ON THE SPACING BETWEEN TERMS OF k-GENERALIZED FIBONACCI SEQUENCES

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ABSTRACT. For $k \geq 2$, the k-generalized Fibonacci sequence $(F_n^{(k)})_n$ is defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and such that each term afterwards is the sum of the k preceding terms. In this paper, we will prove that the number of solutions of the Diophantine equation $F_m^{(k)} - F_n^{(\ell)} = c > 0$ (under weak assumptions) is bounded by an effectively computable constant depending only on c.

1. INTRODUCTION

The problem of studying the spacing between terms of some sequences has attracted the attention of mathematicians for decades. For instance, the equation related to the spacing between perfect powers, is so-called as *Pillai's equation*:

$$(1.1) \qquad m^k - n^\ell = c,$$

for a previously fixed positive constant c. The *Pillai's conjecture* [10] is that for any given $c \geq 1$, the number of positive integer solutions to the Diophantine equation (1.1), with $\min\{k, \ell\} \geq 2$, is finite. To the best of our knowledge, this conjecture remains open (there are several related results, some of them are ineffective, see the nice survey [11]).

We recall that the particular case $c = 1$, was already considered by E. Catalan who, in 1844, conjectured that the only consecutive perfect powers are 8 and 9. Recently, this conjecture was confirmed by P. Mihăilescu [9]. We refer the reader to [1] for a better discussion on this subject.

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [6] together with its very extensive annotated bibliography for additional references and history). It is a simple matter to deduce that if $F_n \neq F_m$, then

$$|F_m - F_n| > \left(\frac{1 + \sqrt{5}}{2}\right)^{\max\{m,n\}-3}.$$

There are several generalizations of Fibonacci numbers in the literature. For instance, the *Fibonomial coefficient* is defined, for $1 \leq k \leq m$, as

$$(1.2) \qquad \begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m \cdots F_{m-k+1}}{F_k \cdots F_1}.$$

Clearly, $(\left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]_F)_m$ is the Fibonacci sequence. In 2010, Luca, Marques and Stănică [7] studied the spacing between Fibonomial coefficients. In particular, they proved that the difference

$$\left| \begin{bmatrix} m \\ k \end{bmatrix}_F - \begin{bmatrix} n \\ \ell \end{bmatrix}_F \right|$$

tends to infinity when (m, k, n, ℓ) are such that $1 \leq k \leq m/2$, $1 \leq \ell \leq n/2$, $(m, k) \neq (n, \ell)$ and $\max\{m, n\}$ tends to infinity in an effective way.

Another known generalization is, for $k \geq 2$, the k-generalized Fibonacci sequence $F^{(k)} := (F_n^{(k)})_{n \geq -(k-2)}$, which is the sequence whose terms satisfy the k-th order recurrence relation

$$(1.3) \qquad F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)},$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$. Clearly for $k = 2$, we obtain the Fibonacci numbers $F_n^{(2)} = F_n$, and for $k = 3$, the Tribonacci numbers $F_n^{(3)} = T_n$.

The aim of this paper is to prove a related result (in the spirit of Pillai) about the spacing between terms of distinct k-generalized Fibonacci sequences. That is, to study the Diophantine equation

$$(1.4) \qquad F_m^{(k)} - F_n^{(\ell)} = c.$$

This equation could be considered as a "Fibonacci version" of Pillai's equation (where, we replace the powers ℓ and k by the respective order of a generalized Fibonacci sequence, that is, by superscripts (ℓ) and (k)). More precisely, our main results are the following

Theorem 1.1. *Let c be a positive integer number. Then, there exists an effectively computable constant $M = M(c)$ such that if (m, n, ℓ, k) is a positive integer solution of Eq. (1.4), with $\ell > k \geq 2$, $n > \ell + 2$ and $m > k + 2$, then $\max\{m, n, \ell, k\} < M$. A suitable choice for M is*

$$(1.5) \qquad M := \max\{c_1, 1.9 \cdot 10^{146} c_2^{24} \log^{27} c_2, 8 \cdot 10^{246}\},$$

where $c_1 := 5\log(c+1) + 2$ e $c_2 := 4\log(c+5)/\log 2$.

