

ON THE INTERSECTION OF TWO DISTINCT  
 $k$ -GENERALIZED FIBONACCI SEQUENCES

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(Received January 31, 2011)

*Abstract.* Let  $k \geq 2$  and define  $F^{(k)} := (F_n^{(k)})_{n \geq 0}$ , the  $k$ -generalized Fibonacci sequence whose terms satisfy the recurrence relation  $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \dots + F_{n-k}^{(k)}$ , with initial conditions  $0, 0, \dots, 0, 1$  ( $k$  terms) and such that the first nonzero term is  $F_1^{(k)} = 1$ . The sequences  $F := F^{(2)}$  and  $T := F^{(3)}$  are the known Fibonacci and Tribonacci sequences, respectively. In 2005, Noe and Post made a conjecture related to the possible solutions of the Diophantine equation  $F_n^{(k)} = F_m^{(l)}$ . In this note, we use transcendental tools to provide a general method for finding the intersections  $F^{(k)} \cap F^{(m)}$  which gives evidence supporting the Noe-Post conjecture. In particular, we prove that  $F \cap T = \{0, 1, 2, 13\}$ .

*Keywords:*  $k$ -generalized Fibonacci numbers, linear forms in logarithms, reduction method

*MSC 2010:* 11B39, 11D61, 11J86

## 1. INTRODUCTION

Several problems in number theory are actually questions about the intersection of two known sequences (or sets). Before giving examples, let us recall some terminology: let  $F := (F_n)_{n \geq 0}$  be the *Fibonacci sequence*,  $\mathbb{P} := \{p: p \text{ prime}\}$ ,  $\mathcal{P} := \{y^t: y, t \in \mathbb{Z}, t > 1\}$  (the perfect powers),  $\mathcal{F} := \{n!: n \in \mathbb{Z}, n \geq 0\}$ ,  $\mathcal{R} := \{a(10^n - 1)/9: 1 \leq a \leq 9, n \in \mathbb{Z}, n > 0\}$  (the *repdigits* or *unidigital numbers*). Below, we cite some results about the intersection of these sets:

- ▷ Erdős and Selfridge [8] proved that  $\mathcal{F} \cap \mathcal{P} = \{1\}$ .
- ▷ In 2000, Luca [25] proved that  $F \cap \mathcal{R} = \{0, 1, 2, 3, 5, 8, 55\}$ .
- ▷ Luca [26] also proved that  $F \cap \mathcal{F} = \{1, 2\}$ .
- ▷ In 2003, Bugeaud et al [4] showed that  $F \cap \mathcal{P} = \{0, 1, 8, 144\}$  (see [28] for a generalization).

▷ Let  $(a_n)_{n \geq 1}$  be the tower given by  $a_1 = 1$  and  $a_n = n^{a_{n-1}}$ , for  $n \geq 2$ . Luca and the author [27] proved that  $\{a_1 + \dots + a_n : n \geq 1\} \cap \mathcal{P} = \{1\}$ .

However, some related questions are still open problems, as for instance the sets  $\mathbb{P} \cap F$  and  $\mathbb{P} \cap \mathcal{R}$  are unknown.

Let  $k \geq 2$  and denote  $F^{(k)} := (F_n^{(k)})_{n \geq 0}$ , the *k-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$(1.1) \quad F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \dots + F_{n-k}^{(k)},$$

with initial conditions  $0, 0, \dots, 0, 1$  ( $k$  terms) and such that the first nonzero term is  $F_1^{(k)} = 1$ .

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k-step Fibonacci numbers*, the *Fibonacci k-sequence*, or *k-bonacci numbers*. Clearly, for  $k = 2$  we obtain the well-known Fibonacci numbers and for  $k = 3$ , Tribonacci numbers.

Recall that Tribonacci numbers have a long history. For the first time, they were studied in 1914 by Agronomoff [1] and subsequently by many others. The name Tribonacci was coined in 1963 by Feinberg [9]. The basic properties of Tribonacci numbers can be found in [18], [24], [36], [38]. For recent papers, we refer the reader to [3], [19], [20], [33] and to the collection [21], [22], [23].

Recently, Alekseyev [2] described how to compute the intersection of two Lucas sequences including the sequences of Fibonacci, Pell, Lucas and Lucas-Pell numbers. In general, we refer the reader to [34], [35], [37] for results on the intersection of two recurrence sequences.

