Uma Comparação do Método de Unificação de Ordem Superior de Huet e Unificação via Cálculos de Substituições Explícitas

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Outline

- Introduction
- Unification Tree Notation
- The $\lambda\sigma$-calculus
- The λs_e-calculus
- Comparing the $\lambda\sigma$- and the λs_e-styles of unification
- Conclusion

Motivation

- Higher-Order terms appear frequently in Mathematics, Logic, Automated Reasoning, etc.
- Higher Order Unification (HOU) is a basic operation extensively used in computational systems based on the λ-calculus such as functional programming languages and proof assistants.
- Explicit substitutions are a refinement of the λ-calculus in which the substitution operation is not treated as a meta-operation but as an operation of the calculus itself.

Simply typed λ-calculus in de Bruijn notation

Definition

The set $\Lambda_{dB}(\mathcal{X})$ of untyped λ-terms in de Bruijn notation:

$$a ::= \underline{\mathtt{n}} \mid X \mid (a\ b) \mid \lambda.a \qquad \text{where } n \in \mathbb{N} \text{ and } X \in \mathcal{X}.$$

The syntax of simply typed λ-calculus in de Bruijn notation:

Types	$A ::= K \mid A \to B$	
Contexts	$\Gamma ::= nil \mid A.\Gamma$	
Terms	$a ::= \underline{\mathtt{n}} \mid X \mid (a\ b) \mid \lambda_A.a$	where $n \in \mathbb{N}$ and $X \in \mathcal{X}$.

The type of the term a is indicated by $\tau(a)$.

Simply typed λ-calculus in de Bruijn notation

Definition

1. Every λ-term in β-normal form (β-nf) has the form

$$\lambda_{A_1} \dots \lambda_{A_n}.(h\ e_1 \dots e_p)$$

   where $n, p \geq 0$, h is a variable (or a constant) called its *head* and $e_1, \dots, e_p$ are λ-terms in β-nf called its *arguments*.
2. A λ-term in β-nf is *rigid* if its head is a constant or a bound variable. If it is a meta-variable, the term is *flexible*.

Simply typed λ-calculus in de Bruijn notation

3 Let $a \in \Lambda_{dB}(\mathcal{X})$ be a λ-term in de Bruijn notation of type $A_1 \to \ldots \to A_m \to B$ with B atomic. The η-*long* form of a β-nf term a, written a', is inductively defined as follows:

- if $a = \lambda_A.b$ then $a' = \lambda_A.b'$.
- if $a = (\underline{\mathtt{n}}\ b_1 \ldots b_q)$ then $a' = \lambda_{A_1} \ldots \lambda_{A_m}.(\underline{\mathtt{n}+\mathtt{m}}\ c_1 \ldots c_q\ \underline{\mathtt{m}}' \ldots \underline{\mathtt{1}}')$, where c_j $(1 \leq j \leq q)$ is the η-long form of the normal form of $U_0^{m+1}(b_j)$.
- if $a = (X\ b_1 \ldots b_q)$ then $a' = \lambda_{A_1} \ldots \lambda_{A_m}.(X\ c_1 \ldots c_q\ \underline{\mathtt{m}}' \ldots \underline{\mathtt{1}}')$, where c_j $(1 \leq j \leq q)$ is the η-long form of the normal form of $U_0^{m+1}(b_j)$.

Unification problems

Definition

A *unification equation* is an equation of the form $a =^? b$ where a and b are λ-terms of the same type and under the same context.
A *unification problem* is a finite set of unification equations.

Examples:

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- $(X_{A \to A}^{A \cdot nil}\ \underline{1}_A^{A \cdot nil}) =^? (\underline{2}_{A \to A}^{A \cdot nil}(Y_{A \to A}^{A \cdot nil}\ \underline{1}_A^{A \cdot nil}))$

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- Solutions: $\sigma_1 = \{X/\lambda_A.(\underline{3}\ \underline{1}), Y/\lambda_A.\underline{1}\}$
  $\sigma_2 = \{X/\lambda_A.(\underline{3}\ \underline{2}), Y/\lambda_A.\underline{1}\}$

Unification problems

- Let $\Delta = A \to A \cdot A \cdot nil$ be a context.
  $\lambda_A.(\underline{2}^{A\cdot\Delta}_{A\to A}\ X^{A\cdot\Delta}_A) =^? \lambda_A.(\underline{2}^{A\cdot\Delta}_{A\to A}\underline{3}^{A\cdot\Delta}_A)$

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- Let $\Delta = A \rightarrow A \cdot A \cdot nil$ be a context. $\lambda_A.(\underline{2}^{A\cdot\Delta}_{A\rightarrow A}\ X^{A\cdot\Delta}_{A}) =^? \lambda_A.(\underline{2}^{A\cdot\Delta}_{A\rightarrow A}\underline{3}^{A\cdot\Delta}_{A})$
- Solution: $X/\underline{2}$
- $\lambda_A.X^{A\cdot\Gamma}_A =^? \lambda_A.\underline{1}^{A\cdot\Gamma}_A$, where Γ is any context, does not have solutions.

The procedure SIMPL

INPUT: A unif. problem P with at least one rigid-rigid equation:

$$\lambda_{A_1}\ldots\lambda_{A_r}.(\underline{\mathtt{n}}\ e^1_1\ldots e^1_{p_1}) =^? \lambda_{A_1}\ldots\lambda_{A_r}.(\underline{\mathtt{m}}\ e^2_1\ldots e^2_{p_2}) \wedge P'$$

where $r, p_1, p_2 \geq 0$ and $n, m > 0$.
WHILE there exists a rigid-rigid equation in P DO
If $n \neq m$ then stop and report a failure status else let $p = p_1 = p_2$ and replace the selected equation by the conjunction

$$\lambda_{A_1}\ldots\lambda_{A_r}.e^1_1 =^? \lambda_{A_1}\ldots\lambda_{A_r}.e^2_1 \wedge\ldots\wedge \lambda_{A_1}\ldots\lambda_{A_r}.e^1_p =^? \lambda_{A_1}\ldots\lambda_{A_r}.e^2_p$$

in P and call the result $\overline{P}$ (the simplified version of P).
DONE.
If there exists a flexible-rigid equation in $\overline{P}$ then return $\overline{P}$ else stop and report a success status.

