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The Fibonacci version of the Brocard-Ramanujan Diophantine equation

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Abstract. In this note, we prove that the Fibonacci version of the Brocard-Ramanujan Diophantine equation $n!+1=m^2$, that is, $F_n\cdots F_1+1=F_m^2$, has no solution in positive integers n,m.

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1. Introduction

In 1876, Brocard [3] and independently Ramanujan [10],[11, p. 327], in 1913, posed the problem of finding all integral solutions of the Diophantine equation

$$n!+1=m^2 \tag{1}$$

which is then known as *Brocard-Ramanujan Diophantine equation.*

The only known solutions to (1) are $(n,m)\in\{(4,5),(5,11),(7,71)\}$. In 1906, Gérardin [6] claimed that, if $m>71$, then m must have at least 20 digits. Gupta [7] stated that calculations of $n!$ up to $n=63$ gave no further solutions. Recently, Berndt and Galway [1] did not find further solutions up to $n=10^9$. We also point out the existence of several variants for this equation, for instance, see [5] and the very recent paper [8].

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* given by $F_0=0,\ F_1=1$ and $F_{n+2}=F_{n+1}+F_n$, for $n\geq 0$. The first few terms are $0,1,1,2,3,5,8,13,21,\ldots$.

In this note, we shall prove the unsolubility of the Fibonacci version of the Brocard-Ramanujan equation, where in equation (1) we replace m,n with their respective Fibonacci numbers and we use the usual notation $n_F!=F_n\cdots F_1$. Actually, our more general result is the following

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Theorem 1.1. *The Diophantine equation*

$$F_nF_{n+1}\cdots F_{n+k-1}+1=F_m^2 \tag{2}$$

has no solution in positive integers n, m, k.

We point out that Luca and Shorey [9] proved, in particular, that if t is any fixed rational number which is not a perfect power of a different rational number, then the equation

$$F_nF_{n+1}\cdots F_{n+k-1}+t=y^m$$

has only finitely many integer solutions $n,\ k,\ y,\ m\geq 2$. However this does not apply to (2) since $t=1$ is a perfect power.

2. The proof of Theorem

2.1. Auxiliary results. Before proceeding further, some results will be needed in order to prove the Theorem.

A *primitive divisor* p of F_n is a prime factor of F_n which does not divide $\prod_{j=1}^{n-1}F_j$. It is known that a primitive divisor p of F_n exists whenever $n\geq 13$. The above statement is usually referred to as the *Primitive Divisor Theorem* (see [2] for the more general version).

The sequence of the *Lucas numbers* is defined by $L_{n+1}=L_n+L_{n-1}$, with $L_0=2$ and $L_1=1$. Let us state some interesting and helpful facts which will be essential ingredients in the proof of Theorem 1.1.

For all $n\geq 1$, we have

(L1) $F_{2n}=F_nL_n$;

(L2) (Binet's formulae) If $\alpha=(1+\sqrt{5})/2$ and $\beta=(1-\sqrt{5})/2$, then

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}\quad\text{and}\quad L_n=\alpha^n+\beta^n.$$

The proof of these properties are well-known and can be found in [12, Chapter 1].

The equation $F_n+1=y^2$ and more generally $F_n\pm 1=y^\ell$ with integer y and $\ell\geq 2$ have been solved in [13] and [4], respectively. The solution for the last equation makes appeal to Fibonacci and Lucas numbers with negative indices which are defined as follows: let $F_n=F_{n+2}-F_{n+1}$ and $L_n=L_{n+2}-L_{n+1}$. Thus, for example, $F_{-1}=1, F_{-2}=-1$, and so on. Bugeaud et al [4, Section 5] used these numbers to give factorizations for $F_m\pm 1$. Let us sketch their method for the convenience of the reader.

Since that the Binet's formulae remain valid for Fibonacci and Lucas numbers with negative indices, one can deduce the following result.

Lemma 2.1. *For any integers a, b, we have*

$$F_a L_b = F_{a+b} + (-1)^b F_{a-b}.$$

Proof. The identity $\alpha = (-\beta)^{-1}$ leads to

$$F_a L_b = \frac{\alpha^a - \beta^a}{\alpha - \beta}(\alpha^b + \beta^b) = F_{a+b} + \frac{\alpha^a\beta^b - \beta^a\alpha^b}{\alpha - \beta} = F_{a+b} + (-1)^b F_{a-b}.$$

□

Lemma 2.1 gives immediately the following factorizations for $F_n \pm 1$, depending on the class of n modulo 4:

$$\begin{array}{rcl} F_{4k}+1 = F_{2k-1}L_{2k+1} & ; & F_{4k}-1 = F_{2k+1}L_{2k-1} \\ F_{4k+1}+1 = F_{2k+1}L_{2k} & ; & F_{4k+1}-1 = F_{2k}L_{2k+1} \\ F_{4k+2}+1 = F_{2k+2}L_{2k} & ; & F_{4k+2}-1 = F_{2k}L_{2k+2} \\ F_{4k+3}+1 = F_{2k+1}L_{2k+2} & ; & F_{4k+3}-1 = F_{2k+2}L_{2k+1} \end{array} \tag{3}$$

Now, we are ready to deal with the proof of the theorem.

2.2. The proof. The equation (2) can be rewritten as

$$F_n \cdots F_{n+k-1} = (F_m - 1)(F_m + 1).$$

By the relations in (3), we have that $F_n \cdots F_{n+k-1} = (F_m - 1)(F_m + 1) = F_a F_b L_c L_d$, where $a < b$, $c < d$ are close to $m/2$. In fact, each of $2a, 2b, 2c, 2d$ is in $\{m-2, m-1, m+1, m+2\}$. By (L1), we have $L_s = F_{2s}/F_s$ and our equation becomes

$$F_n \cdots F_{n+k-1} F_c F_d = F_a F_b F_{2c} F_{2d} \tag{4}$$

A quick computation reveals that we can assume that $n + k - 2 > 12$. Indeed, $F_\ell \cdots F_1 + 1$ is prime for $\ell = 1, ..., 8$ and

$$\begin{array}{rcl} F_9 \cdots F_1 + 1 & = & 599 \cdot 3719, \\ F_{10} \cdots F_1 + 1 & = & 1373 \cdot 89237, \\ F_{11} \cdots F_1 + 1 & = & 181 \cdot 60245821, \\ F_{12} \cdots F_1 + 1 & = & 631 \cdot 2488505671, \end{array}$$

which clearly are not perfect squares. Now, if we assume that $m > 14$, then $2c > \max\{12, b, d\}$. Thus in the right hand side of (4), we have a product of Fibonacci numbers with the largest two being of indices $2c, 2d$ both larger than d. By the Primitive Divisor Theorem, these two indices should be the largest ones in the left hand side also, but these are the consecutive (hence, not both even), indices $n + k - 2$, $n + k - 1$. This is a contradiction. □

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