In a very recent paper, Togbé and the author [29] proved that only finitely many terms of a linear recurrence sequence whose characteristic polynomial has a dominant root can be repdigits. As an application, since the characteristic polynomial of the recurrence in (1.1), namely  $x^k - x^{k-1} - \dots - x - 1$ , has just one root  $\alpha$  such that  $|\alpha| > 1$  (see for instance [39]), hence  $F^{(k)} \cap \mathcal{R}$  is a finite set, for all  $k \geq 2$ . See also the article [32] for some results on the set  $F^{(k)} \cap \mathbb{P}$  and a conjecture on the intersection  $F^{(k)} \cap F^{(m)}$ . We point out that this last intersection is, to the best of our knowledge, not known even in the easiest case  $(k, m) = (2, 3)$ , that is, for numbers that are both Fibonacci and Tribonacci. A possible way to find this intersection is to look at the Fibonacci and Tribonacci sequences modulo  $p^t$ , where  $p$  is a prime number. We refer the reader to [5], [13], [16], [17] for results of this nature. However, this approach seems to be hard to work in practice. This observation prompted the author to look for a more interesting and constructive approach which could be useful in the general case.

It is important to notice that Mignotte (see [31]) showed that if  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  are two linearly recurrence sequences then, under some weak technical as-

sumptions, the equation

$$u_n = v_m$$

has only finitely many solutions in positive integers  $m, n$ . Moreover, all such solutions are effectively computable. Therefore, it seems reasonable to think that  $F^{(k)} \cap F^{(m)}$  is a finite set for all  $k \neq m$ .

The goal of this paper is to apply transcendental tools to provide a method for studying the intersection  $F^{(k)} \cap F^{(m)}$ , for integers  $2 \leq k < m$  and determine completely this set for  $(k, m) = (2, 3)$  (confirming the expectation). More precisely, our result is the following.

**Theorem 1.** *The only solution of the Diophantine equation*

$$(1.2) \quad F_n = T_m$$

in positive integer numbers  $m$  and  $n$  with  $n > 3$ , is  $(n, m) = (7, 6)$ . Hence,  $F \cap T = \{0, 1, 2, 13\}$ .

We organize this paper as follows. In Section 2, we will recall some useful properties such as a result of Matveev on linear forms in three logarithms and the reduction method of Baker-Davenport that we will use in the proof of Theorem 1. In Section 3, we first use Baker's method to obtain a bound for  $n$ , then we completely prove Theorem 1 by means of the Baker-Davenport reduction method.

## 2. AUXILIARY RESULTS

We recall the well-known Binet's formula:

$$(2.1) \quad F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \quad \text{for all } n \geq 0,$$

where  $\varphi = (1 + \sqrt{5})/2$ . It is almost unnecessary to stress that this is a very helpful formula which moreover allows to deduce that

$$\varphi^{n-2} < F_n < \varphi^{n-1} \quad \text{for all } n \geq 1.$$

In 1982, Spickerman [36] found the following "Binet-style" formula for the Tribonacci sequence:

$$(2.2) \quad T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1} \quad \text{for all } n \geq 0,$$

where  $\alpha, \beta, \gamma$  are the roots of  $x^3 - x^2 - x - 1 = 0$ . Explicitly, we have

$$\begin{aligned}\alpha &= \frac{1}{3} + \frac{1}{3}(19 - 3\sqrt{33})^{1/3} + \frac{1}{3}(19 + 3\sqrt{33})^{1/3}, \\ \beta &= \frac{1}{3} - \frac{1}{6}(1 + i\sqrt{3})(19 - 3\sqrt{33})^{1/3} - \frac{1}{6}(1 - i\sqrt{3})(19 + 3\sqrt{33})^{1/3}, \\ \gamma &= \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})(19 - 3\sqrt{33})^{1/3} - \frac{1}{6}(1 + i\sqrt{3})(19 + 3\sqrt{33})^{1/3}.\end{aligned}$$

Another interesting formula due to Spickermann is

$$T_n = \text{Round}\left[\frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}\alpha^n\right],$$

where, as usual,  $\text{Round}[x]$  is the nearest integer to  $x$ .

