RECOGNIZING AUTOMORPHISMS OF POLYNOMIAL ALGEBRAS

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Abstract

We discuss how to recognize whether an endomorphism of a polynomial algebra is an automorphism through three different approaches: Grb̈ner basis, the Jacobian conjecture and test polynomials.

1. Introduction

Let K be a field. To avoid complications, we assume throughout this paper that the characteristic of K is zero. Let $K[X] := K[x_1, \ldots, x_n]$ be the polynomial algebra in n variables over K. Let $F := (f_1, \ldots, f_n) \in (K[X])^n$ be an n-tuple. Obviously, $\phi : p(X) \to p(F)$ is an endomorphism of $K[X]$. On the otherhand, every endomorphism of $K[X]$ may be defined in that way. To slightly abuse the language, sometimes we say that $F := (f_1, \ldots, f_n)$ is an endmorphism of $K[X]$.

The main problem considered in this paper is: given $\phi : X \to F$ an endomorphism of $K[X]$, how to recognize whether ϕ is an automorphism?

We shall discuss this problem in this paper via three different approaches: Gröbner basis, Jacobian conjecture and test polynomials.

The paper is organized as follows: Section 2 introduce the Gröbner basis approach given by van den Essen. In Section 3, we introduce the Jacobian conjecture and present our recent result on the 'positive' and 'negative' case of the conjecture. In Section 4, we give a new appoach on the $n = 2$ Jocobian conjecture via polynomial retracts. Section 5 deals with the test polynomial

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approch. Finally, in the concluding Section 6, we proposed two open problems related to the Jacobian conjecture.

In the sequal we sometimes denote $(x_1, \ldots, x_n)$ by X, $(y_1, \ldots, y_n)$ by Y, $(f_1, \ldots, f_n)$ by F, and $(g_1, \ldots, g_n)$ by G.

2. Gröbner basis approach

In 1990, Arno van den Essen [8] proved the following theorem.

Theorem 1. *Let $\phi : x_i \to f_i$ be an endomorphism of $K[x_1, \ldots, x_n]$. Then ϕ is an automorphism if and only if the reduced Groebner basis of the ideal generated by*

$$\{y_1 - f_1, \ldots, y_n - f_n\}$$

in the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ under the lexicographic ordering

$$x_1 > \ldots > x_n > y_1 > \ldots > y_n$$

is

$$\{x_1 - g_1, \ldots, x_n - g_n\}$$

where $g_i \in K[y_1, \ldots, y_n]$. Moreover, if $X \to F$ is an automorphim, and if we define $G := (g_1, \ldots, g_n)$. Then $Y \to G$ is the inverse automorphism of F.

Note that Theorem 1 gives an algorithm to decide whether an endomorphism of $K[X]$ is an automorphism.

3. The Jacobian conjectrue

If $\phi : X \to F$ is an automorphism of $K[X]$ and $\phi^{-1} : X \to G$ is the inverse. Then

$$G \circ F = X.$$

Hence

$$J(G \circ F) = J(X)$$

where J denotes the usual Jacobian (matrix) operator. By the chain rule, $J(G)(F)J(F) = I$ where I is the identity matrix of order n. Hence

$$J(F) \in GL_n(K[X]).$$

The Jacobian conjecture is that the converse of the above statement is true.

The Jacobian conjecture. *Let $\phi : X \to F$ be an automorphism of $K[X]$. If $J(F) \in GL_n(K[X])$. Then ϕ is an automorphism.*

Formulated by O. Keller [12] in 1939, the conjecture is still open for $n \geq 2$ (the $n = 1$ conjecture is obviously true), to the best of our knowledge.

For arbitrary n, O. Keller [12] himself proved the birational case (i.e., with the addtional condition that $K(X) = K(F)$) in 1939. In 1973, L.A. Campbell [3] proved the Galois case of the conjecture (i.e., with the additional condition that $K(X)/K(F)$ is a Galois extension). In 1980, S.S.-S. Wang [21] proved the quadratic case of the Jacobian conjecture. In 1982, H. Bass, E. Connell and D. Wright [2] reduce the Jacobian conjecture to the cubic homogeneous case. Namely, to solve the conjecture, one only needs to consider the case $F = X + H$ where every monomial in H is cubic (that implies that $J(H)$ is a nilpotent matrix). For $n = 2$, T.T. Moh [14] proved the conjecture for the case $\max\{\deg(f_1), \deg(f_2)\} \leq 100$. For a history and background of the Jacobian conjecture, see [2].

