A Rewriting Characterization of Higher-Order Feasibility via Tuple Interpretations

Ongoing joint work with Patrick Baillot, Ugo dal Lago,
Cynthia Kop, and **Deivid Vale**
June 28, 2023

iCIS | Software Science
Radboud University

Outline

Poly-time in a nutshell

Higher-order Feasibility

BFFs Characterization

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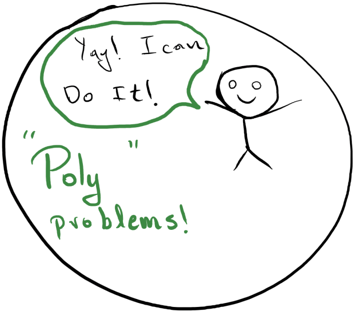

- Ordering a list of size n

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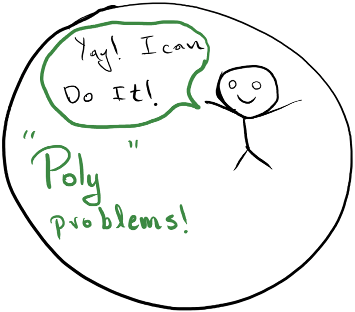

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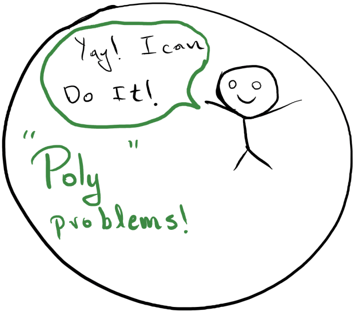
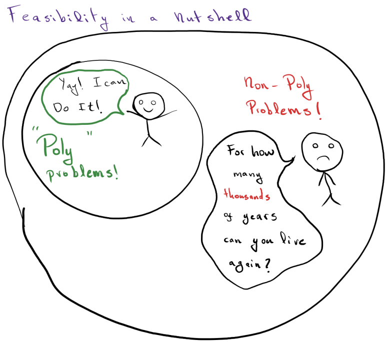


- Ordering a list of size n
- Computing the strongly connected components in a graph
- Adding/multiplying numbers (matrices)

Polytime in a nutshell



- decomposition of integers
- “a lot” of proof-searching algorithms
- automata learning

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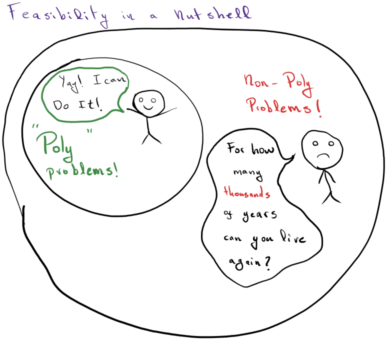

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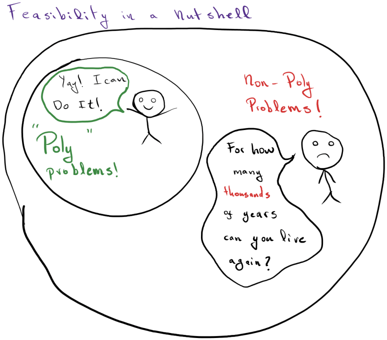