Note that Theorem 1.1 implies, in particular, that the difference

$$\left| F_m^{(k)} - F_n^{(\ell)} \right|$$

tends to infinity when (m, n, ℓ, k) are such that $m > k + 2$, $n > \ell + 2$, $\ell > k > 1$ and $\max\{m, n\}$ tends to infinity in an effective way.

As another application of the method, we solve completely the case $c = 1$ ("Catalan-Fibonacci" version), that is, we find all consecutive numbers among $\bigcup_{k \geq 2} F^{(k)}$.

Theorem 1.2. *The only solution of Diophantine equation*

$$(1.6) \qquad |F_m^{(k)} - F_n^{(\ell)}| = 1,$$

with $\ell > k \geq 2$, $n > \ell + 2$ and $m > k + 2$ is $(m, n, \ell, k) = (10, 8, 4, 2)$. That is,

$$F_8^{(4)} - F_{10}^{(2)} = 56 - 55 = 1.$$

We remark that the hypotheses $n > \ell + 2$ and $m > k + 2$ are necessary to avoid the trivial solutions

$$(m, n, \ell, k) = (k + 2, k + 2, k + 1, k),$$

for all $k \geq 2$.

Let us give a brief overview of our strategy for proving Theorem 1.1. First, we use a Dresden formula [5, Formula (2)] to get an upper bound for a linear form in three logarithms related to equation (1.4). After, we use a lower bound due to Matveev to obtain an upper bound for m and n in terms of ℓ. Very recently, Bravo and Luca solved the equation $F_n^{(k)} = 2^m$ and for that they used a nice argument combining some estimates together with the Mean Value Theorem (this can be seen in pages 72 and 73 of [2]). In our case, we must use two times this Bravo and Luca approach together with a reduction argument due to Dujella and Pethö to prove our main theorem. In the final section, we present a program for checking the "small" cases. The computations in the paper were performed using *Mathematica*®.

We remark some differences between our work and the one by Bravo and Luca. In their paper, the equation $F_n^{(k)} = 2^m$ was studied. By applying a key method, they get directly an upper bound for $|2^m - 2^{n-2}|$. In our case, the equation $F_m^{(k)} - F_n^{(\ell)} = c$ needs a little more work, because it is necessary to apply two times their method to get an upper bound for $|2^{n-2} - 2^{m-2}|$. Moreover, they used a reduction argument due to Dujella and Pethö to solve all small cases. In our work, we use a fast *Mathematica* routine to deal with the "very" small cases.

2. Auxiliary results

In order to avoid unnecessary repetitions, throughout the paper the integers m, n, k, ℓ are supposed to satisfy the conditions in the statement of Theorem 1.1. First, we claim that if (m, n, ℓ, k) is a solution of equation (1.4), then $n < m$. In fact, to obtain a contradiction, suppose that $m \leq n$. Thus, by using that the sequences $(F_n^{(\ell)})_n$ and $(F_n^{(\ell)})_\ell$ are nondecreasing together with (1.4), we obtain $F_m^{(k)} \leq F_n^{(\ell)}$ yielding

$$0 < c = F_m^{(k)} - F_n^{(\ell)} \leq 0.$$

This absurdity gives $n < m$ as desired.

Before proceeding further, we shall recall some facts and properties of these sequences which will be used after.

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and 2 (as can be seen in [12]). Also, in a recent paper, G. Dresden [5, Theorem 1] gave a simplified "Binet-like" formula for $F_n^{(k)}$:

$$(2.1) \qquad F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1},$$

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. Also, it was proved in [3, Lemma 1] that

$$(2.2) \qquad \alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \text{ for all } n \geq 1,$$

where α is the dominant root of $\psi_k(x)$. Also, the contribution of the roots inside the unit circle in formula (2.1) is almost trivial. More precisely, it was proved in [5] that

$$(2.3) \qquad |F_n^{(k)} - g(\alpha, k)\alpha^{n-1}| < \frac{1}{2},$$

where we adopt throughout the notation $g(x, y) := (x-1)/(2 + (y+1)(x-2))$.

Now, we wish to find a lower bound for m in terms of n. In fact, by (1.4) and (2.2),

$$(2.4) \qquad 2^{n-1} > \phi^{n-1} \geq F_n^{(\ell)} = F_m^{(k)} + 1 > \alpha^{m-2} > (\sqrt{2})^{m-2} \text{ and so } 2n > m,$$

where in the last inequality, we used that $\alpha > 3/2 > \sqrt{2}$.