Since  $\alpha^{-2} < \alpha/(\alpha - \beta)(\alpha - \gamma) = 0.33622\dots < \alpha$ , the above identity yields the bounds

$$\alpha^{n-3} < T_n < \alpha^{n+2} \quad \text{for all } n \geq 1.$$

The Fibonacci and Tribonacci numbers can also be computed using the generating functions

$$(2.3) \quad \frac{z}{1 - z - z^2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + 34z^8 + \dots,$$

$$(2.4) \quad \frac{z}{1 - z - z^2 - z^3} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + 81z^8 + \dots$$

In order to prove Theorem 1, we will use a lower bound for a linear form in three logarithms *à la Baker* and such a bound was given by the following result of Matveev [30].

**Lemma 1.** *Let  $\alpha_1, \alpha_2, \alpha_3$  be real algebraic numbers and let  $b_1, b_2, b_3$  be nonzero rational numbers. Define*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

*Let  $D$  be the degree of the number field  $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$  over  $\mathbb{Q}$  and let  $A_1, A_2, A_3$  be positive real numbers which satisfy*

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \text{for } j = 1, 2, 3.$$

*Assume that*

$$B \geq \max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\}.$$

Define also

$$C = 6750000 \cdot e^4(20.2 + \log(3^{5.5} D^2 \log(eD))).$$

If  $\Lambda \neq 0$ , then

$$\log |\Lambda| \geqslant -CD^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)).$$

As usual, in the above statement, the *logarithmic height* of an  $s$ -degree algebraic number  $\alpha$  is defined as

$$h(\alpha) = \frac{1}{s} \left( \log |a| + \sum_{j=1}^s \log \max\{1, |\alpha^{(j)}|\} \right),$$

where  $a$  is the leading coefficient of the minimal polynomial of  $\alpha$  (over  $\mathbb{Z}$ ),  $(\alpha^{(j)})_{1 \leqslant j \leqslant s}$  are the conjugates of  $\alpha$  and, as usual, the absolute value of the complex number  $z = a + bi$  is  $|z| = \sqrt{a^2 + b^2}$ .

After finding an upper bound on  $n$  which is generally too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethö [6]. For a real number  $x$ , we use  $\|x\| = \min\{|x - n| : n \in \mathbb{N}\} = |x - \text{Round}[x]|$  for the distance from  $x$  to the nearest integer.

**Lemma 2.** *Suppose that  $M$  is a positive integer. Let  $p/q$  be a convergent of the continued fraction expansion of  $\gamma$  such that  $q > 6M$  and let  $\varepsilon = \|\mu q\| - M\|\gamma q\|$ , where  $\mu$  is a real number. If  $\varepsilon > 0$ , then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers  $m, n$  with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leqslant m < M.$$

See Lemma 5, a) in [6]. Now, we are ready to deal with the proofs of our results.

### 3. THE PROOF OF THEOREM 1

**3.1. Finding a bound on  $n$ .** By Binet's formulae (2.1) and (2.2) we get

$$\frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} = \frac{\alpha^m}{\alpha'} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}.$$

Let us denote by  $\alpha'$ ,  $\beta'$ ,  $\gamma'$  the values of  $Q(x) = -x^2 + 4x - 1$  at  $x = \alpha, \beta, \gamma$ , respectively. By (2.2) and equation (1.2), we have

$$\frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'} = \frac{(-1)^n \varphi^{-n}}{\sqrt{5}} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}, \quad m, n \geq 1.$$

More precisely,

$$(3.1) \quad \left| \frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'} \right| \leq \left| \frac{\varphi^{-1}}{\sqrt{5}} \right| + 2 \left| \frac{\beta}{\beta'} \right| < 0.67 \quad \text{for any } m, n \geq 1$$

where in the last inequality we have used  $|\beta| = |\gamma| = 0.73735\dots$  and  $|\beta'| = |\gamma'| = 3.84631\dots$

Define

$$\Lambda = \Lambda(m, n) = m \log \alpha - n \log \varphi + \log \left( \frac{\sqrt{5}}{\alpha'} \right).$$

Then

$$\Lambda = \log \left( \frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'} \right),$$

which yields

$$|e^\Lambda - 1| = \left| \frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'} - 1 \right|.$$

On the other hand, from (3.1) we get

$$\left| \varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'} \right| < 0.67 \cdot \sqrt{5} < 1.5.$$

Hence

$$|e^\Lambda - 1| = \frac{1}{\varphi^n} \left| \varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'} \right| < \frac{1.5}{\varphi^n}.$$

