we set $s_n = j_n - 1$.

**Lemma 3** (Truncation)**.** *For $n \geq 1$, we have $P_{n-1}^{(s_n)}(\gamma_n) \neq 0$ and $P_l^{(s_n)}(\gamma_n) = 0$ when $l \geq n$.*

*Proof.* Let us partition the set of these polynomials into infinitely many disjoint sets, of the following way

$$D_0 = \{P_0\} \text{ and } D_m = \{P_{d_m}, P_{d_m+1}, \ldots, P_{d_m+(m-1)}\}$$

where $d_m = m + \frac{(m-1)(m-2)}{2}$, for $m > 0$. Explicitly, the $m$ polynomials in $D_m$ are defined as

$$P_{d_m}(z) = (z-\alpha_1)^{m-1}(z-\alpha_2)^{m-2}\cdots(z-\alpha_{m-2})^2(z-\alpha_{m-1})(z-\alpha_m)$$

and for $j \in [1, m-1]$,

$$P_{d_m+j}(z) = P_{d_m}(z)\prod_{t=1}^{j}(z-\alpha_t)$$

Also, we may deduce that $\gamma_{d_m+k} = \alpha_{m-k}$ and $s_{d_m+k} = k$. Now, by construction of the polynomials, it is enough to prove the lemma for $k = n$. Let us distinguish two cases: the first one, when $P_{n-1}$ and $P_n$ are in $D_m$, for some $m \geq 2$. Thus $P_{n-1} = P_{d_m+k}$ and $P_n = P_{d_m+k+1}$, for some $k \in [0, m-2]$. Therefore we must prove that $P^{(k+1)}_{d_m+k}(\alpha_{m-k-1}) \neq 0$ and $P^{(k+1)}_{d_m+k+1}(\alpha_{m-k-1}) = 0$. The result follows because $\alpha_{m-k-1}$ is a zero of $P_{n-1}$ with multiplicity $k+1$, which means $P^{(s_n)}_{n-1}(\gamma_n) \neq 0$ and on the other hand, $\alpha_{m-k-1}$ is a zero of $P_n(z)$ with multiplicity $k+2$, which implies $P^{(s_n)}_n(\gamma_n) = 0$.

The second case is when $P_{n-1} \in D_{m-1}$ and $P_n \in D_m$, for some $m \geq 1$. In this case $P_n(z) = P_{n-1}(z)(z - \alpha_m)$, where

$$P_{n-1}(z) = (z - \alpha_1)^{m-1} \cdots (z - \alpha_{m-2})^2 (z - \alpha_{m-1})$$

It is easy see that $P^{(s_n)}_{n-1}(\gamma_n) = P_{n-1}(\alpha_m) \neq 0$ and $P^{(s_n)}_n(\gamma_n) = P_n(\alpha_m) = 0$. ☐

**Lemma 4** (Identity)**.** *If* $\sum_{k=0}^{\infty} a_k P_k(z) = \sum_{k=0}^{\infty} b_k P_k(z)$ *for all* $z \in \mathbb{C}$*, then* $a_k = b_k$ *for each* $k \geq 0$*.*

*Proof.* It suffice to prove that if $f(z) := \sum_{k=0}^{\infty} a_k P_k(z) = 0$ for all $z \in \mathbb{C}$, then $(a_k)_{k\geq 0}$ is identically 0. Notice that $a_0 = f(\alpha_1) = 0$. Assuming $a_0, a_1, \dots, a_{n-1}$ are all 0, by Lemma 3, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} a_k P^{(s_{n+1})}_k(\alpha_{\gamma_{n+1}}) \\ &= \sum_{k=0}^{n-1} a_k P^{(s_{n+1})}_k(\gamma_{n+1}) + a_n P^{(s_{n+1})}_n(\alpha_{j_{n+1}}) + \sum_{k=n+1}^{\infty} a_k P^{(s_{n+1})}_k(\alpha_{j_{n+1}}) \\ &= a_n P^{(s_{n+1})}_n(\gamma_{n+1}) \end{aligned}$$

Since $P^{(s_{n+1})}_n(\gamma_{n+1}) \neq 0$, we have $a_n = 0$. Hence the proof will be completed by induction. ☐

Now we are able to prove our first theorem.

