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RICCI FLOW AND THE GEOMETRIZATION OF 3-MANIFOLDS

G. Besson *

Dedicated to Manfredo do Carmo with admiration

1 Introduction

This text is a short account of a joint work with M. Boileau, L. Bessières, S. Maillot and J. Porti on the geometrization of certain 3-dimensional manifolds (see [5]). It relies heavily on R. Hamilton and G. Perelman's works on the Ricci flow which we shall briefly describe at the beginning. Extended notes on these have been published by H.-D. Cao and X.-P. Zhu ([9]), B. Kleiner and J. Lott ([23]) and J. Morgan and G. Tian ([25]). For the completion of the geometrization conjecture, at the moment, at least four schemes have been developped. The key part is the understanding of the non hyperbolic pieces that appear during the long term evolution of the flow. In [30], G. Perelman suggested an approach which relies on a work by T. Shioya and T. Yamaguchi [34] using a refined study of Alexandrov spaces and in particular a stability theorem due to G. Perelman himself and written by V. Kapovitch in [22]. Recently (september 2008) J. Morgan and G. Tian posted on the web (see [26]) their approach which uses much less of the structure of Alexandrov spaces. In june 2008, B. Kleiner and J. Lott announced that they had completed their proof. Finally, the scheme which is presented below is written in detail in [5], and is a trade-off between geometry and topology. The idea is to use the geometrization of Haken manifolds proved by W. Thurston and to reduce the other cases to that one using the outcome of the Ricci flow technique.

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For further informations the reader may also look at the following survey papers [2, 6, 31] and the forthcoming monography [4].

2 H. Poincaré and W. Thurston: the topology and geometry of 3-manifolds

Let M^3 be a closed, connected, orientable 3-dimensional manifold. The famous Poincaré conjecture tells us how to distinguish the sphere among the other 3-dimensional manifolds only from its topology.

Conjecture 2.1 (Poincaré [32], 1904)**.** *If M^3 is simply connected then M is homeomorphic (diffeo) to the 3-sphere S^3.*

This conjecture was published in an issue of the Rendiconti del Circolo Matematica di Palermo ([32]). Let us recall that in dimension 3 the homeomorphism classes and the diffeomorphism classes are the same. While trying to solve this conjecture, prominent mathematicians have introduced quite a few important notions and proved fundamental results in 3-dimensional topology. However, for a long time it was not clear whether it was true or not and several attempts to disprove it were made. The reader may want to read [3] where a more precise historical note is given (this text is in french). In the 70's W. Thurston generalized the Poincaré conjecture allowing the geometric construction of all 3-dimensional manifolds.

Conjecture 2.2 (geometrization conjecture, Thurston [36], 1982)**.** *M^3 can be cut open into geometric pieces.*

The precise meaning of this statement can be checked in [6]. *Grosso modo* it means that M can be cut open along a finite family of incompressible tori so that each piece left carries one of the eight geometries in dimension 3 (see [33]). Among these geometries are the three constant curvature ones: spherical, flat and hyperbolic and five others such as the one given by the Heisenberg group and the Sol group (check the details in [33]). Thurston included the differential geometry in the heart of the study of the 3-dimensional manifolds. Along come into play the analysis; at the beginning of the 80's R. Hamilton launched a new program in order to prove the geometrization conjecture. His approach started a systematic study of the so-called Ricci flow. It is a differential equation on the space of Riemannian metrics whose ω- limit points, if any, ought to be Einstein manifolds i.e., in dimension three, constant curvature metrics. The Ricci flow should homogenize the metric. However, it is not always possible, since not every 3-dimensional manifold carries a metric of constant curvature; then blow-up phenomenon for the (scalar) curvature may occur. Recently, G. Perelman (see [28, 30, 29]) described quantities (one of which is called entropy) which are monotonic along this flow. This allowed him to give a precise geometric description of the singularities that may occur and to perform a surgery in order to remove them. This idea of surgery was introduced by Hamilton in [18]. It was with a *tour de force* that G. Perelman put an end to Hamilton's program.

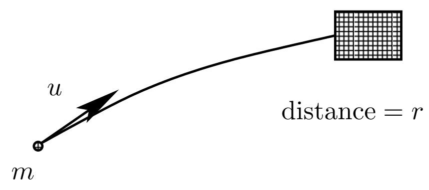
It is worth noticing that the idea of Ricci flow was present in the physics literature, see for example [13].