As another tool to prove Theorem 1.1, we still use a lower bound for a linear form logarithms *à la Baker* and such a bound was given by the following result of Matveev [8].

Lemma 2.1. *Let $\gamma_1, \ldots, \gamma_t$ be real algebraic numbers and let $b_1, \ldots, b_t$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_t)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\} \text{ for } j = 1, \ldots, t.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

As usual, in the above statement, the *logarithmic height* of an s-degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{s}\left(\log |a| + \sum_{j=1}^{s} \log \max\{1, |\gamma^{(j)}|\}\right),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1 \leq j \leq s}$ are the conjugates of α (over $\mathbb{Q}$).

After finding an upper bound on n which is general too large, the next step is to reduce it. For that, our last ingredient is a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethő [4, Lemma 5 (a)]. For a real number x, we use $\| x \| = \min\{|x - n| : n \in \mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 2.2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and let A, B be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon = \| \mu q \| - M \| \gamma q \|$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < A \cdot B^{-k}$$

in positive integers m, n and k with

$$m \leq M \text{ and } k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

See Lemma 5, a.) in [4].

3. The proof of Theorem 1.1

Note that in order to prove the Theorem 1.1, it suffices to show that Eq. (1.4) has no solution when $m > M$ (with M defined as in (1.5)). Thus suppose, towards a contradiction, that (m, n, ℓ, k) is a solution of Eq. (1.4) with $m > M$.

The first step is to find an upper bound for m (and so for n) in terms of ℓ.

For that, we use (2.3) to get

$$|F_m^{(k)} - g(\alpha, k)\alpha^{m-1}| < \frac{1}{2} \text{ and } |F_n^{(\ell)} - g(\phi, \ell)\phi^{n-1}| < \frac{1}{2},$$

where α and ϕ are the dominant roots of the recurrences $(F_m^{(k)})_m$ and $(F_n^{(\ell)})_n$, respectively. Combining these inequalities together with $|F_n^{(\ell)} - F_m^{(k)}| = c$, we obtain

$$(3.1) \qquad |g(\phi, \ell)\phi^{n-1} - g(\alpha, k)\alpha^{m-1}| < c + 1$$

and so

$$(3.2) \qquad \left| \frac{g(\phi, \ell)\phi^{n-1}}{g(\alpha, k)\alpha^{m-1}} - 1 \right| < \frac{c+1}{g(\alpha, k)\alpha^{m-1}} < \frac{4(c+1)}{\alpha^{m-1}} < \frac{1}{\alpha^{m/2}},$$

where we used the facts that $\alpha^{(m-2)/2} > 4(c+1)$ (since $m > c_1$) and $g(\alpha, k) > 1/4$ (since $\alpha > 3/2$, for $k \geq 2$, and $2 + (k+1)(\alpha - 2) < 2$). Thus (3.2) becomes

$$(3.3) \qquad |e^\Lambda - 1| < \frac{1}{\alpha^{m/2}},$$

where $\Lambda := (n-1)\log \phi + \log(g(\phi, \ell)/g(\alpha, k)) - (m-1)\log \alpha$.

Now, we shall apply Lemma 2.1. To this end, take $t := 3$,

$$\gamma_1 := \phi, \ \gamma_2 := \frac{g(\phi, \ell)}{g(\alpha, k)}, \ \gamma_3 := \alpha$$

and

$$b_1 := n - 1, \ b_2 := 1, \ b_3 := m - 1.$$

For this choice, we have $D = [\mathbb{Q}(\alpha, \phi) : \mathbb{Q}] \leq k\ell < \ell^2$. Also $h(\gamma_1) = (\log \phi)/\ell < (\log 2)/\ell < 0.7/\ell$ and similarly $h(\gamma_3) < 0.7/k$. In [2, p. 73], an estimate for $h(g(\alpha, k))$ was given. More precisely, it was proved that

$$h(g(\alpha, k)) < \log(k+1) + \log 4.$$

Analogously,

$$h(g(\phi, \ell)) < \log(\ell+1) + \log 4.$$

Thus

$$h(\gamma_2) \leq h(g(\phi, \ell)) + h(g(\alpha, k)) \leq \log(\ell+1) + \log(k+1) + 2\log 4,$$

where we used the well-known facts that $h(xy) \leq h(x) + h(y)$ and $h(x) = h(x^{-1})$. Also, in [2] was proved that $|g(\alpha_i, k)| < 2$, for all $i = 1, \ldots, k$.