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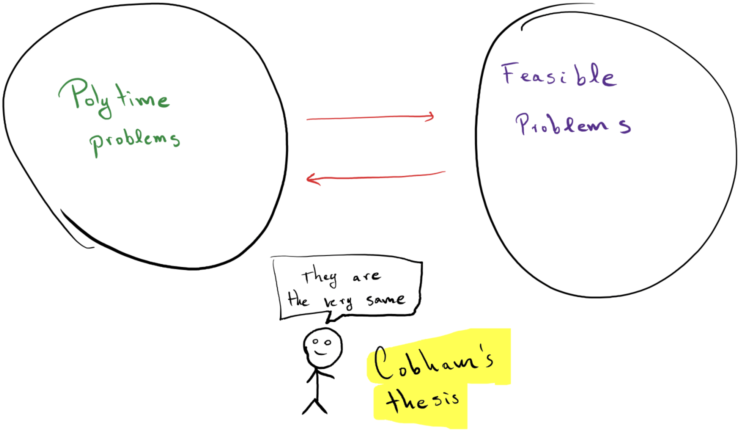
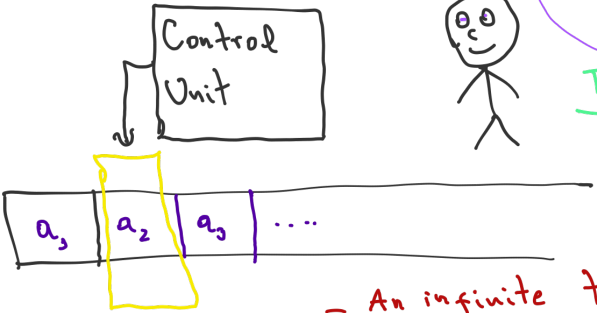
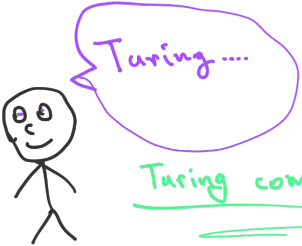

Poly time problems
Feasible Problems
They are the very same
Cobham's thesis

This is rather informal. . .