## 3. Proof of the Theorem 1

We are going to construct the desired entire function by fixing the coefficients in the series $\sum_{k=0}^{\infty} a_k P_k(z)$ recursively, where the sequence $(P_k)_{k\geq 0}$ has been defined in Section 2.

First, as the same notation of Lemma 2, the condition $|a_k| \leq t_k := \frac{1}{L(P_k) m_k!}$ will ensure $\sum_{k=0}^{\infty} a_k P_k(z)$ to be entire with order at most $\rho$.

Next, we will fix the coefficients $a_k$ recursively. For $n \geq 1$, we denote $E_n = E_{\gamma_n, s_n}$ and let the numbers $\beta_n := f^{(s_n)}(\gamma_n) = \sum_{k=0}^{\infty} a_k P^{(s_n)}_k(\gamma_n)$. We are going to choose the value of $a_k$ so that $\beta_n \in E_{\gamma_n, s_n} = E_n$ for all $n \geq 1$.

By Lemma 2, we know that $P^{(s_n)}_l(\gamma_n) = 0$ when $l \geq n$, so $\beta_n$ is actually the finite sum $\sum_{k=0}^{n-1} a_k P^{(s_n)}_k(\gamma_n)$. Notice that $\beta_1 = a_0 P^{(0)}_0(\alpha_1) = a_0$ and $E_1$ is dense, so we can choose a value for $a_0$ in an infinite set $I_0$ such that $0 < |a_0| \leq t_0$ and $\beta_1 \in E_1$. Now suppose that the values of $\{a_0, a_1, \cdots, a_{n-1}\}$ are well fixed respectively in infinite sets $I_k$ such that $0 < |a_k| \leq t_k$ and $\beta_k \in E_k$ for $0 \leq k \leq n-1$. By Lemma 3, we know that $P^{(s_{n+1})}_n(\gamma_{n+1}) \neq 0$, set

$$I_n := \left( \frac{E_n - A_n}{P_n^{(s_{n+1})}(\gamma_{n+1})} \right) \cap B\,(0; t_n) \setminus \{0\},$$

where $A_n := \sum_{k=0}^{n-1} a_k P_k^{(s_{n+1})}(\gamma_{n+1})$. So we can pick a proper value of $a_n$ in the infinite set $I_n$, thus $0 < |a_n| \leq t_n$ and $\beta_n = \sum_{k=0}^{n-1} a_k P_k^{(s_{n+1})}(\gamma_{n+1}) + a_n P_n^{(s_{n+1})}(\gamma_{n+1}) \in E_n$.

So now by induction all the $a_k$ are well chosen such that for all $k \geq 0$ we have $0 < |a_k| \leq t_k$ and $\beta_{k+1} \in E_{k+1}$. Thus $f$ is an entire function satisfying the conditions (a) and (b).

Let $\mathcal{F}_A$ be the set of all entire functions satisfying the conditions (a) and (b). Set $I = I_0 \times I_1 \times \cdots$ and consider the function $\phi : I \to \mathcal{F}_A$ given by $\phi(a_0, a_1, ...) = \sum_{n=0}^{\infty} a_n P_n(z)$. The well definition of $\phi$ follows from proof above, also the Lemma 4 implies that $\phi$ is one-to-one. Hence $\mathcal{F}_A$ is uncountable, since that $I$ is.

There exists an uncountable set $\mathcal{B} := \{\xi\} \cup \{T_{\lambda,s}\}_{\lambda \in \Lambda, s \geq 0}$ algebraically independent over $\overline{\mathbb{Q}}$ (for instance the transcendental basis of the field extension $\mathbb{C}/\mathbb{Q}$). Consider $A' = \{\xi\} \cup A$. Fix $\lambda \in \Lambda$, set $E_{\xi,s}^{\lambda} = \{\alpha T_{\lambda,s} : \alpha \in \overline{\mathbb{Q}} \setminus \{0\}\}$ and $E_{\alpha_n,s}^{\lambda} = E_{\alpha,s}$ for all $\alpha \in A$ and $s \geq 0$. By the all previous discussion, there exists a set $\mathcal{F}_\lambda$ of entire functions satisfying the conditions (b) and (d), as well as the condition (a) for the new set $A'$ (which is still countable). Next, for each $\lambda \in \Lambda$ take a unique function $f_\lambda \in \mathcal{F}_\lambda$. Set $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$, we shall prove that this one is our desired set. In fact, by construction, this set satisfies the conditions (a), (b) and (d). To prove (c), take distinct functions $f_1, ..., f_m \in \bigcup_{s \geq 0} \mathcal{F}^{(s)}$. Therefore $f_j(z) = f_{\lambda_j}^{(s_j)}(z)$ for $j = 1, ..., m$ and for some pairwise distinct pairs $(\lambda_1, s_1), ..., (\lambda_m, s_m) \in \Lambda \times \mathbb{N}_0$. It follows that $f_j(\xi) = \gamma_j T_{\lambda_j, s_j}$ for $j = 1, ..., m$ and some $\gamma$'s $\in \overline{\mathbb{Q}} \setminus \{0\}$. This yields that the numbers $\xi, f_1(\xi), ...., f_m(\xi)$ are algebraically independent and then it holds (c). $\square$