Since $\ell > k$ and $m > n$, we can take $A_1 = A_3 := 0.7\ell$, $A_2 := 2\ell^2 \log(4\ell + 4)$ and $B := m - 1$.

Before applying Lemma 2.1, it remains us to prove that $e^\Lambda \neq 1$. Suppose, towards a contradiction, the contrary, i.e., $g(\alpha, k)\alpha^{m-1} = g(\phi, \ell)\phi^{n-1} \in \mathbb{Q}(\phi)$. So, we can conjugate this relation in $\mathbb{Q}(\phi)$ to get

$$g(\alpha_{s_i}, k)\alpha_{s_i}^{m-1} = g(\phi_i, \ell)\phi_i^{n-1}, \text{ for } i = 1, \ldots, \ell,$$

where α_{s_i} are the ℓ conjugates of α over $\mathbb{Q}(\phi)$. Since $g(\alpha, k)\alpha^{m-1}$ has at most k conjugates (over $\mathbb{Q}$), then each number in the list $\{g(\alpha_{s_i}, k)\alpha_{s_i}^{m-1} : 1 \le i \le \ell\}$ is repeated at least $\ell/k > 1$ times. In particular, there exists $t \in \{2, \ldots, \ell\}$, such that $g(\alpha_{s_1}, k)\alpha_{s_1}^{m-1} = g(\alpha_{s_t}, k)\alpha_{s_t}^{m-1}$. Thus, $g(\phi, \ell)\phi^{n-1} = g(\phi_t, \ell)\phi_t^{n-1}$ and then

$$\left(\frac{7}{4}\right)^{n-1} < \phi^{n-1} = \left|\frac{g(\phi_t, \ell)}{g(\phi, \ell)}\right| |\phi_t|^{n-1} < 8,$$

where we used that $\phi > 2(1 - 2^{-\ell}) \ge 7/4$, $|g(\phi_t, \ell)| < 2 < 8|g(\phi, \ell)|$ and $|\phi_t| < 1$ for $t > 1$. However, the inequality $(7/4)^{n-1} < 8$ holds only for $n = 1, 2, 3, 4$, but this gives an absurdity, since $n > \ell + 1 \ge 3 + 1 = 4$. Therefore $e^\Lambda \ne 1$.

Now, the conditions to apply Lemma 2.1 are fulfilled and hence

$$|e^\Lambda - 1| > \exp(-1.5 \cdot 10^{11} \ell^8 (1 + 2\log \ell) \log(4\ell + 4)(1 + \log(m-1)))$$

Since, $1 + 2\log \ell \le 3\log \ell$, $4\ell + 4 < \ell^{2.6}$ (for $\ell \ge 3$) and $m - 1 < m^{1.1}$, we have that

$$(3.4) \qquad |e^\Lambda - 1| > \exp(-2.64 \cdot 10^{12} \ell^8 \log^2 \ell \log m))$$

By combining (3.3) and (3.4), we get

$$\frac{m}{\log m} < 1.33 \cdot 10^{13} \ell^8 \log^2 \ell,$$

where we used that $\log \alpha > 0.4$. Since the function $x/\log x$ is increasing for $x > e$, it is a simple matter to prove that

$$(3.5) \qquad \frac{x}{\log x} < A \text{ implies that } x < 2A\log A.$$

A proof for that can be found in [2, p. 74].

