Rethinking UnificationTheory

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Let us lean back for a minute and reflect on the motivation for our field. Apart from its theoretical interest, i.e. the structural relationships among and within equational theories, there is the practical motivation, most clearly expressed in Gordon Plotkin´s seminal paper from 1972 [10]: we want to take certain troublesome axioms, like associativity or commutativity, out of the axiom set for an automated deduction system that may lead the system to go astray. Instead - so the proposal - they should be "built-in".

Now the past 30 years - the first workshop in Val D'Ajol was in 1987 - have revealed an astonishing complexity even for those simple axioms - not so astonishing after all, for someone familiar with semigroup theory [3] and more generally the results about equational theories [7, 6].

Historically, the development of unification theory began with the central notion of *a most general unifier* based on the *subsumption order*. A unifier σ is most general, if it subsumes any other unifier τ, that is, if there is a substitution λ with $\tau =_E \sigma\lambda$, where E is an equational theory and $=_E$ denotes equality under E. Since there is in general more than one most general unifier for a unification problem under an equational theory E, called *E-Unification*, we have the notion of a complete and minimal set of unifiers under E for a unification problem $\mathit{\Gamma}$, denoted as $\mu\mathcal{U}\Sigma_E(\Gamma)$. This set is still the basic notion in unification theory today.

But, unfortunately, the subsumption quasi order is not a well founded quasi order, which is the reason why for certain equational theories there are solvable E-unification problems, but the set $\mu\mathcal{U}\Sigma_E(\Gamma)$ does not exist. We say these problems are of type nullary in the unification hierarchy [11]. In order to overcome this problem and also to substantially reduce the number of most general unifiers in nonnullary theories, we introduced the notion of *essential unification.* An *essential unifier*, as introduced by Hoche and Szabo [2], generalizes the notion of a most general unifier with a most pleasant effect: the set of essential unifiers is often much smaller than the set of most general unifiers. Essential unification may even reduce an infinitary theory to an essentially finitary theory. For example the one variable string unification problem is essentially finitary whereas it is infinitary in the usual sense [1]. A most drastic reduction is obtained for idempotent semigroups, or bands as they are called in computer science, which are of type nullary: there exist two unifiable terms s and t, but the set of most general unifiers does not exist. This is in stark contrast to essential unification: the set of essential unifiers for bands always exists and it is finite [2].

The key idea for essential unification is to base the notion of generality not on the standard subsumption order for terms with the associated subsumption order for substitutions, but on the well known *encompassment order* for terms. We also extended this ordering for terms to an order for substitutions and proposed the encompassment order as a more natural relation for minimal and complete sets of E-unifiers, calling them *essential unifiers*, denoted as $e\mathcal{U}\Sigma_E(\Gamma)$. If $\mu\mathcal{U}\Sigma_E(\Gamma)$ exists, then $e\mathcal{U}\Sigma_E(\Gamma) \subseteq \mu\mathcal{U}\Sigma_E(\Gamma)$, i.e. it is always a subset. An interesting effect is, that there are cases of an equational theory E, for which the complete set of most general unifiers does not exist, the *minimal and complete set of essential unifiers* however does exist.

Unfortunately again, the encompassment order is not a well founded quasi ordering, that is, there are still theories with a solvable unification problem, for which a minimal and complete set of essential unifiers can not be obtained.

In a more recent paper [8][9] we therefore proposed a third approach, namely the extension of the well known *homeomorphic embedding of terms* to a *homeomorphic embedding of substitutions (modulo E),* known as equational embedding in the literature, and examined the set of E-unifiers under this ordering using the seminal *tree embedding theorem* or Kruskal's Theorem [4, 5] as it is called.

The main result of this latest approach is, that for any solvable E-unification problem the minimal and complete set of E-unifiers always *exists* and it is even smaller than the set of essential unifiers. Under some additional conditions, called *pure equational embedding*, it is **always finite**.

Our main observation is that for unification theory *subsumption* is just a special case of *encompassment*, which in turn is a special case of *homeomorphic embedding*.

References

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