Example of SIMPL

$$\lambda_{A\to A\to A}\lambda_A.(\underline{2}_{A\to A\to A}\ X_A\ \underline{1}_A) =^? \lambda_{A\to A\to A}\lambda_A.(\underline{2}_{A\to A\to A}\ \underline{3}_A\ (Y_{A\to A}\ \underline{1}))$$

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$$\lambda_{A\to A\to A}\lambda_A.\underline{1}_A =^? \lambda_{A\to A\to A}\lambda_A.(Y_{A\to A}\ \underline{1})$$

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$$\lambda_{A_1} \ldots \lambda_{A_r}.(X\ e^1_1 \ldots e^1_{p_1}) =^? \lambda_{A_1} \ldots \lambda_{A_r}.(\underline{\mathrm{n}}\ e^2_1 \ldots e^2_{p_2}) \tag{1}$$

where $\tau(X) = B_1 \to \ldots \to B_{p_1} \to C$, where $p_1, p_2, r \geq 0$, $n > 0$ and C is atomic.

The procedure MATCH

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  where $\tau(X) = B_1 \to \ldots \to B_{p_1} \to C$, where $p_1, p_2, r \geq 0$, $n > 0$ and C is atomic.
- The procedure MATCH is based on two rules named *imitation* and *projection*.

The imitation rule

The imitation substitution corresponds exactly to the η-long term of the type of X, whose head corresponds to the head of the rigid term:

$$X/\lambda_{B_1}\ldots\lambda_{B_{p_1}}.(\underline{\mathtt{p}_1+\mathtt{n}-\mathtt{r}}\,(X_1\ \underline{\mathtt{p}_1}\ldots\underline{1})\ldots(X_{p_2}\ \underline{\mathtt{p}_1}\ldots\underline{1}))$$

where $X_1,\ldots,X_{p_2}$ are meta-variables with appropriate type and all sub-terms are in η-normal form.

Imitation example

Consider the equation:

$$\lambda_A\lambda_A.(X_{A\to A}\ \underline{1}_A) =^? \lambda_A\lambda_A.(\underline{3}_{A\to A}(Y_{A\to A}(\underline{4}_{A\to A}\ \underline{1}_A)))$$

Generated imitation substitution:

$$X_{A\to A}/\lambda_A.(\underline{2}_{A\to A}\ (X_{1_{A\to A}}\ \underline{1}_A))$$

The projection rule

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- The projections substitutions always have the form $\lambda_{B_1} \ldots \lambda_{B_{p_1}}.(\underline{\mathrm{i}}\ (X_1\ \underline{\mathrm{p}_1} \ldots \underline{1}) \ldots (X_k\ \underline{\mathrm{p}_1} \ldots \underline{1}))$, where $1 \leq i \leq p_1$.

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- This gives at most p_1 possible different projections, one for each argument of X.

Projection example

Consider the equation:

$$\lambda_A \lambda_A.(X_{A\to A}\ \underline{1}_A) =^? \lambda_A \lambda_A.\underline{1}_A$$

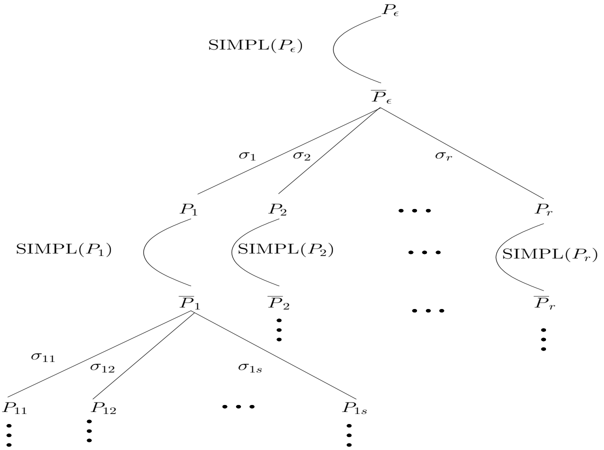
Generated projection substitution:

$$X_{A\to A}/\lambda_A.\underline{1}_A$$

Unification Tree Notation

- The unification tree notation is obtained from the matching tree of Huet by adding labels to the unification problems as well as to the generated substitutions.
- These labels provide information about the position of the unification problems and of the substitutions in the matching tree.
- Facilitates the computation of the solutions.

Visualising the Tree

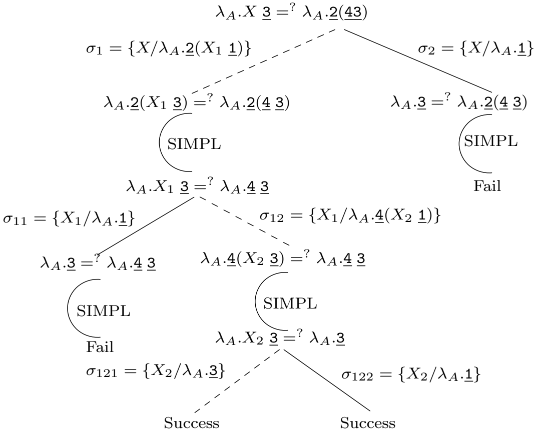


Formal Construction

A unification tree, for a given unification problem P, is given by:

1. Label P with ϵ (the empty position) as a subscript, i.e., P_ϵ.
2. For a node labeled with P_q, its sibling node is labeled with $\overline{P_q}$ whenever the unification problem derives by applying the procedure SIMPL. This step is represented by a curly line in the unification.
3. For a node labeled with P_q containing a flexible-rigid equation, call $\sigma_{q1}, \sigma_{q2}, \ldots, \sigma_{qk}$ the incremental substitutions generated by an application of the procedure MATCH to this equation.
4. The sibling nodes of P_q, written $P_{q1}, \ldots, P_{qk}$ are defined by the composition $P_{qi} := \overline{P_q}\sigma_{qi}$, for $i = 1, \ldots, k$.