Thus, by using (3.5) for $x := m$ and $A := 1.33 \cdot 10^{13} \ell^8 \log^2 \ell$, we have that

$$m < 2(1.33 \cdot 10^{13} \ell^8 \log^2 \ell)\log(1.33 \cdot 10^{13} \ell^8 \log^2 \ell).$$

Now, the inequality $31 + 2\log\log \ell < 29\log \ell$, for $\ell \ge 3$, yields

$$\log(1.33 \cdot 10^{13} \ell^8 \log^2 \ell) < 31 + 8\log \ell + 2\log\log \ell < 37\log \ell.$$

Therefore

$$(3.6) \qquad m < 9.9 \cdot 10^{14} \ell^8 \log^3 \ell.$$

The next step is to find an upper bound for ℓ in terms of k. For that, consider $\ell \le 240$, then the inequality (3.6) yields $m < 1.8 \cdot 10^{36}$, contradiction with the fact that $m > M$. Thus, we may assume that $\ell > 240$. Therefore

$$(3.7) \qquad n < 9.9 \cdot 10^{14} \ell^8 \log^3 \ell < 2^{\ell/2},$$

where we used (3.6) and the fact that $n < m$. Now, we shall use a key argument due to Bravo and Luca [2, p. 72-73]. However, for the sake of completeness and because one needs a slight modification in its final part, we shall present their nice idea.

Setting $\lambda = 2 - \phi$, we deduce that $0 < \lambda < 1/2^{\ell-1}$ (because $2(1 - 2^{-\ell}) < \phi < 2$). So

$$\phi^{n-1} = (2 - \lambda)^{n-1} = 2^{n-1}\left(1 - \frac{\lambda}{2}\right)^{n-1} > 2^{n-1}(1 - (n-1)\lambda),$$

since that the inequality $(1-x)^n > 1 - 2nx$ holds for all $n \geq 1$ and $0 < x < 1$. Moreover, $(n-1)\lambda < 2^{\ell/2}/2^{\ell-1} = 2/2^{\ell/2}$ and hence

$$2^{n-1} - \frac{2^n}{2^{\ell/2}} < \phi^{n-1} < 2^{n-1} + \frac{2^n}{2^{\ell/2}},$$

yielding

(3.8)
$$|\phi^{n-1} - 2^{n-1}| < \frac{2^n}{2^{\ell/2}}.$$

Now, we define for $x > 2(1 - 2^{-\ell})$ the function $f(x) := g(x, \ell)$ which is differentiable in the interval $[\phi, 2]$. So, by the Mean Value Theorem, there exists $\xi \in (\phi, 2)$, such that $f(\phi) - f(2) = f'(\xi)(\phi - 2)$. Thus

(3.9)
$$|f(\phi) - f(2)| < \frac{2\ell}{2^\ell},$$

where we used the bounds $|\phi - 2| < 1/2^{\ell-1}$ and $|f'(\xi)| < \ell$. For simplicity, we denote $\delta = \phi^{n-1} - 2^{n-1}$ and $\eta = f(\phi) - f(2) = f(\phi) - 1/2$. After some calculations, we arrive at

$$2^{n-2} = f(\phi)\phi^{n-1} - 2^{n-1}\eta - \frac{\delta}{2} - \delta\eta.$$

Therefore

$$|2^{n-2} - g(\alpha, k)\alpha^{m-1}| \leq |f(\phi)\phi^{n-1} - g(\alpha, k)\alpha^{m-1}| + 2^{n-1}|\eta| + \left|\frac{\delta}{2}\right| + |\delta\eta|$$

$$\leq c + 1 + \frac{2^n\ell}{2^\ell} + \frac{2^{n-1}}{2^{\ell/2}} + \frac{2^{n+1}\ell}{2^{3\ell/2}},$$

where we used (3.8) and (3.9). Since $n > \ell + 2$, one has that $1 < 2^{n-2}/2^{\ell/2}$ and we rewrite the above inequality as

$$|2^{n-2} - g(\alpha, k)\alpha^{m-1}| < (c+1)\frac{2^{n-2}}{2^{\ell/2}} + \left(\frac{4\ell}{2^{\ell/2}}\right)\frac{2^{n-2}}{2^{\ell/2}} + 2 \cdot \frac{2^{n-2}}{2^{\ell/2}} + \left(\frac{8\ell}{2^\ell}\right)\frac{2^{n-2}}{2^{\ell/2}}.$$

Since the inequalities $4\ell < 8\ell < 2^{\ell/2} < 2^\ell$ hold for all $\ell > 240$ (actually, they hold for $\ell > 13$), then

(3.10)
$$|2^{n-2} - g(\alpha, k)\alpha^{m-1}| < \frac{(c+5) \cdot 2^{n-2}}{2^{\ell/2}} < \frac{2^{n-2}}{2^{\ell/4}},$$

where we used that $2^{\ell/4} > c + 5$. This follows because $\ell > c_2$ (in fact, on the contrary, we can use (3.6) to get $M < m < 9.9 \cdot 10^{14}c_2^8\log^3 c_2$).