Since  $\varphi = 1.61803\dots$ , we have  $1.5/\varphi^n < \varphi^{-n+1}$  and then

$$(3.2) \quad |e^\Lambda - 1| < \varphi^{-n+1}.$$

We claim that  $\Lambda \neq 0$ . In fact, towards a contradiction, suppose that  $\Lambda = 0$  and thus  $\alpha^m \sqrt{5}/\alpha' = \varphi^n$ . Therefore  $\alpha^{2m}/\alpha'^2$  is a quadratic algebraic number. However  $\alpha^{2m}/\alpha'^2 \in \mathbb{Q}(\alpha)$  which is absurd, because  $\alpha$  is a 3-degree algebraic number.

If  $\Lambda > 0$ , then  $\Lambda < e^\Lambda - 1 < \varphi^{-n+2}$  (see (3.2)). If  $\Lambda < 0$ , then  $1 - e^{-|\Lambda|} = |e^\Lambda - 1| < \varphi^{-n+2}$ . Thus, for  $\Lambda < 0$ , we get

$$|\Lambda| < e^{|\Lambda|} - 1 < \frac{\varphi^{-n+1}}{1 - \varphi^{-n+1}} < \varphi^{-n+2},$$

where we have used the fact that  $1 - \varphi^{-n+1} > 1/\varphi$  for all  $n > 3$ .

Hence, we have  $|\Lambda| < \varphi^{-n+2}$  for any  $\Lambda \neq 0$ , which yields

$$(3.3) \quad \log |\Lambda| < -(n-2) \log \varphi.$$

Now, we will apply Lemma 1. Take

$$\alpha_1 = \alpha, \alpha_2 = \varphi, \alpha_3 = \sqrt{5}/\alpha', b_1 = m, b_2 = -n, b_3 = 1.$$

Then  $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\alpha, \varphi)$ ,  $D = 6$  and  $C < 1.2 \cdot 10^{10}$ .

It is easy to verify that  $1/\alpha'$  is a root of  $44x^3 - 2x - 1$  and that  $\sqrt{5}/\alpha'$  is a root of  $1936x^6 - 880x^4 + 100x^2 - 125$ . Since  $\sqrt{5}/\alpha'$  is a 6-degree algebraic number, its minimal polynomial (over  $\mathbb{Z}$ ) is  $1936x^6 - 880x^4 + 100x^2 - 125$ . Using direct calculation, we verify that the absolute value of every root of the minimal polynomial is less than 1. Hence  $h(\alpha_3) < (\log 1936)/6 < 1.262$ . Next, we have  $h(\alpha_1) = (\log \alpha)/3 = 0.204$  and  $h(\alpha_2) = (\log \varphi)/2 < 0.241$ . We then take  $A_1 = 1.22$ ,  $A_2 = 1.45$  and  $A_3 = 7.58$ . Since (1.2) implies  $n > m$ , we have

$$\max \{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\} = \max\{m, 1.2n\} = 1.2n =: B.$$

Hence, Lemma 1 yields

$$(3.4) \quad \log |\Lambda| > -6.8 \cdot 10^{12} \log(82n).$$

Combining the estimates (3.3) and (3.4), we get

$$6.8 \cdot 10^{12} \log(82n) > (n-2) \log \varphi,$$

and this inequality implies  $n < 6 \cdot 10^{14}$  and, by the trivial estimate  $m < n$ , we have  $m < 6 \cdot 10^{14}$ . In order to improve the estimates, we use the bounds on  $F_n$  and  $T_m$  together with Equation (1.2) to obtain  $\alpha^{m-3} < T_m = F_n < \varphi^{n-1}$ , which yields

$m < 0.8n + 2.2$ . Hence,  $m < 4.8 \cdot 10^{14}$ . Similarly,  $\varphi^{n-2} < F_n = T_m < \alpha^{m+2}$  yields  $n < 1.3m + 4.6$ .

**3.2. Reducing the bound.** The next goal is to reduce the bound on  $m$ . For that, let us suppose, without loss of generality, that  $\Lambda > 0$  (the other case can be handled in a similar way by considering  $0 < \Lambda' = -\Lambda$ ).