Example



_ _ _ _ dashed lines denote imitation steps
——— solid lines denote projection steps

The $\lambda\sigma$-calculus

The syntax of typed $\lambda\sigma$-calculus is given by

Types	A	$::=$	$K \mid A \rightarrow B$
Contexts	Γ	$::=$	$nil \mid A \cdot \Gamma$
Terms	a	$::=$	$\underline{1} \mid X \mid (a\ b) \mid \lambda_A.a \mid a[s]$
Substitutions	s	$::=$	$id \mid \uparrow \mid a \cdot s \mid s \circ s$

The $\lambda\sigma$-calculus

The typing rules:

(var) $A.\Gamma \vdash \underline{1} : A$

(lambda) $$\frac{A.\Gamma \vdash a : B}{\Gamma \vdash \lambda_A.a : A \to B}$$

(app) $$\frac{\Gamma \vdash a : A \to B \quad \Gamma \vdash b : A}{\Gamma \vdash (a\ b) : B}$$

(clos) $$\frac{\Gamma \vdash s \rhd \Gamma' \quad \Gamma' \vdash a : A}{\Gamma \vdash a[s] : A}$$

(id) $\Gamma \vdash id \rhd \Gamma$

(shift) $A.\Gamma \vdash \uparrow \rhd \Gamma$

(cons) $$\frac{\Gamma \vdash a : A \quad \Gamma \vdash s \rhd \Gamma'}{\Gamma \vdash a.s \rhd A.\Gamma'}$$

(comp) $$\frac{\Gamma \vdash s'' \rhd \Gamma'' \quad \Gamma'' \vdash s' \rhd \Gamma'}{\Gamma \vdash s' \circ s'' \rhd \Gamma'}$$

(meta) $\Gamma \vdash X : A$, where Γ is any context.

The $\lambda\sigma$-calculus

(Beta)	$(\lambda.a)b$	$\longrightarrow$	$a\,[b \cdot id]$
(App)	$(a\ b)[s]$	$\longrightarrow$	$(a\,[s])\,(b\,[s])$
(Abs)	$(\lambda.a)[s]$	$\longrightarrow$	$\lambda(a\,[\underline{1} \cdot (s \circ \uparrow)])$
(Clos)	$(a\,[s])[t]$	$\longrightarrow$	$a\,[s \circ t]$
(VarCons)	$\underline{1}\,[a \cdot s]$	$\longrightarrow$	a
(Id)	$a[id]$	$\longrightarrow$	a
(Assoc)	$(s \circ t) \circ u$	$\longrightarrow$	$s \circ (t \circ u)$
(Map)	$(a \cdot s) \circ t$	$\longrightarrow$	$a\,[t] \cdot (s \circ t)$
(IdL)	$id \circ s$	$\longrightarrow$	s
(IdR)	$s \circ id$	$\longrightarrow$	s
(ShiftCons)	$\uparrow \circ (a \cdot s)$	$\longrightarrow$	s
(VarShift)	$\underline{1} \cdot \uparrow$	$\longrightarrow$	id
(SCons)	$\underline{1}[s] \cdot (\uparrow \circ s)$	$\longrightarrow$	s
(Eta)	$\lambda.(a\ \underline{1})$	$\longrightarrow$	b if $a =_\sigma b[\uparrow]$

Unification in the $\lambda\sigma$-calculus

Motivation:

- Reduce substitution to grafting.
- Remain closer to implementations.
- Development of a programming language based on ES that includes HOU in a lower level.

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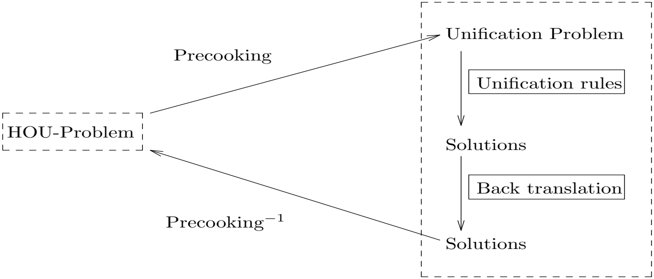
- Reduce substitution to grafting.
- Remain closer to implementations.
- Development of a programming language based on ES that includes HOU in a lower level.
- Possible drawback:
  - Inclusion of a non-trivial equational theory (??).

Unification in Explicit Substitutions Calculi



Language of the λ-calculus
substitution

Language of the explicit substitutions calculus.
grafting

The $\lambda\sigma$-unification rules (part I)

The $\lambda\sigma$-simplification rules:

Dec-λ $$\frac{P \wedge \lambda_A.e_1 =^?_{\lambda\sigma} \lambda_A.e_2}{P \wedge e_1 =^?_{\lambda\sigma} e_2}$$

Dec-App $$\frac{P \wedge (\underline{\mathtt{n}}\ e^1_1 \ldots e^1_p) =^?_{\lambda\sigma} (\underline{\mathtt{n}}\ e^2_1 \ldots e^2_p)}{P \wedge e^1_1 =^?_{\lambda\sigma} e^2_1 \wedge \ldots \wedge e^1_p =^?_{\lambda\sigma} e^2_p}$$

Dec-Fail $$\frac{P \wedge (\underline{\mathtt{n}}\ e^1_1 \ldots e^1_{p_1}) =^?_{\lambda\sigma} (\underline{\mathtt{m}}\ e^2_1 \ldots e^2_{p_2})}{\mathit{Fail}}, \text{ if } \mathtt{m} \neq \mathtt{n}.$$

The SIMPL$_{\lambda\sigma}$ procedure

`INPUT`: A unification problem P_q (in the language of the $\lambda\sigma$-calculus) with at least one rigid-rigid equation.
`OUTPUT`: A terminal (failure or success) status or an equivalent unification problem $\overline{P_q}$ without rigid-rigid equations and containing at least one flexible-rigid equation.