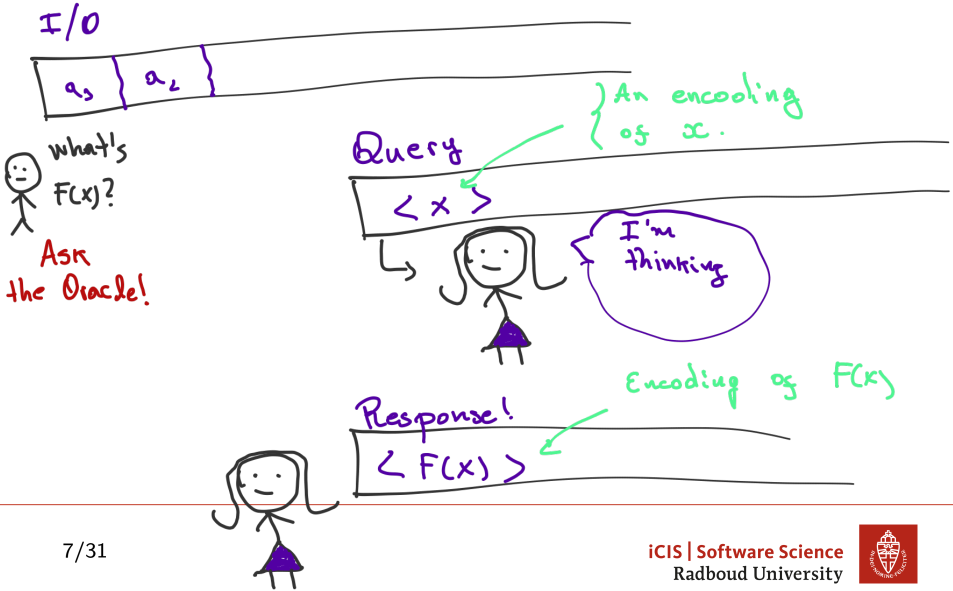
Turing computation

- An infinite tape
- The head moves L or R
- It can change the content of a cell

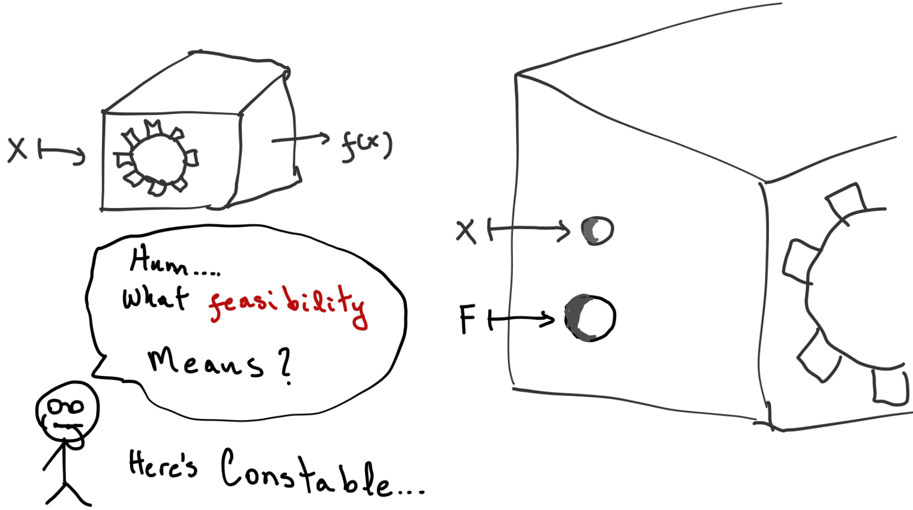
The "head"

Wait, machines have heads!

Hey Oracle! Compute this F at x for me!



Higher-Order what? "Feasibility!!!"



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Constable problem

Constable (1973) posed the problem of finding a **natural analogue** of polynomial time (P) for (type-2) functionals:

$$(\mathbb{N} \to \mathbb{N})^k \times \mathbb{N}^\ell \to \mathbb{N}$$

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- almost all **reasonable** models of deterministic computation are **polynomially** related
- both P and FP have good closure properties

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Good candidate? Let's bring. . . **BFFs**

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$$\texttt{TIME}_M(\vec{f}, \vec{x}) \leq P(\vec{f}, \vec{x})$$

Goal

Our goal is to characterize **BFFs** via **higher-order rewriting** and **tuple interpretations**.

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$$\mathbb{R}_{\mathtt{map}} := \begin{cases} \mathtt{map}\ F\ \mathsf{nil} \to \mathsf{nil} \\ \mathtt{map}\ F\ x :: q \to (F\ x)\ \mathtt{map}\ F\ q \end{cases}$$

Tuple Interpretations 101

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$$(\!|\sigma|\!) = \mathcal{C}_\sigma \times \mathcal{S}_\sigma$$

$$(\!|\mathsf{nat}|\!) = \langle \text{cost}, \text{number of s's} \rangle$$

$$\mathcal{J}_{\mathsf{0}} = \left\langle \; \boxed{0} \; , 1 \right\rangle \qquad \mathcal{J}_{\mathsf{s}} = \left\langle \; \boxed{\boldsymbol{\lambda} x.0} \; , \boldsymbol{\lambda} x.x + 1 \right\rangle$$

Tuple Interpretations 101

$$(\!|\sigma|\!) = \mathcal{C}_\sigma \times \mathcal{S}_\sigma$$

$$(\!|\mathsf{list}|\!) = \langle \mathsf{cost}, (\mathsf{length},\ \mathsf{maximum\ element}) \rangle$$

$$\mathcal{J}_{\mathsf{nil}} = \left\langle\ 0\ , (0, 0) \right\rangle$$

$$\mathcal{J}_{\mathsf{cons}} = \left\langle\ \boldsymbol{\lambda} x.\boldsymbol{\lambda} q.0\ , \boldsymbol{\lambda} xq.(q_{\mathsf{l}} + 1, \max(x, q_{\mathsf{m}})) \right\rangle$$

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How to characterize BFFs by Rewriting?

In order to capture BFFs we need to:

(**Soundness**) Show that if a TRS $\mathbb{R}$ **satisfying certain conditions** computes a type-2 functional $\alpha : (\mathbb{N} \longrightarrow \mathbb{N})^k \times \mathbb{N}^\ell \longrightarrow \mathbb{N}$

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then there exists a TRS $\mathbb{R}$ **satisfying the same certain conditions** that computes α.

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  - the cost int. of each oracle call is 1 and the size is polynomially bounded

How to characterize BFFs by Rewriting?