Equivalently, we have

(3.11)
$$|1 - g(\alpha, k)\alpha^{m-1}2^{-(n-2)}| < \frac{1}{2^{\ell/4}}.$$

For applying Lemma 2.1, it remains us to prove that the left-hand side of (3.11) is nonzero, or equivalently, $2^{n-2} \neq g(\alpha, k)\alpha^{m-1}$. To obtain a contradiction, we suppose the contrary, i.e., $2^{n-2} = g(\alpha, k)\alpha^{m-1}$. By conjugating the previous relation in the splitting field of $\psi_k(x)$, we obtain $2^{n-2} = g(\alpha_i, k)\alpha_i^{m-1}$, for $i = 1, \ldots, k$. However, when $i > 1$, $|\alpha_i| < 1$ and $|g(\alpha_i, k)| < 2$. But this leads to the following absurdity

$$2^{n-2} = |g(\alpha_i, k)||\alpha_i|^{m-1} < 2,$$

since $n > 4$. Therefore $g(\alpha, k)\alpha^{m-1}2^{-(n-2)} \neq 1$ and then we are in position to apply Lemma 2.1. For that, take $t := 3$,

$$\gamma_1 := g(\alpha, k), \ \gamma_2 := \alpha, \ \gamma_3 := 2$$

and

$$b_1 := 1, \ b_2 := m - 1, \ b_3 := -(n - 2).$$

By some calculations made in Section 2, we see that $A_1 := k \log(4k + 4)$, $A_2 = A_3 := 0.7$ are suitable choices. Moreover $D = k$ and $B = m - 1$. Thus
(3.12)
$$|1 - g(\alpha, k)\alpha^{m-1}2^{-(n-2)}| > \exp(-D' \cdot k^3(1 + \log k)(1 + \log(m - 1)) \log(4k + 4)),$$

where we can take $D' = 0.75 \cdot 10^{11}$. Combining (3.11) and (3.12) together with a straightforward calculation, we get

(3.13) $$\ell < 2.16 \cdot 10^{12} k^3 \log^2 k \log m.$$

On the other hand, $m < 9.9 \cdot 10^{14} \ell^8 \log^3 \ell$ (by (3.6)) and so

(3.14) $$\log m < \log(9.9 \cdot 10^{14} \ell^8 \log^3 \ell) < 41 \log \ell,$$

where we used that $35 + 3 \log \log \ell < 33 \log \ell$. Turning back to inequality (3.13), we obtain

$$\frac{\ell}{\log \ell} < 8.9 \cdot 10^{13} k^3 \log^2 k$$

which implies, by (3.5), that

$$\ell < 2(8.9 \cdot 10^{13} k^3 \log^2 k) \log(8.9 \cdot 10^{13} k^3 \log^2 k).$$

Since $\log(8.9 \cdot 10^{13} k^3 \log^2 k) < 47 \log k$, we finally get the inequality

(3.15) $$\ell < 8.4 \cdot 10^{15} k^3 \log^3 k.$$

Now, if $k \leq 1640$, then $\ell < 2 \cdot 10^{28}$ (by (3.15)). Thus, by (3.6), one has that $m < 7.1 \cdot 10^{246}$ which is not possible, because $m > M$.

Therefore, we may suppose that $k > 1640$. The inequality $\ell < 8.9 \cdot 10^{15} k^3 \log^3 k$ together with (3.6) yield

$$\begin{aligned} m &< 9.9 \cdot 10^{14}(8.9 \cdot 10^{15} k^3 \log^3 k)^8 \log^3(8.9 \cdot 10^{15} k^3 \log^3 k) \\ &< 1.9 \cdot 10^{146} k^{24} \log^{27} k < 2^{k/2}, \end{aligned}$$

where the last inequality holds only because $k > 1640$. Now, we use again the key argument of Bravo and Luca to conclude that

