INTRODUCTORY NOTES ON PARTIAL ISOMORPHISMS

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Throughout this text, only relational signatures are considered, and all signatures are assumed to be finite. We do not need to think of a signature as a linguistic object, but as a similarity type of structures. We assume that the arities of the predicate symbols in a signature are positive natural numbers, and that these symbols are ordered in a sequence. A structure for a given signature is a domain equiped with a sequence of predicates that match the signature's sequence. In contexts in which we are considering only one fixed signature we will, in general, not be explicit about it. In those contexts, if the fixed signature is denoted by Σ, then all structures are tacitly assumed to be Σ-structures, and all formulas are assumed to be in the first-order language derived from Σ, unless otherwise stated. All structures are assumed to be non-empty. We assume that all signatures contain one predicate constant, the equality symbol $=$.

A finite sequence of elements of a set X with length k is a function from $s : \{1, ..., k\} \to X$. If s and t are finite sequences, then the concatenation of s and t is denoted by $s\hat{\ }t$. If a is an element of a set X then a is identified with the sequence of elements of X with length 1 which maps 1 to a. We say that the finite sequence s with length k is contained in the string t with length l if there is a strictly increasing function $i : \{1, ..., k\} \to \{1, ..., l\}$, such that $s = t \circ i$. If s is a finite sequence we say that s is a tuple; if s is a tuple with length k, then we say that s is a k-tuple.

Definition 1. We say that f is a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ iff $\mathcal{A}$ and $\mathcal{B}$ are structures, $\overline{a}$ and $\overline{b}$ are n-tuples in $\mathcal{A}$ and $\mathcal{B}$, respectively, $p \in \omega$, f is a bijection, $dom(f)$ is contained in $\mathcal{A}$, $range(f)$ is contained in $\mathcal{B}$, $dom(f)$ is finite, the elements in the n-tuple $\overline{a}$ are in $dom(f)$, the elements in the n-tuple $\overline{b}$ are in $range(f)$, f maps the n-tuple $\overline{a}$ to the n-tuple $\overline{b}$, and either $p = 0$ and f is an isomorphism from the substructure whose domain is $dom(f)$ onto the substructure whose domain is $range(f)$, or $p > 0$ and the *back-and-forth* property holds:

(1) *(Forth)* If a is in $\mathcal{A}$ then there are a b in $\mathcal{B}$ and a function g such that g is an extension of f and $g : (\mathcal{A}, \overline{a}\hat{\ }a) \approx_{p-1} (\mathcal{B}, \overline{b}\hat{\ }b)$.

(2) *(Back)* If b is in $\mathcal{B}$ then there are a a in $\mathcal{A}$ and a function g such that g is an extension of f and $g : (\mathcal{A}, \overline{a}\hat{\ }a) \approx_{p-1} (\mathcal{B}, \overline{b}\hat{\ }b)$.

We denote the above defined predicate by

$$f : (\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b}).$$

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Definition 2. We say that $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ are p-equivalent iff there is an f such that $f : (\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b})$. We denote this relation by

$$(\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b}).$$

Remark 3. If f is a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$, and $\overline{a'}$ is a tuple in $\mathcal{A}$ that is contained in $\overline{a}$, then f is a p-partial isomorphism between $(\mathcal{A}, \overline{a'})$ and $(\mathcal{B}, \overline{b'})$, where $\overline{b'}$ is the image of $\overline{a'}$ under f. If g is a restriction of f, $\overline{a'}$ is contained in $\overline{a}$ and the elements in $\overline{a'}$ are in $dom(g)$, then g is a p-partial isomorphism between $(\mathcal{A}, \overline{a'})$ and $(\mathcal{B}, \overline{b'})$, where $\overline{b'}$ is the image of $\overline{a'}$ under f. In particular, if $(\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b})$, then the empty function is a p-partial isomorphism between $(\mathcal{A}, \emptyset)$ and $(\mathcal{B}, \emptyset)$. Furthermore, if there is a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$, then there is one whose domain is constituted exactly by the elements in the n-tuple $\overline{a}$, and that maps $\overline{a}$ to $\overline{b}$.

Remark 4. For each $p \in \omega$, if f is a $p+1$-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$, then f is a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$. Therefore, if f is a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$, and $q < p$, then f is a q-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$.

Definition 5. The quantificational rank of a first order formula φ is defined, by induction, as the greatest number of nested quantifiers in φ: If φ is atomic then its quantificational rank is 0; if φ is a conjunction (disjunction) then its quantificational rank is the greatest between the quantificational ranks of its conjuncts (disjuncts); if φ is $\neg\psi$ then its quantificational rank is the quantificational rank of ψ; if φ is $\exists x\psi$ then its quantificational rank is the quantificational rank of ψ plus 1.