Before going further, it is worth noting some interesting consequences of the Theorem 4 which give generalizations for classical results on this subject. The suitable choice of $A$, $E_{\alpha,s}$ are noted in parentheses.

**Corollary 1** (Generalization of the first Stäckel's theorem)**.** *For each countable subset $\Sigma \subseteq \mathbb{C}$ and each dense subset $T \subseteq \mathbb{C}$ there is a hypertranscendental entire function $f$ such that $f^{(s)}(\Sigma) \subseteq T$ for $s \geq 0$. ($A = \Sigma$, $E_{\alpha,s} = T$)*

**Corollary 2** (Generalization of the second Stäckel's theorem)**.** *Let $A \subseteq \mathbb{C}$ be countable and dense in $\mathbb{C}$, then there is a hypertranscendental entire function $f$ such that $f^{(s)}(A) \subseteq A$, for $s \geq 0$.($E_{\alpha,s} = A$)*

**Corollary 3** (Generalization of the Faber's theorem)**.** *There exists a hypertranscendental entire function such that $f^{(s)}(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}(i)$, for $s \geq 0$.($A = \overline{\mathbb{Q}}$, $E_{\alpha,s} = \mathbb{Q}(i)$)*

## 4. Applications to exceptional sets: proof of the Theorem 2

4.1. **An overview on exceptional sets.** Weierstrass (see [11]) initiated the question of investigating the set of algebraic numbers where a given transcendental entire function $f$ takes algebraic values. For an entire function $f$, we define the *exceptional set* of $f$ as follows

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \ : \ f(\alpha) \in \overline{\mathbb{Q}}\}$$

The study of exceptional sets started in 1886 with a letter of Weierstrass to Strauss. This study was later developed by Strauss, Stäckel, Faber. Further results are due to van der Poorten, Gramain, Surroca and others (see [5] and [19]).

Some exceptional sets...

**Example 1.** *Any finite set* $\{\alpha_1, ..., \alpha_n\} \subseteq \overline{\mathbb{Q}}$ *is exceptional. In fact, if* $f_1(z) = e^{(z-\alpha_1)\cdots(z-\alpha_k)}$*, then the Hermite-Lindemann theorem implies* $S_{f_1} = \{\alpha_1, \ldots, \alpha_k\}$.

**Example 2.** *The empty set is also exceptional. Indeed, if* $f_2(z) = e^z + e^{z+1}$*, the Lindemann-Weierstrass theorem implies* $S_{f_2} = \emptyset$.

**Example 3.** *Some infinite sets are also known to be exceptional. For instance, if* $f_3(z) = 2^z, f_4(z) = e^{i\pi z}$*, then* $S_{f_3} = S_{f_4} = \mathbb{Q}$*, by the Gelfond-Schneider theorem.*

We point out that is not known an elementary function[1] with exceptional set is either $\mathbb{Z}$ or $\mathbb{N}$. For giving such examples, we appeal to Schanuel's conjecture, one of the main open problems in transcendental number theory.

**Conjecture 1** (Schanuel)**.** *If* $z_1, \ldots, z_n$ *are complex numbers linearly independent over* $\mathbb{Q}$*, then among the numbers* $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$*, at least* $n$ *are algebraically independent.*

This conjecture was introduced in the 1960's by Schanuel in a course given by Lang [8]. Several classical consequences of this conjecture, together with an elegant exposition of it, can be found in [17, Chapter 10, Section 7G]. Very recent consequences can be found in [2], [14] and [22].