(3.16) $$|2^{m-2} - g(\phi, \ell)\phi^{n-1}| < \frac{2^{m-2}}{2^{k/4}},$$

because $k > c_2$ (on the contrary, we substitute $k \leq c_2$ respectively in (3.15) and (3.6) to obtain an upper bound for m less than M). Combining (3.10), (3.16) and (3.1), we get

$$\begin{aligned} |2^{n-2} - 2^{m-2}| &\leq |2^{n-2} - g(\alpha, k)\alpha^{n-1}| + |g(\alpha, k)\alpha^{n-1} - g(\phi, \ell)\phi^{n-1}| \\ &\quad + |2^{m-2} - g(\phi, \ell)\phi^{n-1}| \\ &< \frac{2^{n-2}}{2^{\ell/4}} + c + 1 + \frac{2^{m-2}}{2^{k/4}} < \frac{3 \cdot 2^{m-2}}{2^{k/4}}, \end{aligned}$$

since $n < m$, $k < \ell$, $m > k + 1$ and $c + 1 < 2^{k/2}$ (on the contrary, $k \le 2\log(c + 1)/\log 2 < c_2$). Therefore

(3.17)
$$|2^{n-m} - 1| < \frac{3}{2^{k/4}}.$$

Since $n \le m - 1$, then
$$\frac{1}{2} \le 1 - 2^{n-m} = |2^{n-m} - 1| < \frac{3}{2^{k/4}}.$$

Thus $2^{k/4} < 6$ yielding $k \le 10$, but this leads to an absurdity, since $k > 1640$. With this contradiction, we complete the proof of Theorem 1.1. $\qquad\square$

4. The proof of Theorem 1.2

First, we claim that $n < m$. To derive a contradiction, suppose that $m \ge n$. Then Eq. (1.6) gives $F_n^{(\ell)} \le F_m^{(k)} + 1$. However $F_m^{(k)} + 1 < F_m^{(k+1)}$, for $m > k + 2$. In fact, since $(F_n^{(\ell)})_\ell$ is nondecreasing, then it suffices to prove this inequality for $m = k + 3$. This holds because
$$F_{k+3}^{(k+1)} = 2^{k+1} - 1 > 2^{k+1} - 2 = F_{k+3}^{(k)} + 1.$$
Thus, we obtain the following absurdity
$$F_n^{(\ell)} \le F_m^{(k)} + 1 < F_m^{(k+1)} \le F_n^{(\ell)},$$
where we used that the sequences $(F_n^{(\ell)})_n$ and $(F_n^{(\ell)})_\ell$ are nondecreasing. Therefore, $m > n$ as claimed and we can follow the proof of Theorem 1.1. Summarizing, the previous theorem (for $c = 1$) ensures that the possible solutions (m, n, k, ℓ) of Eq. (1.6) must satisfy
$$m < 8 \cdot 10^{246},$$
where we used that $c_1 < 5.47$, $c_2 < 2.74$ and then $1.9 \cdot 10^{146} c_2^{24} \log^{27} c_2 < 6.4 \cdot 10^{156}$.

Since this upper bound on $\max\{m, n, \ell, k\}$ is too large, we need to use Lemma 2.2.

We recall that $2 \le k \le 1640$, then $\ell < 2 \cdot 10^{28}$ and $n < m < 7.1 \cdot 10^{246}$. In order to use the Lemma 2.2, we rewrite (3.11) as
$$|e^\Theta - 1| < \frac{1}{2^{\ell/4}},$$
where $\Theta := (m - 1)\log\alpha - (n - 2)\log 2 + \log g(\alpha, k)$. Recall that we proved that $e^\Theta \ne 1$ (the paragraph below (3.11)) and so $\Theta \ne 0$.

If $\Theta > 0$, then $\Theta < e^\Theta - 1 < 1/2^{\ell/4}$. In the case of $\Theta < 0$, we use $1 - e^{-|\Theta|} = |e^\Theta - 1| < 1/2^{\ell/4}$ to get $e^{|\Theta|} < 1/(1 - 2^{-\ell/4})$. Thus
$$|\Theta| < e^{|\Theta|} - 1 < \frac{2^{-\ell/4}}{1 - 2^{-\ell/4}} < 2^{-\ell/4 + 1.5},$$
where we used that $1/(1 - 2^{-\ell/4}) < 2^{1.5}$, for $\ell \ge 3$. Summarizing, the further arguments work for $\Theta > 0$ and $\Theta < 0$ in a very similar way.