3 Differential Geometry

Among the various notions of curvature of a Riemannian manifold, the Ricci curvature is the most interesting one, since it is a symmetric bilinear form on each tangent space and hence of the same nature than the metric itself. As a consequence, the question of whether the Ricci curvature determines the metric is in an "optimal" form. An intuitive understanding of this curvature is not completely clear at the moment and the definition of a rough version of it, for pretty general metric spaces, may provide insights (see for example [27]). A precise definition can be found in [10] or [14]. Let $u \in T_m(M)$ be a unit vector and (u, e_2, e_3) an orthonormal basis of $T_m(M)$. Then,

$$\mathrm{Ricci}(u,u) = \sigma(u,e_2) + \sigma(u,e_3)\,,$$

where σ denotes the sectional curvature. This definition is not enlightening we may look at the picture below,



the volume element at distance r from m, in the direction of a unit vector u, has an asymptotic expansion given by,

$$d\,\mathrm{vol} = (1 - \frac{r^2}{6}\mathrm{Ricci}_m(u,u) + o(r^2))d\,\mathrm{vol}_{\mathrm{eucl}}$$

where $d\,\mathrm{vol}_{\mathrm{eucl}}$ is the Euclidean volume element written in normal coordinates (see [14]). From this, it is clear that Ricci is a bilinear form on $T_m(M)$ that is an object of the same nature than the Riemannian metric. Let us recall that an Einstein manifold is a Riemannian manifold whose Ricci curvature is constant, that is a constant multiple of the Riemannian metric, Ricci $= \lambda g$. For more information about such manifolds see [1].

3.1 The scalar curvature

The scalar curvature is the simplest and the weakest, since it is a smooth function. More precisely, at each point, it is the trace of the bilinear form Ricci with respect to the Euclidean structure g. It turns out that in dimension 3 and along the Ricci flow, that we shall define below, it controls the sectional curvature. This is a key fact which is a version of the maximum principle for parabolic equations.

4 The Ricci flow by R. Hamilton

This is an evolution equation on the Riemannian metric g whose expected effect is to make the Riemannian manifold become Einstein, that is with constant sectional curvature in dimension 3. This expectation is however far too optimistic. Nevertheless, this ”flow” turns out to be sufficiently efficient to prove both Poincaré and Thurston’s conjectures. The inspiration for this beautiful idea is explained in [**?**] and [8]. Let (M, g_0) be a Riemannian manifold. We are looking for a family of Riemannian metrics depending on a parameter $t \in \mathbf{R}$, such that $g(0) = g_0$ and,

$$\frac{dg}{dt} = -2\mathrm{Ricci}_{g(t)} \, .$$

The coefficient 2 is completely irrelevant whereas the minus sign is crucial. It is possible to consider this as a differential equation on the space of Riemannian metrics (see [8]), it is however difficult to use this point of view for practical purposes. A more efficient approach is to look at it in local coordinates in order to understand the structure of this equation ([**?**]). It turns out to be a non-linear heat equation, which is schematically like

$$\frac{\partial}{\partial t} = \Delta_{g(t)} + Q \, .$$

Here Δ is the Laplacian associated to the evolving Riemannian metric $g(t)$. The minus sign in the definition of the Ricci flow ensures that this heat equation is not backward and thus it has solutions for any initial data, at least for small time. The expression encoded in Q is quadratic in the curvatures. Such equations are called **reaction-diffusion** equations. The diffusion term is Δ; indeed if Q is equal to zero then it is an honest (time dependent) heat equation whose effect is to spread the initial temperature density. The reaction term is Q; if Δ was not in this equation then the prototype would be the ordinary differential equation,

$$f' = f^2 \, ,$$

for a real valued function f. It is well-known that it blows up for a finite value of t. These two effects are opposite and the main question is to decide which one will be the winner. This turns out to depend on the dimension.

4.1 Dimension 2

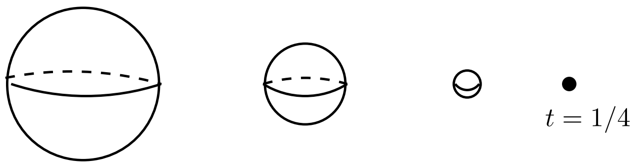
It is shown in [19] (see also[11]) that in 2 dimensions the diffusion wins, that is, starting from any (smooth) Riemannian metric on a compact and orientable surface, the flow converges, after rescaling, towards a constant curvature metric (even in the same conformal class). Full details are given in [11].