We have recently reduced the Jacobian conjecture to the so-called 'positive case' and sovled the 'negative case'. In order to present these results, first note that it is well-knownthat in orger to solve the Jacobian conjecture, we only need to consider the case $K = \mathbf{C}$, the field of complex numbers and we only need to prove that $F : \mathbf{C}^n \to \mathbf{C}^n$ is injective; see, for instance, [2]. In fact, by the following well-known fact, we only need to consider the case $K = \mathbf{R}$, the field of real numbers.

Let $F := (f_1, \dots, f_n) : \mathbf{C}^n \to C^n$ be a differentiable map. Then naturally F may be viewed as a map $\overline{F}$ from $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$. Moreover, F is injective if and only if $\overline{F}$ is injective and $\det(J(\overline{F})) = (\det(J(F))^2$.

Then in 1995 we proved the following two theorems in [22]:

Theorem 2. *To solve the Jacobian conjecture, one only needs to consider the case $f_i = x_i + H_i^{(2)} + H_i^{(3)} + H_i^{(4)} \in \mathbf{R}[x_1, \ldots, x_n]$ where $H_i^{(j)}$ are homogeneous of degree j and all coefficients in f_i are nonnegative.*

Theorem 3. *Let $f_i = x_i - H_i \in \mathbf{R}[x_1, \ldots, x_n]$ where $J(f_1, \ldots, f_n) \in GL_n(\mathbf{R}[x_1, \ldots, x_n])$, $ord(H_i) \geq 2$ and all coefficients of H_i are nonnegative. Then $\phi : x_i \to f_i$ is an automorphism.*

4. Polynomial retracts and the $n = 2$ Jacobian conjecture

In this section we focus on the $n = 2$ Jacobian conjecture with a new approach via polynomial retracts. Let $K[x, y]$ be the polynomial algebra in two variables over a field K of characteristic 0. A subalgebra R of $K[x, y]$ is called a *retract* if it satisfies any of the following equivalent conditions:

(R1) There is an idempotent endomorphism (a *retraction*, or *projection*) φ of $K[x, y]$ that $\varphi(K[x, y]) = R$.

(R2) There is a homomorphism $\varphi : K[x, y] \to R$ that fixes every element of R.

(R3) $K[x, y] = R \oplus I$ for some ideal I of the algebra $K[x, y]$.

(R4) $K[x, y]$ is a projective extension of R in the category of K-algebras. In other words, there is a splitting exact sequence $1 \to I \to K[x, y] \to R \to 1$, where I is the same ideal as in (R3) above.

Examples: K; $K[x, y]$; any subalgebra of the form $K[p]$, where $p \in K[x, y]$ is a *coordinate* polynomial (i.e., $K[p, q] = K[x, y]$ for some polynomial $q \in K[x, y]$). There are other, less obvious, examples of retracts: if $p = x + x^2y$, then $K[p]$ is a retract of $K[x, y]$, but p is not coordinate since it has a fiber $\{p = 0\}$ which is reducible, and therefore is not isomorphic to a line.

The very presence of several equivalent definitions of retracts shows how natural these objects are.

In [5], Costa has proved that every proper retract of $K[x, y]$ (i.e., a one different from K and $K[x, y]$) has the form $K[p]$ for some polynomial $p \in K[x, y]$, i.e., is isomorphic to a polynomial K-algebra in one variable. A natural problem now is to characterize somehow those polynomials $p \in K[x, y]$ that generate a retract of $K[x, y]$. Since the image of a retract under any automorphism of $K[x, y]$ is again a retract, it would be reasonable to characterize retracts up to an automorphism of $K[x, y]$, i.e., up to a "change of coordinates". We give an answer to this problem in [19] as follows

Theorem 4. *Let $K[p]$ be a retract of $K[x, y]$. There is an automorphism ψ of $K[x, y]$ that takes the polynomial p to $x + y \cdot q$ for some polynomial $q = q(x, y)$. A retraction for $K[\psi(p)]$ is given then by $x \to x + y \cdot q$; $y \to 0$.*

Our proof of this result is based on the famous Abhyankar-Moh Theorem of embeddings of the line in the plane [1].