Assume that **Dec-**λ is applied eagerly.
`WHILE` there exists a rigid-rigid equation in P_q `DO`

1. Apply **Dec-Fail**, if possible.
2. Apply **Dec-App**, and if the resulting unification problem contains a flexible-rigid equation, call it $\overline{P_q}$ and give $\overline{P_q}$ as result, else stop and report a success status.

`DONE`.

The SIMPL$_{\lambda\sigma}$ procedure

Theorem

The application of the procedure SIMPL$_{\lambda\sigma}$ to any unification problem P (in the language of the $\lambda\sigma$-calculus) always terminates.

Proof.[Sketch] Applications of the simplification rules decrease the size of the terms in the equations. □

The SIMPL and SIMPL$_{\lambda\sigma}$ correspondence

Theorem

If P is a unification problem in the pure λ-calculus and P_F its precooked image, then:

1. *$SIMPL(P)$ fails $\Leftrightarrow$ $SIMPL_{\lambda\sigma}(P_F)$ fails.*
2. *$SIMPL(P)$ stops and reports a success status $\Leftrightarrow$ $SIMPL_{\lambda\sigma}(P_F)$ stops and reports a success status;*
3. *$SIMPL(P)$ returns a unification problem containing at least one flexible-rigid equation $\Leftrightarrow$ $SIMPL_{\lambda\sigma}(P_F)$ returns a unification problem containing at least one flexible-rigid equation.*

Solved forms

Definition (DHK00)

A unification problem P is in $\lambda\sigma$-*solved form* if all its meta-variables are of atomic type and it is a conjunction of nontrivial equations of the following forms:

- **Solved**: $X =^?_{\lambda\sigma} a$ where the meta-variable X does not appear anywhere else in P and a is in **Eta**-long form. Such an equation is said to be *solved* in P and the variable X is also said to be solved.
- **Flexible-flexible**: $X[a_1.\cdots.a_p.\uparrow^n] =^?_{\lambda\sigma} Y[b_1.\cdots.b_q.\uparrow^m]$, where $X[a_1.\cdots.a_p.\uparrow^n]$ and $Y[b_1.\cdots.b_q.\uparrow^m]$ are **Eta**-long terms and the equation is not solved.

The $\lambda\sigma$-unification rules (part II)

Exp-λ
$$\frac{P}{\exists Y : (A.\Gamma \vdash B), P \wedge X =^?_{\lambda\sigma} \lambda_A Y}$$
if $(X : \Gamma \vdash A \to B) \in \mathcal{TV}ar(P)$, $Y \notin \mathcal{TV}ar(P)$, and X is not a solved variable.

Normalise
$$\frac{P \wedge e_1 =^?_{\lambda\sigma} e_2}{P \wedge e_1' =^?_{\lambda\sigma} e_2'}$$ if e_1 or e_2 is not in long form,
where e_1' (resp. e_2') is the long form of e_1 (resp. e_2) if e_1 (resp. e_2) is not a solved variable and e_1 (resp. e_2) otherwise.

Replace
$$\frac{P \wedge X =^?_{\lambda\sigma} t}{\{X \mapsto t\}(P) \wedge X =^?_{\lambda\sigma} t}$$ if $X \in \mathcal{TV}ar(P), X \notin \mathcal{TV}ar(t)$ and if t is a constant then $t \in \mathcal{TV}ar(P)$.

The $\lambda\sigma$-unification rules (part III)

$$\textbf{Exp-App}\ \frac{P \wedge X[a_1.\cdots.a_p.\uparrow^n] =^?_{\lambda\sigma} \underline{\mathtt{m}}(b_1,\ldots,b_q)}{P \wedge X[a_1.\cdots.a_p.\uparrow^n] =^?_{\lambda\sigma} \underline{\mathtt{m}}(b_1,\ldots,b_q) \wedge \bigvee_{r \in R_p \cup R_i} \exists H_1 \ldots \exists H_k, X =^?_{\lambda\sigma} \underline{\mathtt{r}}(H_1,\ldots,H_k)}$$

if X has an atomic type and is not solved.
where $H_1, \ldots, H_k$ are variables of appropriate types, not occurring in P, with the contexts $\Gamma_{H_i} = \Gamma_X$, R_p is the subset of $\{1, \ldots, p\}$ such that $\underline{\mathtt{r}}(H_1, \ldots, H_k)$ has the right type, $R_i =$if $m \geq n+1$ then $\{m-n+p\}$ else $\emptyset$.

The procedure MATCH$_{\lambda\sigma}$

`INPUT`: A unification system P_q with at least one flexible-rigid equation.
`OUTPUT`: A disjunction of equivalent unification systems, written $P_{q1} \vee \ldots \vee P_{qk}$.
Assume that the rule **Dec-**λ is applied eagerly.

1. Apply **Exp-**λ and **Replace** as much as possible to the selected equation and call P'_q the resulting unification system.
2. Apply **Exp-App** and **Replace** and **Normalise** to P'_q and call $P_{q1} \vee \ldots \vee P_{qk}$ the resulting unification problem.

