Practical Reasoning, Inconsitency and Modelling

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Topics

Next Topic

PSAT Formal Definition

- $x_1, \ldots, x_n$: atomic propositions
- $\varphi_1, \ldots, \varphi_k$: classical propositional formulas
- $\{P(\varphi_i) = p_i,\ 1 \leq i \leq k\}$: set of probabilistic constraints (PSAT instance)
- $W = \{w_1, \ldots, w_{2^n}\}$: possible worlds (valuations)
- $\pi : W \to [0, 1]$: probability mass
- $\pi(\varphi_i) = \sum\{\pi(w_j) | w_j \models \varphi_i\}$
- **Question:** Is there a π such that $\pi(\varphi_i) = p_i$, $1 \leq i \leq k$?

Results obtained for PSAT

- Theoretical study; normal form
  - Bridge Logic-Probability via Linear Algebra
  - Exponentially-sized linear programs
  - Logic probabilistic inference as optimization
- Polynomial reduction to SAT, NP-completeness
- 4 different algorithms
  - Phase transition detected for all algorithms
  - Opens sourse implementations: `psat.sourceforge.net`
- Applications to problems with hard-soft constraints
- Papers: IJCAI 2011, SAT 2013, AIJ 2015, AMAI 2015
- Extensions: JSBC 2015

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- Compare the inconsistency measure of incoherent agents (formal epistemology)
- Is any such $\mathcal{I}$ a possible inconsistency measure?

Dealing with Inconsistency

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- Hunter proposes postulates for inconsistency measures in classical bases
- Thimm extended those postulates to probabilistic logic, buth in an inconsistent way!
- We want to analyse and repair (consolidade) those postulates

Next Topic

1. Probabilistic Satisfiability (PSAT)
2. Measuiring Logic-Probabilistic Inconsistency
   - Classical Measurements
3. Distances
4. Extended Logic-Probabilistic Inference
   - Classical Inference
   - Extended Inference

Consistency Measurements and the Consistency Postulate

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Postulate (Consistency (Hunter 2006))

$\mathcal{I}(\Delta) = 0$ iff Δ is consistent

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- But Γ seems "more inconsistent"
- Δ has a single Minimal Inconsistent Subset, Γ has two

Measurements Based on Minimal Inconsistent Subsets (MIS)

$$\mathcal{I}_{MIS}(\Delta) = |MIS(\Delta)| = |\{\Psi | \Psi \textit{isa} \text{ MIS in } \Delta\}|$$

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$$\mathcal{I}_{MIS^C}(\Delta) = \sum_{\Psi \in MIS(\Delta)} \frac{1}{|\Psi|}$$

- $\mathcal{I}_{MIS^C}(\Delta) = 1/2 \qquad \mathcal{I}_{MIS^C}(\{P(\bot) = 0, 1\}) = 1.$

The Independence Postulate

- MISs are seen as "causing" inconsistencies.
- Formulas not in any MIS in Δ do not take part in Δ's inconsistency
- Those formulas are called *free* in Δ
- Adding a free formula in a base should not alter its inconsistency measurement

Postulate (Independence (Thimm 2013) after (Hunter 2006))

If α is free in Δ, then $\mathcal{I}(\Delta) = \mathcal{I}(\Delta \setminus \{\alpha\})$

Continuity Postulate

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- Map vectors to theories: Given $\Delta = \{P(\varphi_i) = p_i | 1 \leq i \leq k\}$, let $\Lambda_\Delta : [0, 1]^k \to \mathbb{K}$ such that $\Lambda_\Delta([q_1 \dots q_k]) = \{P(\varphi_i) = q_i | 1 \leq i \leq k\}$.

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The function $\mathcal{I} \circ \Lambda_\Delta : [0,1]^k \to [0,\infty)$ is continuous for all $\Delta \in \mathbb{K}$.

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- Classical measurements **do not** satisfy continuity!!!

The Incompatibility of Inconsistency Postulates

Theorem (De Bona and Finger 2015)

There is no inconsistency measurement that jointly satisfies the **consistency**, **continuity** *and* **independence** *postulates*

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- MISs do not capture the totality of existing conflicts
- Based on how probabilities are changed, a different notion of conflict may guarantee the compatibility of postulates

Widening as weakening

$$\Delta_1 = \{P(x) \in [0.4, 0.6], P(y) = 0.7\}$$
$$\Delta_2 = \{P(x) \in [0.3, 0.7], P(y) \in [0.6, 0.7]\}$$

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- Δ_2 is a **dominant consolidation** of Δ_1 if it is a consolidation that is a minimal widening