Theorem 4 yields another characterization of retracts of $K[x, y]$ (see [19]):

Proposition 5. *A polynomial $p \in K[x, y]$ generates a retract of $K[x, y]$ if and only if there is an endomorphism of $K[x, y]$ that takes p to x.*

Although the form to which any retract can be reduced by Theorem 4 might seem rather general, it is in fact quite restrictive, and has an interesting application to the $n = 2$ Jacobian conjecture.

Now we formulate the following conjecture in [19].

Conjecture 6. *If $p, q \in K[x, y]$ with $J(p, q) \in GL_2(K[x, y])$, then $K[p]$ is a retract of $K[x, y]$.*

Proposition 7. (see [19]) *Conjecture 6 is equivalent to the $n = 2$ Jacobian conjecture.*

Another application of retracts to the $n=2$ Jacobian conjecture (somewhat indirect though) is based on the "φ^∞-trick" familiar in combinatorial group theory (see [13]). For an endomorphism φ of $K[x,y]$ denote by $\varphi^\infty(K[x,y]) = \bigcap_{k=1}^{\infty} \varphi^k(K[x,y])$ the *stable image* of φ. Then we have:

Theorem 8. (see [19]) *Let φ be an endomorphism of $K[x,y]$. If the Jacobian matrix of φ is invertible, then either φ is an automorphism, or $\varphi^\infty(K[x,y]) = K$.*

Our proof of Theorem 8 is based on a recent result of Formanek [10].

Obviously, if φ fixes a polynomial $p \in K[x,y]$, then $p \in \varphi^\infty(K[x,y])$. Therefore, we have ([18]):

Proposition 9. *Suppose φ is an endomorphism of $K[x,y]$ with invertible Jacobian matrix. If $\varphi(p) = p$ for some non-constant polynomial $p \in K[x,y]$, then φ is an automorphism.*

This yields the following reformulation of the Jacobian conjecture: if φ is an endomorphism of $K[x,y]$ with invertible Jacobian matrix, then for some automorphism α, the mapping $\alpha \cdot \varphi$ fixes a non-constant polynomial.

5. Test polynomials

In this section we introduce another approach to recognize automorphisms of polynomial algebras via test polynomials. Let A be an algebraic object. An element $a \in A$ is called a *test element* for automorphisms of A if for any endomorphism ϕ of A such that $\phi(a) = a$, then ϕ is an automorphism. This definition was explicitly given by V.Shpilrain [18] in 1994, but the history of test elements goes back to Nielsen in 1918 and Dicks in 1982. A classical result of Nielsen [17] states that an endomorphism $x \to f;\ y \to g$ of the free group F_2 with two generators x, y is an automorphism if and only if $[f,g]$ is conjugate to $[x,y]$. Hence the commutator $[x,y] = xyx^{-1}y^{-1}$ is a test elements F_2. Dicks

[6] proved a similar result for the free associative algebra $K\langle x, y\rangle$ of rank two: an endomorphism $(x, y) \to (f, g)$ of $K\langle x, y\rangle$ is an automorphism if and only if $[f, g] = \alpha[x, y]$ where $\alpha \in K^*$. Hence $[x, y] = xy - yx$ is a test polynomial of $K\langle x, y\rangle$. Obviously any element in a proper retract of an algebraic object is not a test element. For a free algebraic object A generated (freely) by n elements, define the *rank* of an element $a \in A$ as the minimum munber $m \leq n$ such that a belongs to a free subobjects generated (freely) by m free generators of A. It is easy to see that a test element of a free algebraic object A generated freely by n elements must have maximum rank n. Naturally one may ask the question to determine all test elements of an algebraic object. This problem has been solved for both finitely generated free groups and Lie algebras. Turner [20] proved that test elements of a finitely generated free group are precisely those elements not contained in a proper retract of the group. Very recently, we have obtained a similar result for free Lie algebras in [16].

Theorem 10. *Test elements of a finitely generated free Lie algebra are precisely those elements not contained in a proper retract of the Lie algebra*

The proofs of the above results on test elements of free groups and Lie algebras rely heavily on the fact that every subgroup (subalgebra, repectively) of a free group (free Lie algebra, respectively) is again a free group (free Lie algebra, respectively). In polynomial and free associative algebra cases, the problem is much harder, since obviously there are subalgebras of $K[X]$ ($K\langle X\rangle$, respectively) that are not polynomial algebras (free associative algebras, respectively).