The procedure MATCH$_{\lambda\sigma}$

Definition

Let X/a be a substitution generated in the pure λ-calculus by Huet's algorithm. We say that the equation $Y =^?_\xi b$ *corresponds (or is associated)* to the substitution X/a if X and Y are two meta-variables of the same type and the terms a and b have the same headings, where $\xi \in \{\lambda\sigma, \lambda s_e\}$.

Correspondence from MATCH to MATCH$_{\lambda\sigma}$

Theorem

Let $\lambda_{A_1}\ldots\lambda_{A_r}.(X\ e^1_1\ldots e^1_{p_1}) =^? \lambda_{A_1}\ldots\lambda_{A_r}.(\underline{\mathrm{n}}\ e^2_1\ldots e^2_{p_2})$ *be a flexible-rigid equation in* η*-long form in the pure* λ*-calculus where* $p_1, p_2, r \geq 0$ *and* $\tau(X) = B_1 \to \ldots \to B_{p_1} \to B$ *with* B *atomic. Then, for each substitution generated by the procedure MATCH, when applied to this equation, there exists a corresponding equation in the* $\lambda\sigma$*-calculus generated by the procedure MATCH*$_{\lambda\sigma}$ *to the precooked version of the given equation.*

Correspondence from MATCH$_{\lambda\sigma}$ to MATCH

Theorem

For each new generated equation by the rule **Exp-App**, *when applied to a flexible-rigid equation which is in the image of the precooking translation, there exists a corresponding substitution in the pure λ-calculus in the following sense: for each element in R_p there exists a corresponding substitution in the pure λ-calculus and, if $R_i \neq \emptyset$ then there exists an imitation in the pure λ-calculus for the inverse of the precooking translation applied to this equation.*

The MATCH and MATCH$_{\lambda\sigma}$ correspondence

Theorem

Let eq be a flexible-rigid equation in η-long form in the pure λ-calculus and eq_F its precooked image. Then, MATCH applied to eq generates a substitution σ if and only if MATCH$_{\lambda\sigma}$ applied to eq_F generates a substitution equivalent to σ.

The Main Procedure

`INPUT`: A unification system P_ϵ.
`OUTPUT`: A success or a failure status and in the former case the solutions are the solved equations whose left-hand side corresponds to the meta-variables of the initial problem. If the initial problem is non-unifiable the algorithm may not terminate.

1. If P_q contains a rigid-rigid equation, then apply SIMPL$_{\lambda\sigma}$ and go to the next step, else if P_q contains a non-solved flex-rig equation then rename it to $\overline{P_q}$ and go to the next step.
2. Apply MATCH$_{\lambda\sigma}$ to $\overline{P_q}$ and let $P_{q1} \vee \ldots \vee P_{qr}$ be the resulting unification problem.
3. If the current unification problem contains a unification system not in solved form then select it and go to step 1, else stop and report a success status.

The λs_e-grammar

Terms of the λs_e-**calculus** are given by:

$$\Lambda s_e \ ::= \ \underline{\mathtt{n}} \mid X \mid \Lambda s_e \Lambda s_e \mid \lambda.\Lambda s_e \mid \Lambda s_e \sigma^j \Lambda s_e \mid \varphi^i_k \Lambda s_e,$$

where $n, j, i \geq 1$, $k \geq 0$ and $X \in \mathcal{X}$.

The typing rules:

(var) $A.\Gamma \vdash \underline{1} : A$

(varn) $\dfrac{\Gamma \vdash \underline{\mathtt{n}} : B}{A.\Gamma \vdash \underline{\mathtt{n}+1} : B}$

(app) $\dfrac{\Gamma \vdash a : A \rightarrow B \quad \Gamma \vdash b : A}{\Gamma \vdash \ (a\ b) : B}$

(lambda) $\dfrac{A.\Gamma \vdash a : B}{\Gamma \vdash \lambda_A.a : A \rightarrow B}$

(sigma) $\dfrac{\Gamma_{\geq i} \vdash b : B \quad \Gamma_{<i}.B.\Gamma_{\geq i} \vdash a : A}{\Gamma \vdash a\sigma^i b : A}$

(phi) $\dfrac{\Gamma_{\leq k}.\Gamma_{\geq k+i} \vdash a : A}{\Gamma \vdash \varphi^i_k a : A}$

(meta) $\Gamma \vdash X : A$, where Γ is any context.

(σ-generation)	$(\lambda.a)\ b$	$\longrightarrow$	$a\,\sigma^1\, b$
(σ-λ-transition)	$(\lambda.a)\sigma^i b$	$\longrightarrow$	$\lambda.(a\sigma^{i+1} b)$
(σ-app-transition)	$(a_1\ a_2)\sigma^i b$	$\longrightarrow$	$(a_1\sigma^i b)(a_2\sigma^i b)$
(σ-destruction)	$\underline{\mathtt{n}}\sigma^i b$	$\longrightarrow$	$\begin{cases} \underline{\mathtt{n-1}} & \text{if} \quad n > i \\ \varphi_0^i\ b & \text{if} \quad n = i \\ \underline{\mathtt{n}} & \text{if} \quad n < i \end{cases}$
(φ-λ-transition)	$\varphi_k^i(\lambda.a)$	$\longrightarrow$	$\lambda.(\varphi_{k+1}^i a)$
(φ-app-transition)	$\varphi_k^i(a_1\ a_2)$	$\longrightarrow$	$(\varphi_k^i a_1)(\varphi_k^i a_2)$
(φ-destruction)	$\varphi_k^i\underline{\mathtt{n}}$	$\longrightarrow$	$\begin{cases} \underline{\mathtt{n+i-1}} & \text{if} \quad n > k \\ \underline{\mathtt{n}} & \text{if} \quad n \leq k \end{cases}$
(Eta)	$\lambda.(a\ \underline{1})$	$\longrightarrow$	b if $a =_{s_e} \varphi_0^2 b$