**Example 4.** *Assume that Schanuel's conjecture is true. If* $f_5(z) = \sin(\pi z)e^z$*,* $f_6(z) = 2^{3^z}$ *and* $f_7(z) = 2^{2^{2^{z-1}}}$*, then* $S_{f_5} = S_{f_6} = \mathbb{Z}$ *e* $S_{f_7} = \mathbb{N}$.

Summarizing, the sets $\emptyset$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$ (take $\Sigma = T = \overline{\mathbb{Q}}$ in first Stäckel's theorem) and all finite sets are exceptional. But, what are all the possible exceptional sets?

Before answering this question, observe that the exceptional sets of a function $f$ and its derivative $f'$, can be distincts. For instance, if $f(z) = 2^z$, then $S_f = \mathbb{Q}$. However, $f'(z) = 2^z \log 2$ and thus $S_{f'} \cap S_f = \emptyset$ (since $\log 2$ is transcendental). This fact motives a more general definition where multiplicities are included: let $f$ be an entire function. We define *the exceptional set with multiplicity of* $f$ to be

$$M_f = \{(\alpha, t) \in \overline{\mathbb{Q}} \times \mathbb{N}_0 \ : \ f^{(t)}(\alpha) \in \overline{\mathbb{Q}}\}$$

**Example 5.** *If* $f(z) = e^z + \sum 10^{-n!}$*,* $g(z) = e^z + e^{z+1}$ *and* $h(z) = e^z$*, then* $M_f = \{0\} \times \mathbb{N}$*,* $M_g = \emptyset$ *and* $M_h = \{0\} \times \mathbb{N}_0$.

A relation between $S_f$ and $M_f$ is given in the next result

**Proposition 1.** *If* $M_f = A \times N$*, then* $S_{f^{(t)}} = A$ *for all* $t \in N$.

*Proof.* If $t \in N$ and $\alpha \in \overline{\mathbb{Q}}$, then $\alpha \in S_{f^{(t)}}$, if and only if $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$. Since that $M_f = A \times N$ and $t \in N$, then $f^{(t)}(\alpha) \in \overline{\mathbb{Q}}$ if and only if $\alpha \in A$. □

In view of the previous proposition, we can restate our question: what are the possible subsets of $\overline{\mathbb{Q}} \times \mathbb{N}_0$ which are exceptional sets with multiplicity of a transcendental function?

How about the previous question where we replace transcendental fucntions by hypertranscendental functions? Recall that by a hypertranscendental function, we

[1] A function built from a finite number of exponentials, logarithms, constants, one variable, and $n$th roots through composition and combinations using the four elementary operations $(+, -, \times, \div)$. By allowing these functions (and constants) to be complex numbers, trigonometric functions and their inverses become included in the elementary functions

mean a function which does not satisfy any algebraic differential equations. Clearly, hypertranscendental functions are transcendental. The exponential function $e^z$ gives an example of a transcendental function which is not hypertranscendental and the well-known zeta ($\zeta(z)$) and gamma function ($\Gamma(z)$) are hypertranscendental, see [16]. Moreover (see [16]), sums, products, differences, quotients and compositions of hypotranscendental functions are again hypotranscendental, e.g., the function $\sin(e^{e^{1/z}} - 2^{\pi \log z})$ is hypotranscendental.

In view of that, we note that all the previous functions $f_i$, with $i \in [1, 7]$ are hypotranscendental. Hence it arises a very stronger question: what are the possible exceptional sets with multiplicity of hypertranscendental functions?

All this mistery finishes by the Theorem 5: every $A \times N \subseteq \overline{\mathbb{Q}} \times \mathbb{N}_0$ is the exceptional set with multiplicity of uncountable many hypertranscendental entire functions with order of growth as small as we wish. In particular, when $N = \mathbb{N}_0$, $A \subseteq \overline{\mathbb{Q}}$, Theorem 5 and Proposition 1 yield

**Corollary 4.** *If $A \subseteq \overline{\mathbb{Q}}$, then there is an uncountable set, $\mathcal{F}_A$, of hypertranscendental entire functions such that, if $f \in \mathcal{F}_A$, then*

$$S_{f^{(t)}} = A \text{ for } t \geq 0$$

*Moreover, the set*

$$\{f^{(n)}(\alpha) \ : \ \alpha \in \overline{\mathbb{Q}} \backslash A, \ n \geq 0 \text{ and } f \in \mathcal{F}_A\} \tag{4.1}$$

*is algebraically independent.*

Thus, all that remains is to prove the Theorem 5.