In order to capture BFFs we need to:

- show that every TRS **satisfying certain conditions** represent a BFF
- show that every BFF can be embedded as a TRS

Higher-Order Rewriting with Oracles

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- for each $n, m \in \mathbb{N}$ such that $m = f(n)$, there exists exactly one rule $\mathtt{f} \ulcorner \mathtt{n} \urcorner \rightarrow \ulcorner \mathtt{m} \urcorner$ in $\mathbb{R}$.

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We write $\mathbb{R}_{\mathtt{F},f,\mathtt{G}}$ for the infinite TRS consisting of the rules in $\mathbb{R}$ together with the rules defining f by way of $\mathtt{S}_f$ and the rule:

$$\mathtt{G}\ x \rightarrow \mathtt{F}\ \mathtt{S}_f\ x$$

First-Order Rewriting Computability

Definition (Type-1 Computability)

Let $(\mathbb{F}, \mathbb{R})$ be a TRS and $\mathtt{f} \in \Sigma$. We say that the symbol $\mathtt{f}$ $\mathbb{R}$-**computes** a type-1 function $f : \mathbb{N} \longrightarrow \mathbb{N}$ **whenever**

$$\mathtt{f} \ulcorner \mathtt{n} \urcorner \twoheadrightarrow \ulcorner \mathtt{m} \urcorner \text{ iff } f(n) = m.$$

Higher-Order Rewriting Computability

Definition (Type-2 Computability)
We say that in a finite TRS $\mathbb{R}$ the function symbol $\texttt{F} : (\mathsf{nat} \Rightarrow \mathsf{nat}) \Rightarrow \mathsf{nat} \Rightarrow \mathsf{nat}$ **computes** the type-2 functional $\alpha : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N} \longrightarrow \mathbb{N}$ iff

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- for every type-1 function f in $\mathbb{N}^{\mathbb{N}}$,
- the TRS $\mathbb{R}_{\texttt{F},f,\texttt{G}}$ is such that the symbol G computes $\alpha(f)$.

Polynomial tuple interpretations give BFF!

Theorem

Let $(\mathbb{F}, \mathbb{R})$ *be a finite TRS such that the symbol* $\texttt{F} \in \Sigma$ *computes the type-2 functional* $\alpha : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N} \longrightarrow \mathbb{N}$.

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then α *is in* $\texttt{BFF}$*.*

One tuple for the oracles that know it all!

$$\mathtt{G}\ x \rightarrow \mathtt{F}\ \mathtt{S}_f\ x$$

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$$\mathtt{G}\ x \to \mathtt{F}\ \mathtt{S}_f\ x$$

$$\mathcal{J}_{\mathtt{S}_f} = \left\langle \underbrace{(\boldsymbol{\lambda} x.1)}_{\text{cost of oracle}}, \underbrace{\boldsymbol{\lambda} x. \max_{y \leq x} f(y)}_{\text{size of oracle}} \right\rangle$$

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$$\begin{aligned} [\![\mathtt{S}_f\ \ulcorner \mathtt{n} \urcorner]\!] &= \left\langle (\boldsymbol{\lambda} x.1), \mathcal{J}^{\mathsf{s}}_{\mathtt{S}_f} \right\rangle \cdot \langle 0, n \rangle \\ &= \left\langle 1, \mathcal{J}^{\mathsf{s}}_{\mathtt{S}_f}(n) \right\rangle \\ &\succ \langle 0, m \rangle \end{aligned}$$

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$$\begin{aligned}
[\![\mathtt{G}\ x]\!] &= \left\langle 1 + \mathcal{J}^{\mathsf{c}}_{\mathtt{F}}(\left\langle \mathcal{J}^{\mathsf{c}}_{\mathtt{S}_f}, \mathcal{J}^{\mathsf{s}}_{\mathtt{S}_f} \right\rangle, x), \mathcal{J}^{\mathsf{s}}_{\mathtt{F}}(\mathcal{J}^{\mathsf{s}}_{\mathtt{S}_f}, x) \right\rangle \\
&\succ \left\langle \mathcal{J}^{\mathsf{c}}_{\mathtt{F}}(\left\langle \mathcal{J}^{\mathsf{c}}_{\mathtt{S}_f}, \mathcal{J}^{\mathsf{s}}_{\mathtt{S}_f} \right\rangle, x), \mathcal{J}^{\mathsf{s}}_{\mathtt{F}}(\mathcal{J}^{\mathsf{s}}_{\mathtt{S}_f}, x) \right\rangle \\
&= [\![\mathtt{F}]\!] \cdot [\![\mathtt{S}_f]\!] \cdot [\![x]\!] \\
&= [\![\mathtt{F}\ \mathtt{S}_f\ x]\!]
\end{aligned}$$