(σ-σ-transition)	$(a \sigma^i b)\, \sigma^j\, c \longrightarrow (a\, \sigma^{j+1}\, c)\, \sigma^i\, (b\, \sigma^{j-i+1}\, c)$	if $i \le j$
(σ-φ-transition 1)	$(\varphi^i_k\, a)\, \sigma^j\, b \longrightarrow \varphi^{i-1}_k\, a$	if $k < j < k+i$
(σ-φ-transition 2)	$(\varphi^i_k\, a)\, \sigma^j\, b \longrightarrow \varphi^i_k (a\, \sigma^{j-i+1}\, b)$	if $k+i \le j$
(φ-σ-transition)	$\varphi^i_k (a\, \sigma^j\, b) \longrightarrow (\varphi^i_{k+1}\, a)\, \sigma^j\, (\varphi^i_{k+1-j}\, b)$	if $j \le k+1$
(φ-φ-transition 1)	$\varphi^i_k\, (\varphi^j_l\, a) \longrightarrow \varphi^j_l\, (\varphi^i_{k+1-j}\, a)$	if $l+j \le k$
(φ-φ-transition 2)	$\varphi^i_k\, (\varphi^j_l\, a) \longrightarrow \varphi^{j+i-1}_l\, a$	if $l \le k < l+j$

λs_e-unification rules (part I)

Dec-λ
$$\frac{P \wedge \lambda_A.e_1 =^? \lambda_A.e_2}{P \wedge e_1 =^? e_2}$$

Dec-App
$$\frac{P \wedge \underline{\mathtt{n}}(e_1^1, \ldots, e_p^1) =^? \underline{\mathtt{n}}(e_1^2, \ldots, e_p^2)}{P \wedge e_1^1 =^? e_1^2 \wedge \ldots \wedge e_p^1 =^? e_p^2}$$

App-Fail
$$\frac{P \wedge \underline{\mathtt{n}}(e_1^1, \ldots, e_{p_1}^1) =^? \underline{\mathtt{m}}(e_1^2, \ldots, e_{p_2}^2)}{Fail}, \text{ if } \mathtt{m} \neq \mathtt{n}.$$

λs_e-unification rules (part II)

Exp-λ

$$\frac{P}{\exists Y : (A.\Gamma \vdash B), P \wedge X =^?_{\lambda\sigma} \lambda_A Y}$$

if $(X : \Gamma \vdash A \rightarrow B) \in \mathcal{TV}ar(P)$, $Y \notin \mathcal{TV}ar(P)$, and X is not a solved variable.

Replace

$$\frac{P \wedge X =^?_{\lambda s_e} t}{\{X/t\}(P) \wedge X =^?_{\lambda s_e} t}$$ if $X \in \mathcal{TV}ar(P), X \notin \mathcal{TV}ar(t)$ and if $t \in \mathcal{X} \Rightarrow t \in \mathcal{TV}ar(P)$.

Normalise

$$\frac{P \wedge e_1 =^?_{\lambda s_e} e_2}{P \wedge e_1' =^?_{\lambda\sigma} e_2'}$$ if e_1 or e_2 is not in long form.

where e_1' (resp. e_2') is the long form of e_1 (resp. e_2), if e_1 (resp. e_2) is not solved and e_1 (resp. e_2) otherwise.

λs_e-unification rules (part III)

Exp-App

$$\frac{P \wedge \psi_{i_p}^{j_p} \ldots \psi_{i_1}^{j_1}(X, a_1, \ldots, a_p) =^?_{\lambda s_e} \underline{\mathtt{m}}(b_1, \ldots, b_q)}{P \wedge \psi_{i_p}^{j_p} \ldots \psi_{i_1}^{j_1}(X, a_1, \ldots, a_p) =^?_{\lambda\sigma} \underline{\mathtt{m}}(b_1, \ldots, b_q) \wedge \bigvee_{r \in R_p \cup R_i} \exists H_1 \ldots \exists H_k, X =^?_{\lambda s_e} \underline{\mathtt{r}}(H_1, \ldots, H_k)}$$

if $\psi_{i_p}^{j_p} \ldots \psi_{i_1}^{j_1}(X, a_1, \ldots, a_p)$ is the skeleton of a λs_e normal term, and X has an atomic type and is not solved, where $H_1, \ldots, H_k$ are meta-variables of appropriate types, not occurring in P, with the contexts $\Gamma_{H_i} = \Gamma_X$, R_p is the subset of $\{i_1, \ldots, i_p\}$ of superscripts of the σ operator such that $\underline{\mathtt{r}}(H_1, \ldots, H_k)$ has the right type, $R_i = \bigcup_{k=0}^{p}$ if $i_k \geq m + p - k - \Sigma_{l=k+1}^{p} j_l > i_{k+1}$ then $\{m + p - k - \Sigma_{l=k+1}^{p} j_l\}$ else $\emptyset$, where $i_0 = \infty$ and $i_{p+1} = 0$.

SIMPL$_{\lambda s_e}$

`INPUT`: A unification problem P_q with at least one rigid-rigid equation.
`OUTPUT`: A terminal (failure or success) status or an equivalent unification problem $\overline{P_q}$ without rigid-rigid equations and containing at least one flexible-rigid equation.

Assume that **Dec-λ** is applied eagerly.
`WHILE` there exists a rigid-rigid equation in P_q `DO`:

1. Apply **Dec-App-λ** or **App-Fail**.
2. Apply **Dec-App** and, if the resulting unification problem contains a flexible-rigid equation, call it $\overline{P_q}$ and give $\overline{P_q}$ as result, else stop and report a success status.

`DONE`.