The polynomial $x_1^2 + \ldots + x_n^2$ is the first example of test polynomial for $\mathbf{R}[x_1, \ldots, x_n]$; it was given by van den Essen and V.Shpilrain [9]. However, in [7] we showed that it is not a test polynomials for $\mathbf{C}[x_1, \ldots, x_n]$. Therefore, whether a polynomial in $K[X]$ is a test polynomial depends on the properties of the ground field K.

In [7] we obtained some test polynomials for both $K[X]$ and $K\langle X\rangle$. Unfortunately, all the test polynomials of $K[X]$ we know can only recognize linear automorphisms. Hence we may ask whether there exists a 'nontrial test polynomial' for the polynomial algebra. On the otherhand, for free associative algebras, we obtained in [7] some 'non-trivial' test polynomials.

Theorem 11. $[x_1, x_2] \dots [x_{2n-1}, x_{2n}]$ *is a test polynomial for the free associative algebra* $K[x_1, \dots, x_{2n}]$ *recognizing the automorphisms that fix all but one or two of variables.*

In [7], a *test vector space* W of $K\langle X\rangle$ is defined as follows. An endomorphism $X \to F$ of $K\langle X\rangle$ is an automorphism if and only if $w(F) \in W$ for every $w(X) \in W$ and $w(F)$ is not the zero polynomial.

Then we determined in [7] all test vector space of a free associative algebra.

Theorem 12. i) *W is a test vector space of $K\langle x, y\rangle$ if and only if W is spanned on a finite set of powers $[x, y]^k$ of the commutator $[x, y]$.*

ii) *For $n > 2$ there is no test vector space in $K\langle x_1, \dots, x_n\rangle$.*

The proofs of both Theorem 11 and 12 are based on the result of Dicks in [6] and non-commutative algebra technics developed in P.M. Cohn [4].

The problems to determine all test polynomials of $K[x_1, \dots, x_n]$ or $K\langle x_1, \dots, x_n\rangle$ remains open, even for the case $n = 2$.

6. Two open problems

In this section, we propose two open problems which are closely related to the Jacobian conjecture.

$p \in K[x_1, \dots, x_n]$ is called a *coordinate polynomial* if there exsit $p_2, \dots, p_n \in K[x_1, \dots, x_n]$ such that $K[p, p_2, \dots, p_n] = K[x_1, \dots, x_n]$. Or, equivalently, there exists an automorphism ϕ of $K[x_1, \dots, x_n]$ such that $\phi(p) = x_1$.

Problem 13. *Is it true that any endomorphism ϕ of $K[x_1, \dots, x_n]$ taking coordinate polynomials to coordinate polynomials, is actually an automorphism?*

In [9], van den essen and Shpilrain give a positive answer to the problem 13 for the case $n = 2$. Moreover, they observed that in general, $J(\phi) \in GL_n(K[x_1, \dots, x_n]$.

Problem 14. *Let $F = (F_1, \dots, F_n) \in (K[X_1, \dots, X_n])^n$ with each F_i irreducible and let $p_{ij} \in K[t_1, \dots, t_{n-1}]$ with zero constant terms for $i, j = 1, \dots, n$ such that the $n \times n$ matrix*

$$(F_j(p_{i1}, \dots, p_{in}))_{j=1,\dots,n}^{i=1,\dots,n} = \begin{pmatrix} 0 & t_1 & t_2 & \dots & t_{n-2} & t_{n-1} \\ t_1 & 0 & t_2 & \dots & t_{n-2} & t_{n-1} \\ t_1 & t_2 & 0 & \dots & t_{n-2} & t_{n-1} \\ & \dots & & \dots & & \dots \\ t_1 & t_2 & t_3 & \dots & 0 & t_{n-1} \\ t_1 & t_2 & t_3 & \dots & t_{n-1} & 0 \end{pmatrix}.$$

Is $X \to F$ an automorphism of $\mathbf{K}[X_1, \dots, X_n]$?

In [15], McKay, Moh and Wang give a positive answer to Problem 14 for the case $n = 2$. Then in [11], Jelonek proved that $J(F) \in GL_n(K[x_1, \dots, x_n])$.

Note in some sense, Problem 13 and 14 are 'dual problems'.

Remark. The author was notified by Arno van den Essen that Problem 13 and 14 have recently been solved with positive solutions by Z. Jenolek.

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