Theorem 6. (Fraïssé Theorem, first part) *If $(\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b})$, $\overline{a}$ and $\overline{b}$ are n-tuples, and φ is a formula of quantificational rank $\leq p$ with free variables among x_1,..., x_n, then $\mathcal{A} \models \varphi[\overline{a}]$ iff $\mathcal{B} \models \varphi[\overline{b}]$.*

Proof. Let f denote a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$.

If φ is atomic, then, since f is an isomorphism between a (finite) substructure that contains $\overline{a}$ and a (finite) substructure that contains $\overline{b}$,

$$\mathcal{A} \models \varphi[\overline{a}] \text{ iff } \mathcal{B} \models \varphi[\overline{b}].$$

If φ is a conjunction, or a disjunction, or a negation, then the equivalence $\mathcal{A} \models \varphi[\overline{a}]$ iff $\mathcal{B} \models \varphi[\overline{b}]$ follows from the induction hypothesis.

Now, suppose that φ is $\exists x_{n+1}\psi$, that the quantificational rank of φ is at most p, and that $\mathcal{A} \models \varphi[\overline{a}]$. Therefore, there is an element a in $\mathcal{A}$ such that $\mathcal{A} \models \psi[\overline{a}\,\hat{}\,a]$. Since f is a p-partial isomorphism, and $p > 0$, it follows that there is an element b in $\mathcal{B}$ and a function g that is an extension of f, and such that

$$g : (\mathcal{A}, \overline{a}\,\hat{}\,a) \approx_{p-1} (\mathcal{B}, \overline{b}\,\hat{}\,b).$$

By induction hypothesis, if ϕ is a formula with quantificational rank $\leq p-1$ with free variables among x_1,...,x_{n+1}, then

$$\mathcal{A} \models \phi[\overline{a}\hat{\ }a] \text{ iff } \mathcal{B} \models \phi[\overline{b}\hat{\ }b].$$

Since the quantificational rank of ψ is at most $p-1$, it follows that

$$\mathcal{B} \models \psi[\overline{b}\hat{\ }b], \text{ and hence } \mathcal{B} \models \varphi[\overline{b}].$$

The converse implication follows from the symmetry of the relation of p-equivalence with respect to $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$. □

Remark 7. The proof given above shows that, given a natural number p, for each formula φ with quantificational rank p, every p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ is a 0-partial isomorphism between $(\mathcal{A}', \overline{a})$ and $(\mathcal{B}', \overline{b})$, where $\mathcal{A}'$ is the expansion of $\mathcal{A}$ with the predicate $\varphi^{\mathcal{A}}$, and similarly for $\mathcal{B}'$.

Let us define, by induction in q, a family of sets $(\Gamma(i,q))_{i\in\omega}$:

If $q = 0$, then consider all the finitely many atomic formulas in the variables x_1,..., x_i. Denote this plurality of atomic formulas by ϕ_1,..., ϕ_m. Let $\Gamma(i,0)$ be the set of all conjunctions of the form $\phi_1^* \wedge ... \wedge \phi_m^*$, where ϕ_j^* is either ϕ_j or its negation, $\neg\phi_j$. This is the set of all *state descriptions* based on ϕ_1,..., ϕ_m.

If $q = r+1$, then the family of sets $(\Gamma(i,q))_{i\in\omega}$ is defined in terms of the family $(\Gamma(i,r))_{i\in\omega}$. Denote the formulas in $\Gamma(i+1,r)$ by ϕ_1,..., ϕ_k. Consider the formulas $\exists x_{i+1}\phi_1$,..., $\exists x_{i+1}\phi_k$, and let $\Gamma(i, r+1)$ be the set of all state descriptions based on $\exists x_{i+1}\phi_1$,..., $\exists x_{i+1}\phi_k$, that is, the set of all conjunctions of k conjuncts, such that the jth conjunct is either $\exists x_{i+1}\phi_j$ or its negation, $\neg\exists x_{i+1}\phi_j$.