MATCH$_{\lambda s_e}$

INPUT: A unification system P_q with at least one flexible-rigid equation.
OUTPUT: A disjunction of equivalent unification systems, written $P_{q1} \vee \ldots \vee P_{qk}$.
Assume that **Dec-λ** is applied eagerly.

1. Apply **Exp-λ** and **Replace** as much as possible to the selected flexible-rigid equation and call P'_q the resulting unification system.
2. Apply **Exp-App** and **Replace** and **Normalise** to P'_q and call $P_{q1} \vee \ldots P_{qr}$ the resulting unification problem.

The Main Procedure

INPUT: A unification system P_ϵ.
OUTPUT: A success or a failure status and in the former case the solutions are the solved equations whose left-hand side corresponds the meta-variables of the initial problem. If the initial problem is non-unifiable the algorithm may not terminate.

1. If P_q contains a rigid-rigid equation then apply SIMPL$_{\lambda s_e}$ to it, else if P_q contains a non-solved flexible-rigid equation then rename it to $\overline{P_q}$ and go to the next step.
2. Apply MATCH$_{\lambda s_e}$ to $\overline{P_q}$ and let $P_{q1} \vee \ldots \vee P_{qr}$ be the resulting unification problem.
3. If the current unification problem contains a unification system not in solved form then select it and go to step 1, else stop and report a success status.

Corresponding equations

Definition

Let $X =^?_{\lambda\sigma} a$ and $X =^?_{\lambda s_e} a'$ be two flexible-rigid equations in the $\lambda\sigma$- and λs_e-calculus respectively. These equations are said to be *corresponding (or associated)* if a and a' have the same heading.

Translating λs_e-terms into $\lambda\sigma$-terms

Definition

The operator $T : \Lambda_{\lambda s_e} \to \Lambda_{\lambda\sigma}$ is defined inductively as:

1. $T(X) = X$
2. $T(\underline{n}) = \underline{1}[\uparrow^{n-1}]$
3. $T(a\ b) = T(a)\ T(b)$
4. $T(\lambda.a) = \lambda.T(a)$
5. $T(a\sigma^i b) = T(a)[\underline{1}.\underline{2}.\cdots.\underline{i-1}.T(b)[\uparrow^{i-1}].\uparrow^{i-1}]$, where $i \geq 1$.
6. $T(\varphi^i_k(a)) = T(a)[\underline{1}.\underline{2}.\cdots.\underline{k}.\uparrow^{k+i-1}]$, where $k \geq 0$ and $i \geq 1$.

If $r = 0$ in the list $\underline{1}.\cdots.\underline{r}$, then it is to be interpreted as the empty list. In addition, $\uparrow^0 = id$.

Preservation of types by T

Theorem

Let Γ be a context, A a type and a a term in the language of the λs_e-calculus such that $\Gamma \vdash a : A$. Then $\Gamma \vdash T(a) : A$.

λs_e-skeleton

Definition (**Ayala & Kamareddine 2001**)

Let t be a λs_e-normal term whose root operator is either σ or φ and let X be its leftmost innermost meta-variable. Denote by $\psi_{i_k}^{j_k}$ the k-th operator following the sequence of operators σ and φ, considering only left arguments of the σ operators, in the innermost outermost ordering. Additionally, if $\psi_{i_k}^{j_k}$ corresponds to an operator φ then j_k and i_k denote its superscripts and subscripts, respectively, and if $\psi_{i_k}^{j_k}$ corresponds to an operator σ then $j_k = 0$ and i_k denote its superscript. Let a_k denote the corresponding right argument of the k-th operator if $\psi_{i_k}^{j_k} = \sigma^{i_k}$ and the empty argument if $\psi_{i_k}^{j_k} = \varphi_{i_k}^{j_k}$. The *skeleton* of t, written as $sk(t)$, is $\psi_{i_p}^{j_p} \ldots \psi_{i_1}^{j_1}(X, a_1, \ldots, a_p)$.

Correspondence from MATCH$_{\lambda s_e}$ to MATCH$_{\lambda\sigma}$

Theorem

Let $\psi^{j_p}_{i_p} \ldots \psi^{j_1}_{i_1}(X, a_1, \ldots, a_p) =^?_{\lambda s_e} (\underline{\mathtt{m}}\ b_1 \ldots b_q)$ *be a flexible-rigid equation in the* λs_e*-calculus, where* X *has atomic type. Then, for each equation generated by the rule* **Exp-App**$_{\lambda s_e}$ *there exists a corresponding equation in the* $\lambda\sigma$*-calculus generated by the rule* **Exp-App**$_{\lambda\sigma}$ *for the equation*

$$T(\psi^{j_p}_{i_p} \ldots \psi^{j_1}_{i_1}(X, a_1, \ldots, a_p)) =^?_{\lambda\sigma} T(\underline{\mathtt{m}}\ b_1 \ldots b_q)$$