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- Widening to $[0, 1]$ has the effect of deleting a condition, so every base has a consolidation
- Δ_2 is a **dominant consolidation** of Δ_1 if it is a consolidation that is a minimal widening
- A probabilistic condition in Δ is **innocuous** if it belongs to every dominant consolidation of Δ

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An infinite number of measurements satisfy consistency, *i*-independence and continuity

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Measurements as Distances

- Idea: a measurement is the smallest distance between a base and one of its consolidations
- Distance in vectorial spaces are typically continuous
- For every $\Delta = \{P(\varphi_i) = p_i | 1 \leq i \leq k\}$, there is $q = [q_1\, q_2\, \ldots\, q_k]$ s.t. $\{P(\varphi_i) = q_i | 1 \leq i \leq k\}$ is consistent
- Let $\Delta[q]$ denote Δ with probabilities $q = [q_1\, q_2\, \ldots\, q_k]$
- Define the inconsistency measure of $\Delta = \Delta[p]$ as the smallest distance between p and q such that $\Delta[q]$ is consistent

Distances via ℓ-norms

Definition

Let $k, \ell \geq 1 \in \mathbb{Z}$. A distance ℓ-norm between $p = [p_1 \dots p_k]$ and $q = [q_1 \dots q_k]$:

$$d_\ell^k(p,q) = \sqrt[\ell]{\sum_{i=1}^{k} |p_i - q_i|^\ell}$$

- $\mathcal{I}_\ell(\Delta) = \min\{d_\ell^{|\Delta|}(p,q) | \Delta = \Delta[p], \Delta[q] \text{ consistent}\}$

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Theorem (De Bona and Finger 2015)

Every inconsistency measure based on ℓ-norm distance satisfy consistency, i-independence and continuity

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- $\mathcal{I}_\infty(\Delta_A) = 0.1 \qquad \mathcal{I}_\infty(\Delta_B) = 1/20 = 0.05$.

Two measurements lead to linear programs

- Minimize $d_\ell^k(p, q)$, such that each $P(\varphi) = q$ yield a linear restriction

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- Minimize $d_\ell^k(p, q)$, such that each $P(\varphi) = q$ yield a linear restriction
- Only $\ell = 1$, $\ell = \infty$ lead to linear programs using column generation
- $\mathcal{I}_1$ and $\mathcal{I}_\infty$ can be computed with greater efficiency
- Open problem: how to compute $\mathcal{I}_2$ with quadratic programming and column generation?

Final comment on Inconsistency Measures

- Inconsistency measures are related to a topic in the foundations of probability and Formal Epistemology: Dutch Books
- A Dutch Book is a bet which is guaranteed to yield a loss
- No loss is guaranteed iff laws of probabilities are obeyed
- Higher losses are associated with more inconsistent bases
- Different bets correspond to different inconsistency measures
- Details in [De Bona and Finger 2015]

Next Topic

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Logic Probabilistic Inference

Problem (Probabilistic Inference)

Given a PSAT instance $\Sigma = \{P(\alpha_i) = p_i\}$ and a target formula α, find the largest interval of probabilities $[\underline{p}, \bar{p}]$ for which α is consistent with Σ.

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Problem (Optimization version)

$$\begin{array}{rl} \textit{min/max} & P_\pi(\alpha) \\ \textit{subject to} & P_\pi(\alpha_i) = p_i \\ & \pi \geq 0 \quad \sum \pi_i = 1 \end{array}$$

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Note: $\max P(\alpha) = \min P(\neg\alpha)$

Inference Under Consistency

Problem (Phase 1: PSAT succeeds)

$$
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\textit{find} & \pi \\
\textit{such that} & P_\pi(\alpha_i) = p_i \\
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Problem (Phase 1: Linear algebra)

$$\begin{array}{rl} \textit{min} & c \cdot \pi \qquad \color{blue}{[= 0]} \\ \textit{subject to} & A \cdot \pi = p \\ & \pi \geq 0 \quad \sum \pi_i = 1 \quad c_j \in \{0, 1\} \\ & \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\ & c_j = 1 \textit{ iff column } A^j \textit{ is } \Gamma\textit{-inconsistent} \end{array}$$

Consistent Inference

Problem (Phase 2: Linear algebra)

$$
\begin{array}{rl}
\textit{min} & c \cdot \pi \qquad [\textit{min } P(\alpha)] \\
\textit{subject to} & A \cdot \pi = p \\
& \pi \geq 0 \quad \sum \pi_i = 1 \quad c_j \in \{0, 1\} \\
& \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\
& c_j = 1 \textit{ iff column } A^j \textit{ is } \alpha \wedge \Gamma\textit{-consistent}
\end{array}
$$