Remark 8. The sets in the family $(\Gamma(i,q))_{i,q\in\omega}$ are finite, and their cardinalities can be calculated with the help of the following equation: $|\Gamma(i,q+1)| = 2^{|\Gamma(i+1,q)|}$. The free variables of the formulas in $\Gamma(i,q)$ are x_1,..., x_i, and their quantificational rank is q. No variable in a formular in $\Gamma(i,q)$ occurs both free and bound. The disjunction of the formulas in $\Gamma(i,q)$ is valid, and any two of them are incompatible. Therefore, for all numbers n and p, all pairs $(\mathcal{A}, \overline{a})$, where $\mathcal{A}$ is a structure, and $\overline{a}$ is an n-tuple in $\mathcal{A}$, there is one, and only one, formula $\psi \in \Gamma(n,p)$ such that $\mathcal{A} \models \psi[\overline{a}]$. We say that the pair $(\mathcal{A}, \overline{a})$ *is in the state described by* ψ of quantificational rank p, in n free variables.

Theorem 9. (Fraïssé Theorem, second part) *For all numbers n and p, all pairs $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$, where $\mathcal{A}$ and $\mathcal{B}$ are structures, and $\overline{a}$ and $\overline{b}$ are n-tuples in $\mathcal{A}$ and $\mathcal{B}$, respectively, if there is a formula $\psi \in \Gamma(n,p)$ such that $\mathcal{A} \models \psi[\overline{a}]$ and $\mathcal{B} \models \psi[\overline{b}]$, then $(\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b})$. On the other hand, if $(\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b})$, then there is a formula $\psi \in \Gamma(n,p)$ such that $\mathcal{A} \models \psi[\overline{a}]$ and $\mathcal{B} \models \psi[\overline{b}]$.*

Proof. The first statement is proved by induction in p. Assume that there is a formula $\psi \in \Gamma(n, p)$ such that

$$\mathcal{A} \models \psi[\overline{a}] \text{ and } \mathcal{B} \models \psi[\overline{b}].$$ [1]

If $p = 0$, then the pairs $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ are in the same atomic state, in n free variables, that is, the pairs $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ bear the same atomic relations. This means that the function f whose domain is constituted by the elements in $\overline{a}$, which maps $\overline{a}$ to $\overline{b}$ is an isomorphism of the corresponding substructures[2] and

$$(\mathcal{A}, \overline{a}) \approx_0 (\mathcal{B}, \overline{b}).$$

If $p = q+1$, then, by remark 3, it is necessary and sufficient to show that the function f whose domain is constituted by the elements in $\overline{a}$, which maps $\overline{a}$ to $\overline{b}$ is a p-partial isomorphism between $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$. It is enough to prove the *back-and-forth* property:

(1) If a is in $\mathcal{A}$, then there is a unique formula $\phi \in \Gamma(n+1, q)$ such that
$$\mathcal{A} \models \phi[\overline{a}\hat{\ }a].$$
Therefore, $\mathcal{A} \models \exists x_{n+1}\phi[\overline{a}]$. Each state description in $\Gamma(n, p)$ contains an occurrence of either $\exists x_{n+1}\phi$ or $\neg\exists x_{n+1}\phi$. If ψ is the unique state description in $\Gamma(n, p)$ such that $\mathcal{A} \models \psi[\overline{a}]$, then, since
$$\mathcal{A} \models \exists x_{n+1}\phi[\overline{a}],$$
the formula $\exists x_{n+1}\phi$ cannot occur negated in ψ. By hypothesis, we also have that $\mathcal{B} \models \psi[\overline{b}]$, and hence that
$$\mathcal{B} \models \exists x_{n+1}\phi[\overline{b}].$$
This means that there is a b in $\mathcal{B}$ such that
$$\mathcal{B} \models \phi[\overline{b}\hat{\ }b].$$
We showed that for some $\phi \in \Gamma(n+1, q)$, both
$$\mathcal{A} \models \phi[\overline{a}\hat{\ }a] \text{ and } \mathcal{B} \models \phi[\overline{b}\hat{\ }b].$$
By induction hypothesis,
$$(\mathcal{A}, \overline{a}\hat{\ }a) \approx_q (\mathcal{B}, \overline{b}\hat{\ }b),$$
which means that there is a function g that maps $\overline{a}\hat{\ }a$ to $\overline{b}\hat{\ }b$, and such that g is a q-partial isomorphism. It follows at once that g is an extension of f, and this item of the *back-and-forth* property is proved.

(2) If b is in $\mathcal{B}$, then the obvious changes in the above reasoning show that there is an element a in $\mathcal{A}$ and a q-partial isomorphism g between $(\mathcal{A}, \overline{a}\hat{\ }a)$ and $(\mathcal{B}, \overline{b}\hat{\ }b)$. It follows at once that g is an extension of f, and this item of the *back-and-forth* property is also proved.

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[1] In this case we say that the pairs $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ are in the same state of quantificational rank p, in n free variables.