First-Order typed based interpretation

$$(\mathbb{F}, \mathbb{R})_f := \begin{cases} \mathtt{f}(0, y) \to y \\ \mathtt{f}(\mathsf{s}(x), y) \to \mathtt{f}(x, \mathsf{c}(y, y)) \end{cases}$$

$$\begin{array}{rclrcl} [\![0]\!] & = & 1 & [\![\mathsf{s}(x)]\!] & = & 4x + 1 \\ [\![\mathsf{c}(x, y)]\!] & = & x + y & [\![\mathtt{f}(x, y)]\!] & = & x + xy^2 + y \end{array}$$

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How many steps to normalize $t = f(\mathsf{s}^{100}(0), 0)$?

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$$\mathtt{f}(\mathsf{s}^{100}(0), 0) \quad \to \quad \mathtt{f}(\mathsf{s}^{99}(0), \underbrace{\mathsf{c}(0, 0)}_{c_0})$$

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$$
\begin{aligned}
\mathtt{f}(\mathsf{s}^{100}(0), 0) &\to \mathtt{f}(\mathsf{s}^{99}(0), \underbrace{\mathsf{c}(0, 0)}_{c_0}) \\
&\to \mathtt{f}(\mathsf{s}^{98}(0), \underbrace{\mathsf{c}(\mathsf{c}_0, \mathsf{c}_0)}_{\mathsf{c}_1})
\end{aligned}
$$

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& \vdots & \\
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The Size Explosion Problem

How many steps to normalize $t = f(\mathsf{s}^{100}(0), 0)$?

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Is the cost of $f(\mathsf{s}^n(0), 0)$ linear in n? c_{n-1} **is exponential in n!**

First-Order type-based interpretation

$$0 :: \mathsf{nat} \quad s :: \mathsf{nat} \Rightarrow \mathsf{nat} \quad c :: \mathsf{nat} \times \mathsf{nat} \Rightarrow \mathsf{nat} \quad f :: \mathsf{nat} \times \mathsf{nat} \Rightarrow \mathsf{nat}$$

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Lemma (Subterm Lemma)

Let $(\mathbb{F}, \mathbb{R})$ *be a term rewriting system admitting a CPI. Then there is a second-order polynomial interpretation* P *such that for every type-1 functional* $f : \mathbb{N} \longrightarrow \mathbb{N}$*, data term* $\ulcorner \mathtt{n} \urcorner : \mathsf{nat}$*, and context* C*:*

$$\textit{\textbf{if}}\ \mathtt{F}\ \mathtt{S}_f\ \ulcorner \mathtt{n} \urcorner \twoheadrightarrow C[\mathtt{S}_f\ \ulcorner \mathtt{m} \urcorner]$$

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$$\textbf{\textit{then}}\ |\ulcorner \mathtt{m} \urcorner| \leq P(|f|, |\ulcorner \mathtt{n} \urcorner|).$$

Polynomial tuple interpretations give BFF!

To prove this theorem we needed an interesting strategy:

- show that polynomial interpretations induce polynomial bounds to the runtime complexity of terms $\mathtt{G}\ \ulcorner\mathtt{n}\urcorner$

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To prove this theorem we needed an interesting strategy:

- show that polynomial interpretations induce polynomial bounds to the runtime complexity of terms $\mathtt{G}\ \ulcorner \mathtt{n} \urcorner$
- fix the size-explosion problem computing with **graph rewriting**
- show that OTMs can simulate graph rewriting with polynomial time overhead

Overview

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Thank you!