Example

Consider the unification problem
$\varphi^4_1(((\varphi^3_7 X)\sigma^5 a)\sigma^3 b) =^?_{\lambda s_e} (\underline{6}\ b_1 \ldots b_q)$
The **Exp-App**$_{\lambda s_e}$ rule generates the equation $X =^?_{\lambda s_e} (\underline{4}\ H_1 \ldots H_q)$.
The $\lambda\sigma$-normal form of $T(\varphi^4_1(((\varphi^3_7 X)\sigma^5 a)\sigma^3 b))$ is computed by:
$T(\varphi^4_1(((\varphi^3_7 X)\sigma^5 a)\sigma^3 b)) =$
$T(((\varphi^3_7 X)\sigma^5 a)\sigma^3 b)[\underline{1}.\uparrow^4] =$
$T((\varphi^3_7 X)\sigma^5 a)[\underline{1}.\underline{2}.T(b)[\uparrow^2].\uparrow^2][\underline{1}.\uparrow^4] \rightarrow^*_\sigma$
$T((\varphi^3_7 X)\sigma^5 a)[\underline{1}.\underline{5}.T(b)[\uparrow^5].\uparrow^5] =$
$T(\varphi^3_7 X)[\underline{1}.\underline{2}.\underline{3}.\underline{4}.T(a)[\uparrow^4].\uparrow^4])[\underline{1}.\underline{5}.T(b)[\uparrow^5].\uparrow^5] \rightarrow^*_\sigma$
$T(\varphi^3_7 X)[\underline{1}.\underline{5}.T(b)[\uparrow^5].\underline{6}.T(a)[\uparrow^6].\uparrow^6] =$
$X[\underline{1}.\underline{2}.\underline{3}.\underline{4}.\underline{5}.\underline{6}.\underline{7}.\uparrow^9][\underline{1}.\underline{5}.T(b)[\uparrow^5].\underline{6}.T(a)[\uparrow^6].\uparrow^6] \rightarrow^*_\sigma$
$X[\underline{1}.\underline{5}.T(b)[\uparrow^5].\underline{6}.T(a)[\uparrow^6].\underline{7}.\underline{8}.\uparrow^{10}]$

Example

The rule **Exp-App**$_{\lambda\sigma}$ generates the corresponding equation $X =^?_{\lambda\sigma} \underline{4}(Y_1 \ldots Y_q)$ which corresponds to the selection of the de Bruijn index $\underline{6}$ inside the explicit substitution $[\underline{1}.\underline{5}.T(b)[\uparrow^5].\underline{6}.T(a)[\uparrow^6].\underline{7}.\underline{8}.\uparrow^{10}]$.

Translating $\lambda\sigma$-terms into λs_e-terms

Definition

The operator $L : \Lambda_{\lambda\sigma-\mathtt{terms}} \rightarrow \Lambda_{\lambda s_e}$ is defined inductively as:
$L(X) = X$
$L(\underline{1}[\uparrow^{m-1}]) = \underline{\mathtt{m}}$, where $m \in \mathbb{N}$
$L(a\ b) = L(a)\ L(b)$
$L(\lambda.a) = \lambda.L(a)$
$L(a[a_1.a_2.\cdots.a_p.\uparrow^n]) =$
$\sigma^1 \ldots \sigma^{p-1}\sigma^p\varphi_p^{n+1}(L(a), L(a_p), L(a_{p-1}), \ldots, L(a_2), L(a_1))$, where $a_1.a_2.\cdots.a_p.\uparrow^n$ is a substitution in $\lambda\sigma$-normal form, and $n, p \geq 0$.

Preservation of types by L

Theorem

Let Γ be a context, A a type and a a term in the language of the $\lambda\sigma$-calculus such that $\Gamma \vdash a : A$. Then $\Gamma \vdash L(a) : A$.

Correspondence from MATCH$_{\lambda\sigma}$ to MATCH$_{\lambda s_e}$

Theorem

Let $X[a_1.\cdots.a_p.\uparrow^n] =^?_{\lambda\sigma} (\underline{\mathrm{m}}\ b_1 \ldots b_q)$ *be a flexible-rigid equation in the* $\lambda\sigma$*-calculus, where* X *has atomic type and* $a_1.\cdots.a_p.\uparrow^n$ *is a* $\lambda\sigma$*-normal substitution. Then, for each equation generated by the rule* **Exp-App**$_{\lambda\sigma}$ *there exists a corresponding equation in the* λs_e*-calculus generated by the rule* **Exp-App**$_{\lambda s_e}$ *for the equation*

$$L(X[a_1.\cdots.a_p.\uparrow^n]) =^?_{\lambda s_e} (\underline{\mathrm{m}}\ L(b_1)\ldots L(b_q))$$

Correspondence between MATCH$_{\lambda s_e}$ and MATCH$_{\lambda\sigma}$

Theorem

Let P be a unification problem in the simply typed λ-calculus, and P_ξ its precooking translation to the ξ-calculus of explicit substitutions, where $\xi \in \{\lambda\sigma, \lambda s_e\}$. Then $P_{\lambda\sigma}$ is unifiable if and only if $P_{\lambda s_e}$ is unifiable. Moreover, whenever unifiers exist, they are associated.

$\lambda\sigma$ and λs_e correspondence

Corollary

Let $P_{\lambda\sigma}$ be a unification problem in the $\lambda\sigma$-calculus of explicit substitutions, and $L(P_{\lambda\sigma})$ its translation to the λs_e-calculus of explicit substitutions. Then $P_{\lambda\sigma}$ is unifiable if and only if $L(P_{\lambda\sigma})$ is unifiable. Moreover, whenever unifiers exists, they are associated.

Conclusion

- In this work we compared the $\lambda\sigma$- and the λs_e-styles of unification.
- To do so, we presented the *unification tree* notation which allows a clear presentation of the Huet's algorithm in de Bruijn notation.
- This notation was applied to unification problems in de Bruijn notation, but it can be applied to λ-terms with names with minor modifications.

Conclusion

- We compared the classical method of Huet for HOU and the one of Dowek, Hardin and Kirchner for the $\lambda\sigma$-calculus.
- We described the counterpart of the procedures SIMPL and MATCH, called SIMPL$_{\lambda\sigma}$ and MATCH$_{\lambda\sigma}$.
- We concluded that there exists a correspondence between the substitutions generated by Huet's algorithm and the graftings generated by the $\lambda\sigma$-HOU algorithm for unification problems which are in the image of the precooking translation.
- This comparison was extended to the λs_e-HOU algorithm.

Conclusion

- We concluded that the $\lambda\sigma$- and the λs_e-HOU algorithms generate associated graftings.