We know that  $0 < \Lambda < \varphi^{-n+2}$  and therefore

$$0 < m \log \alpha - n \log \varphi + \log \left( \frac{\sqrt{5}}{\alpha'} \right) < \varphi^{-m+2}.$$

Dividing by  $\log \varphi$ , we get

$$(3.5) \quad 0 < m\hat{\gamma} - n + \mu < 5.45 \cdot \varphi^{-m},$$

with  $\hat{\gamma} = \log \alpha / \log \varphi$  and  $\mu = \log(\sqrt{5}/\alpha') / \log \varphi$ .

Surely  $\hat{\gamma}$  is an irrational number (actually, this number is transcendental by the Gelfond-Schneider theorem: if  $\alpha$  and  $\beta$  are algebraic numbers with  $\alpha \neq 0$  or  $1$ , and  $\beta$  is irrational, then  $\alpha^\beta$  is transcendental). So, let us denote by  $p_n/q_n$  the  $n$ th convergent of its continued fraction.

In order to reduce our bound on  $m$ , we will use Lemma 2. For that, taking  $M = 4.8 \cdot 10^{14}$ , we have that

$$\frac{p_{33}}{q_{33}} = \frac{53739149317980067}{42436582738078750},$$

and then  $q_{33} > 6M$ . Moreover, we get

$$\|\mu q_{33}\| - M \|\hat{\gamma} q_{33}\| > 0.028 =: \varepsilon.$$

Thus all the hypotheses of Lemma 2 are satisfied and we take  $A = 5.45$  and  $B = \varphi$ . It follows from Lemma 2 that there is no solution of the inequality in (3.5) (and then for the Diophantine equation (1.2)) in the range

$$\left[ \left[ \frac{\log(Aq_{33}/\varepsilon)}{\log B} \right] + 1, M \right] = [91, 4.8 \cdot 10^{14}].$$

Therefore  $m \leq 90$  and then  $n \leq 120$ . To conclude, we use the formulas in (2.3) and (2.4) together with the Mathematica command

```
Intersection[CoefficientList[Series[x/(1-x-x^2),x,0,120],x],
CoefficientList[Series[x/(1-x-x^2-x^3),x,0,90],x]]
```

to find the possible solutions. Fastly, Mathematica returns us the set  $\{0, 1, 2, 13\}$  as its answer. This completes the proof.



#### 4. FINAL REMARKS AND A CONJECTURE

We point out that the method in proof of Theorem 1 is quite general and that it can be used to work on the intersection of two arbitrary  $k$ -generalized Fibonacci sequences. In fact, in a similar fashion, we found the set  $F^{(k)} \cap F^{(m)}$  for  $4 \leq k < m \leq 10$ . These cases suggest that the following statement (which is Conjecture 1 in [32]) should be true.

**Conjecture 1.** *Let  $k < m$  be positive integer numbers. Then*

$$F^{(k)} \cap F^{(m)} = \begin{cases} \{0, 1, 2, 13\}, & \text{if } (k, m) = (2, 3), \\ \{0, 1, 2, 4, 504\}, & \text{if } (k, m) = (3, 7), \\ \{0, 1, 2, 8\}, & \text{if } k = 2 \text{ and } m > 3, \\ \{0, 1, 2, \dots, 2^{k-1}\}, & \text{otherwise.} \end{cases}$$

When working on these cases it may be helpful that the polynomials  $\psi_k(x) := x^k - x^{k-1} - \dots - x - 1$  are irreducible over  $\mathbb{Q}[x]$  with just one zero outside the unit circle. That single zero is located between  $2(1 - 2^{-k})$  and 2 (as seen in [39]). Also, in a recent paper, G. Dresden [7, Theorem 1] gave a simplified “Binet-like” formula for  $F_n^{(k)}$ :

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}$$

for  $\alpha_1, \dots, \alpha_k$  being the roots of  $\psi_k(x) = 0$ . There are many other ways of representing these  $k$ -generalized Fibonacci numbers, as can be seen in [10], [11], [12], [14], [15]. Also, it was proved in [7, Theorem 2] that

$$F_n^{(k)} = \text{Round} \left[ \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right],$$

where  $\alpha$  is the dominant root of  $\psi_k(x)$ .

**Acknowledgement.** The author would like to express his gratitude to the anonymous referee for carefully examining this paper and providing a number of important comments and suggestions. He is also grateful to Gregory Dresden for nice talks on the subject. The author is financially supported by CNPq and FEMAT.

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