4.2. **Proof of the Theorem 5.** Suppose that $A$, $\overline{\mathbb{Q}} \backslash A$, $N$ and $\mathbb{N}_0 \backslash N$ are all infinite sets, thus we can enumerate $\overline{\mathbb{Q}} = \{\alpha_0, \alpha_1, \ldots\}$ and $\mathbb{N}_0 = \{s_0, s_1, \ldots\}$ where $A = \{\alpha_2, \alpha_4, ..., \alpha_{2n}, ...\}$ and $N = \{s_2, s_4, ..., s_{2n}, ...\}$. Consider $\{T_{\lambda,m,l} : \lambda \in \Lambda \text{ and } (m, l) \in \mathbb{N}_0 \times \mathbb{N}_0\}$ an uncountable set and algebraically independent and set $A_{\lambda,m,l} = \{\gamma T_{\lambda,m,l} : \gamma \in \overline{\mathbb{Q}} \backslash \{0\}\}$ a dense subset of $\mathbb{C}$. For $\lambda \in \Lambda$, define

$$E^{\lambda}_{\alpha_n, s_k} = \begin{cases} \mathbb{Q}(i), & \text{se} \quad (n, k) \in (2\mathbb{Z})^2 \\ A_{\lambda,n,k}, & \text{se} \quad (n, k) \notin (2\mathbb{Z})^2 \end{cases}$$

Now by Theorem 4, there exists an uncountable set $\mathcal{F}_\lambda$ of hypertranscendental entire functions $f$ with $f^{(l_{2k})}(\alpha_{2m}) \in \mathbb{Q}(i)$ and $f^{(l)}(\alpha_m) \in A_{\lambda,m,l}$, for each $(\alpha_m, l) \notin A \times N$. Therefore it is plain that $M_f = A \times N$. For all $\lambda \in \Lambda$, we take only one function $f_\lambda \in \mathcal{F}_\lambda$. Set $\mathcal{F}_{A,N} = \{f_\lambda\}_{\lambda \in \Lambda}$, so $M_{f_\lambda} = A \times N$ for all $\lambda \in \Lambda$. Also, for all pairwise distincts ternaries $(\lambda_1, \alpha_{n_1}, t_1), ..., (\lambda_k, \alpha_{n_k}, t_k)$ with $(\alpha, t)'s \notin A \times N$ and $\lambda' s \in \Lambda$, the numbers $f^{(t_1)}_{\lambda_1}(\alpha_{n_1}), ..., f^{(t_k)}_{\lambda_k}(\alpha_{n_k})$ lie respectively in $A_{\lambda_1,n_1,t_1}, ..., A_{\lambda_k,n_k,t_k}$ hence they are algebraically independent.

For the case that $A$ is finite, we can suppose $A = \{\alpha_1, ..., \alpha_m\}$. Take $E^{\lambda}_{\alpha_k, s_{2l}} = \mathbb{Q}(i)$ for any $k \in [1, m]$ and any $l \geq 0$, denote $E^{\lambda}_{\alpha_k, l} = A_{\lambda,k,l}$ for each $(\alpha_k, l) \in \overline{\mathbb{Q}} \times \mathbb{N}_0 \setminus A \times N$. Then for this case we proceed as in the proof above. The other possibilities are solved of the same way. □

Returning to the exceptional sets, we still have the following last corollary

**Corollary 5.** *Let $P(z_1, ..., z_n)$ be a non-constant polynomial with algebraic coefficients. If $f_1, ..., f_n \in \bigcup_{s \geq 0} \mathcal{F}^{(s)}_A$, then*

$$S_{P(f_1,...,f_n)} = A \tag{4.2}$$

**Proof** In the case $A = \overline{\mathbb{Q}}$ the result follows easily. If there is $\alpha \in \overline{\mathbb{Q}}\backslash A$, then by (4.1) the numbers $f_1(\alpha), ...., f_n(\alpha)$ are algebraically independent, therefore $P(f_1, ..., f_n)(\alpha) \in \overline{\mathbb{Q}}$ if and only if $\alpha \in A$. In other words $S_{P(f_1,...,f_n)} = A$. $\square$

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, BRASÍLIA, DF, BRAZIL
*E-mail address*: diego@mat.unb.br