Thus, to avoid unnecessary repetitions we shall consider only the case $\Theta > 0$. For that, we have
$$0 < (m - 1)\log\alpha - (n - 2)\log 2 + \log g(\alpha, k) < (\sqrt[4]{2})^{-\ell}$$
and then

(4.1)
$$0 < (m - 1)\gamma_k - (n - 2) + \mu_k < 1.45 \cdot (\sqrt[4]{2})^{-\ell},$$

with $\gamma_k := \log \alpha^{(k)} / \log 2$ and $\mu_k := \log g(\alpha^{(k)}, k) / \log 2$. Here, we added the super-script to α for emphasizing its dependence on k.

We claim that γ_k is irrational, for any integer $k \geq 2$. In fact, if $\gamma_k = p/q$, for some positive integers p and q, we have that $2^p = (\alpha^{(k)})^q$ and as before we can conjugate this relation by some automorphism of the Galois group of the splitting field of $\psi_k(x)$ over $\mathbb{Q}$ to get $2^p < |(\alpha_i^{(k)})^q| < 1$, for $i > 1$, which is an absurdity, since $p \geq 1$. Let $q_{n,k}$ be the denominator of the n-th convergent of the continued fraction of γ_k. Taking $M_k := 1.9 \cdot 10^{146} k^{24} \log^{27} k \leq M_{1640} < 7.1 \cdot 10^{246}$, we use *Mathematica* to get

$$\min_{2 \leq k \leq 1640} q_{650,k} > 6 \cdot 10^{308} > 6 M_{1640}.$$

Also

$$\max_{2 \leq k \leq 1640} q_{650,k} < 2 \cdot 10^{1112}.$$

Define $\epsilon_k := \| \mu_k q_{650,k} \| - M_k \| \gamma_k q_{650,k} \|$, for $2 \leq k \leq 1640$, we get

$$\min_{2 \leq k \leq 1640} \epsilon_k > 5.2 \cdot 10^{-169}.$$

Note that the conditions to apply Lemma 2.2 are fulfilled for $A = 1.45$ and $B = \sqrt[4]{2}$, and hence there is no solution to inequality (4.1) (and then no solution to the Diophantine equation (1.4)) for m and ℓ satisfying

$$m < M_k < 7.1 \cdot 10^{246} \text{ and } \ell \geq \frac{\log(A q_{650,k} / \epsilon_k)}{\log B}.$$

Since $m < M_k$ (for $2 \leq k \leq 1640$), then

$$\ell < \frac{\log(A q_{650,k} / \epsilon_k)}{\log B} \leq \frac{\log(1.45 \cdot 2 \cdot 10^{1112} / 5.2 \cdot 10^{-169})}{\log \sqrt[4]{2}} < 17014.18\ldots.$$

Therefore $2 \leq k \leq 1640$ and $k < \ell \leq 17014$. Now, by applying (3.6), we obtain $n < m < 6.5 \cdot 10^{51}$.

To deal with these remaining cases, we prepared the following Mathematica routine

```
nn = 17014;
f = 2^(Range[nn] + 1) - 3;
f[[1]] = Infinity;
cnt = 0;
seq = Table[
  Join[2^(Range[i - 2] + 1), {2^i - 1}, {2^(i + 1) - 3}], {i, 1, nn}];
seq[[1]] = {1};
done = False;
While[! done, fMin = Min[f];
 pMin = Flatten[Position[f, fMin]];
 f[[pMin[[1]]]] = fMin + 1;
 sMin = Flatten[Position[f, fMin + 1]];
 If[Length[sMin] > 1, Print[{fMin + 1, sMin}]];
 Do[k = sMin[[1]];
  s = Plus @@ seq[[k]];
  seq[[k]] = RotateLeft[seq[[k]]];
  seq[[k, k]] = s;
  f[[k]] = s, {i, Length[sMin]}];
```

```
 cnt++;
 done = (fMin > 6.5*10^(51))]; cnt
```

It returns us $\{56, \{2, 4\}\}$ which corresponds to the only solution $(m, n, \ell, k) = (10, 8, 4, 2)$. The calculations in this paper took roughly 8 days on 2.5 GHz Intel Core i5 4GB Mac OSX. The proof is complete. □

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