[2] Notice that f is one-to-one, therefore a bijection, because $x_i = x_j$, for $i, j \leq n$, are atomic formulas, and $\overline{a}$ and $\overline{b}$ share exactly the same atomic relations.

The second statement follows from the first part of Fraïssé's Theorem: if $(\mathcal{A},\overline{a}) \approx_p (\mathcal{B},\overline{b})$, then for any formula φ with quantificational rank p and free variables x_1,..., x_n (such as the formulas in $\Gamma(n,p)$), $\mathcal{A} \models \varphi[\overline{a}]$ iff $\mathcal{B} \models \varphi[\overline{b}]$. □

Corollary 10. *The structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent iff for all $p \in \omega$, the empty function is a p-partial isomorphism between $(\mathcal{A},\emptyset)$ and $(\mathcal{B},\emptyset)$.*

Theorem 11. (Hintikka Normal Form Theorem) *If φ is a formula with quantificational rank at most p, and free variables among x_1,..., x_n, then φ is equivalent to a disjunction of formulas in $\Gamma(n,p)$. Given the formula φ, one such disjunction can be effectively found.*

Proof. Under the conditions of the statement, if $\psi \in \Gamma(n,p)$ then

$$\models \forall\overline{x}(\psi \to \varphi) \text{ or } \models \forall\overline{x}(\psi \to \neg\varphi),$$

and both can occur in case ψ is unsatisfiable. In fact, suppose that it is not the case that $\models \forall\overline{x}(\psi \to \varphi)$ or $\models \forall\overline{x}(\psi \to \neg\varphi)$. Therefore, there are structures $\mathcal{A}$ and $\mathcal{B}$, and n-tuples $\overline{a}$ in $\mathcal{A}$ and $\overline{b}$ in $\mathcal{B}$, such that

$$\mathcal{A} \models \neg(\psi \to \varphi)[\overline{a}] \text{ and } \mathcal{B} \models \neg(\psi \to \neg\varphi)[\overline{b}].$$

Therefore, $\mathcal{A} \models \psi[\overline{a}]$ and $\mathcal{A} \models \neg\varphi[\overline{a}]$, and $\mathcal{B} \models \psi[\overline{b}]$ and $\mathcal{B} \models \varphi[\overline{b}]$.

Since

$$\mathcal{A} \models \psi[\overline{a}] \text{ and } \mathcal{B} \models \psi[\overline{b}],$$

it follows from Fraïssé's Theorem, second part, that

$$(\mathcal{A},\overline{a}) \approx_p (\mathcal{B},\overline{b}).$$

However, $\mathcal{A} \models \neg\varphi[\overline{a}]$, $\mathcal{B} \models \varphi[\overline{b}]$ and φ is a formula with quantificational rank at most p. This is impossible, by Fraïssé's Theorem, first part, and it follows that

$$\models \forall\overline{x}(\psi \to \varphi) \text{ or } \models \forall\overline{x}(\psi \to \neg\varphi).$$

In order to find Hintikka's normal form, we can apply the following procedure: For each $\psi \in \Gamma(n,p)$,

search for a proof of $\forall\overline{x}(\psi \to \varphi)$ *and* for a proof of $\forall\overline{x}(\psi \to \neg\varphi)$.

By the above argument, at least one of them must be found, and when one of them is found stop searching. If the proof found is a proof of $\forall\overline{x}(\psi \to \varphi)$ then keep this ψ; if the proof found is a proof of $\forall\overline{x}(\psi \to \neg\varphi)$, then discard ψ. In the end, we are left with a set of $\psi's$ (the ones not discarded). Make the disjuction of this set, and call this disjunction ϕ.

Now, we can prove that

$$\models \forall\overline{x}(\varphi \leftrightarrow \phi).$$

In fact, since each disjunct in ϕ implies φ, it follows that ϕ implies φ. On the other hand, suppose that there are a structure $\mathcal{A}$ and an n-tuple $\overline{a}$ in $\mathcal{A}$, such that

$$\mathcal{A} \models \varphi[\overline{a}] \text{ and } \mathcal{A} \models \neg\phi[\overline{a}].$$

Since the disjunction of all formulas in $\Gamma(n,p)$ is valid, there is a $\psi \in \Gamma(n,p)$, which was discarded in the above procedure, and such that $\mathcal{A} \models \psi[\overline{a}]$. However, since ψ was discarded, a proof of

$$\forall \overline{x}(\psi \to \neg\varphi)$$

was found, and this formula is valid. Therefore, $\mathcal{A} \models \neg\varphi[\overline{a}]$, which is a contradiction.