Consistent Inference

Problem (Phase 2: Linear algebra)

$$\begin{array}{rl} \textit{min} & c \cdot \pi \qquad [\textit{min } P(\alpha)] \\ \textit{subject to} & A \cdot \pi = p \\ & \pi \geq 0 \quad \sum \pi_i = 1 \quad c_j \in \{0,1\} \\ & \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\ & c_j = 1 \textit{ iff column } A^j \textit{ is } \alpha \wedge \Gamma\textit{-consistent} \end{array}$$

Problem (Phase 2: Linear algebra)

$$\begin{array}{rl} \textit{min} & c \cdot \pi \qquad [\textit{max } P(\alpha)] \\ \textit{subject to} & A \cdot \pi = p \\ & \pi \geq 0 \quad \sum \pi_i = 1 \quad c_j \in \{0,1\} \\ & \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\ & c_j = 1 \textit{ iff column } A^j \textit{ is } \neg\alpha \wedge \Gamma\textit{-consistent} \end{array}$$

Next Topic

Extended Inference Under Inconsistency

Problem (Phase 1: P-UnSAT)

$$
\begin{array}{rl}
\textit{min} & c \cdot \pi \qquad {\color{red}[> 0]} \\
\textit{subject to} & A \cdot \pi = p \\
& \pi \geq 0 \quad \sum \pi_i = 1 \quad c_j \in \{0, 1\} \\
& \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\
& c_j = 1 \textit{ iff column } A^j \textit{ is } \Gamma\textit{-inconsistent}
\end{array}
$$

Extended Inference: minimize distance from consistency

Problem (Phase 2: Linear algebra)

$$
\begin{array}{rl}
\textit{min} & \|\varepsilon\|_\ell \qquad [\textit{min } P(\alpha)] \\
\textit{subject to} & \varepsilon = A \cdot \pi - p \\
 & \pi \geq 0 \quad \sum \pi_i \leq 1 \\
 & \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\
 & \pi_j > 0 \textit{ if column } A^j \textit{ is } \alpha \wedge \Gamma\textit{-consistent}
\end{array}
$$

Extended Inference: minimize distance from consistency

Problem (Phase 2: Linear algebra)

$$
\begin{array}{rl}
\textit{min} & \|\varepsilon\|_\ell \qquad [\textit{min } P(\alpha)] \\
\textit{subject to} & \varepsilon = A \cdot \pi - p \\
& \pi \geq 0 \quad \sum \pi_i \leq 1 \\
& \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\
& \pi_j > 0 \textit{ if column } A^j \textit{ is } \alpha \wedge \Gamma\textit{-consistent}
\end{array}
$$

- $\ell = 1, \infty$: linear program, column generation

Extended Inference: minimize distance from consistency

Problem (Phase 2: Linear algebra)

$$\begin{array}{rl}
\textit{min} & \|\varepsilon\|_\ell \qquad [\textit{min } P(\alpha)] \\
\textit{subject to} & \varepsilon = A \cdot \pi - p \\
 & \pi \geq 0 \quad \sum \pi_i \leq 1 \\
 & \Sigma = (\Gamma, \Psi = \{P(y_i) = p_i\}), \quad a_{ij} = v_j(y_i), 1 \leq j \leq 2^n \\
 & \pi_j > 0 \textit{ if column } A^j \textit{ is } \alpha \wedge \Gamma\textit{-consistent}
\end{array}$$

- $\ell = 1, \infty$: linear program, column generation
- $\ell = 2$: quadratic program of exponential size

Inference of Conditional Probabilities

Problem (Conditional Model)

$$
\begin{array}{rl}
\textit{min/max} & P_\pi(\alpha|\beta) \\
\textit{subject to} & P_\pi(\alpha_i|\beta_i) = p_i \\
 & \pi \geq 0 \quad \sum \pi_i = 1 \quad P(\beta) > 0
\end{array}
$$

Inference of Conditional Probabilities

Problem (Conditional Model)

$$\begin{array}{rl} \textit{min/max} & P_\pi(\alpha|\beta) \\ \textit{subject to} & P_\pi(\alpha_i \wedge \beta_i) - p_i \cdot P(\beta_i) = 0 \\ & \pi \geq 0 \quad \sum \pi_i = 1 \quad P(\beta) > 0 \end{array}$$

- In the consistent case, can be solved with linear program and column generation
- In the inconsistent case: can be **approximated** with a linear program and column generation using $\|\varepsilon\|_\ell$
  - $\ell = 1, \infty$
- This is sometimes called the OPSAT problem