□

Remark 12. If the satisfiability of the formulas $\psi \in \Gamma(n,p)$ were decidable, then the validity of formulas with quantificational rank at most p, and free variables x_1,..., x_n, would be decidable also. In fact, since the disjunction of *all* formulas in $\Gamma(n,p)$ is valid, a disjunction ϕ of *some* formulas in $\Gamma(n,p)$ is valid iff the remaining state descriptions in $\Gamma(n,p)$ that are not disjuncts of ϕ are unsatisfiable.

Definition 13. An extensional function O with predicate arguments in Σ (extensional function for short) of type $\to n$ is a function that assigns, for each structure $\mathcal{A}$, an n-ary predicate $\mathrm{O}^{\mathcal{A}}$. We say that O is $L_{\omega\omega}$-definable iff there is a formula φ in $L_{\omega\omega}$ such that (*i*) x_1,..., x_n are the free variables of φ, and (*ii*) for each structure $\mathcal{A}$ and each n-tuple $\overline{a}$ in $\mathcal{A}$,

$$\mathrm{O}^{\mathcal{A}}(\overline{a}) \text{ iff } \mathcal{A} \models \varphi[\overline{a}].$$

Definition 14. An extensional function O of type $\to n$ is preserved under p-equivalence iff for all structures $\mathcal{A}$ and $\mathcal{B}$, all n-tuples $\overline{a}$ in $\mathcal{A}$ and $\overline{b}$ in $\mathcal{B}$,

$$\text{If } (\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b}) \text{ then } (\mathrm{O}^{\mathcal{A}}(\overline{a}) \text{ iff } \mathrm{O}^{\mathcal{B}}(\overline{b}))$$

Theorem 15. (Fraïssé) *An extensional function* O *of type* $\to n$*, with* $n > 0$*, is* $L_{\omega\omega}$*-definable iff there is a* $p \in \omega$ *such that* O *is preserved under* p*-equivalence. More precisely, if* $p \in \omega$*, then an extensional function* O *of type* $\to n$ *is* $L_{\omega\omega}$*-definable by a formula* φ *with quantificational rank* p *and free variables* x_1*,...,* x_n *iff* O *is preserved under* p*-equivalence.*

Proof. Let us prove the second, and more precise assertion. Assume that O is preserved under p-equivalence, for some $p \in \omega$. Consider the set

$$\Delta^p(\mathrm{O}) = \{\psi \in \Gamma(n,p) : \exists(\mathcal{C}, \overline{c}); (\mathcal{C} \models \psi[\overline{c}]) \wedge \mathrm{O}^{\mathcal{C}}(\overline{c})\}.$$

It follows that O is defined by the disjunction of $\Delta^p(\mathrm{O})$, which is a formula φ in $L_{\omega\omega}$ with quantificational rank p. Indeed, given a structure $\mathcal{A}$ and an n-tuple $\overline{a}$ in $\mathcal{A}$, if $\mathcal{A} \models \varphi[\overline{a}]$ then, for some $\psi \in \Delta^p(\mathrm{O})$,

$$\mathcal{A} \models \psi[\overline{a}].$$

However, since $\psi \in \Delta^p(\mathrm{O})$, there is a couple $(\mathcal{C}, \overline{c})$ such that

$$\mathcal{C} \models \psi[\overline{c}] \text{ and } \mathrm{O}^{\mathcal{C}}(\overline{c}).$$

Since $\mathcal{A} \models \psi[\overline{a}]$ and $\mathcal{C} \models \psi[\overline{c}]$, it follows from Fraïssé's Theorem that

$$(\mathcal{C}, \overline{c}) \approx_p (\mathcal{A}, \overline{a}).$$

Therefore, $\mathrm{O}^{\mathcal{A}}(\overline{a})$, because O is preserved under p-equivalence and we already have that $\mathrm{O}^{\mathcal{C}}(\overline{c})$.

Now, assume that O is defined by a formula in $L_{\omega\omega}$, which is denoted by φ and has quantificational rank p, for some $p \in \omega$. If $(\mathcal{A}, \overline{a}) \approx_p (\mathcal{B}, \overline{b})$, then, since φ and has quantificational rank p,

$$\mathcal{A} \models \varphi[\overline{a}] \text{ iff } \mathcal{B} \models \varphi[\overline{b}],$$

which means that $\mathrm{O}^{\mathcal{A}}(\overline{a})$ iff $\mathrm{O}^{\mathcal{B}}(\overline{b})$, and O is preserved under p-equivalence. □