4.2 Examples in dimension 3

The following examples can be easily computed.

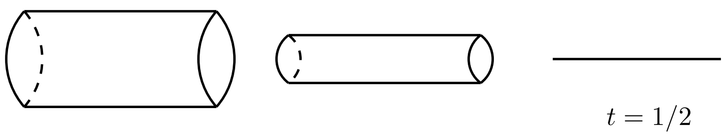
1. Flat tori, $g(t) \equiv g_0$; (it is said to be an eternal solution).

2. Round sphere $g(t) = (1 - 4t)g_0$; (ancient solution).



3. Hyperbolic space $g(t) = (1 + 4t)g_0$; (immortal solution).

4. Cylinder $g(t) = (1 - 2t)g_{S^2} \oplus g_{\mathbf{R}}$.



Two features deserve to be emphasised. For the round sphere the flow stops in finite positive time but has an infinite past. For the hyperbolic manifolds, on the contrary, the flow has a finite past but an infinite future. We find these aspects in the core of the proofs of the two conjectures. Indeed, for the

Poincaré conjecture one can show that starting from any Riemannian metric on a simply-connected 3-manifold the flow stops in finite time whereas for Thurston's conjecture one ought to study the long term behaviour of the evolution.

5 Evolution of curvatures

From the above equation we derive the evolution of the various Riemannian quantities. The most important ones are the curvatures.

5.1 The curvature operator

Let us recall that the curvature operator is a symmetric endomorphism of $\Lambda^2(M)$. If we denote it by Rm, it satisfies the following evolution equation

$$\frac{\partial \mathrm{Rm}}{\partial t} = \Delta \mathrm{Rm} + \mathrm{Rm}^2 + \mathrm{Rm}^\sharp \,, \tag{1}$$

where Rm^2 denotes the square of the endomorphism, $\mathrm{Rm} \circ \mathrm{Rm}$, and $\mathrm{Rm}^\sharp$ a quadratic expression. We shall see later that this can be expressed easily in terms of the eigenvalues of Rm.

5.2 The scalar curvature

Taking traces leads to the evolution equation for the scalar curvature R,

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ricci}|^2 \,. \tag{2}$$

It is a scalar equation of reaction-diffusion type. If we use the fact that the scalar curvature is a trace of the Ricci curvature we get, in dimension 3, the following inequality.

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{3} R^2 \,,$$

and we shall apply the maximum principle which is a comparison between the P.D.E. and the O.D.E obtained by forgetting the Laplacian term.

5.3 Curvature and blow-up

The main feature of these equations is summarised in the following result which describes the behaviour of the Riemannian metric when the flow approaches the end of the maximal interval of definition (when finite).

Theorem 5.1. *Unique solution exists on a maximal interval* $[0, T)$. *Moreover,*

$$T < +\infty \Longrightarrow \lim_{t \to T} (\sup_{x \in M} |\mathrm{Rm}(x,t)|) = +\infty \,.$$

These basic facts about the Ricci flow, most of which are valid in any dimension, are very nicely described in [11].

6 The maximum principles

This is a very powerful analytic technique for parabolic equations. However we are working in a geometric context and most of the time with systems of equations rather than scalar ones. Hence if we apply blindly maximum principles we loose the geometry. R. Hamilton settled in [17] a very neat control of the curvature tensor by the scalar curvature. I mentioned above, the maximum principles amount to comparing the solutions of the P.D.E to the solutions of the O.D.E. obtained by forgetting the Laplacian term (see [11]).

6.1 Scalar maximum principle

Equation (2) is scalar and the 0-th order term is non negative. We may then use a maximum principle for the inequality $\frac{\partial R}{\partial t} \geq \Delta R$ to get the following straightforward theorem.

Theorem 6.1. $R_{\min}(t) = \min\{R(x,t);\, x \in M\}$ *is non decreasing.*

This shows that the Ricci flow favours positive curvature. it pushes the scalar curvature, more precisely its minimum, towards non negative values.

6.2 Maximum principles for systems

There is a fairly general situation where one considers a bundle $\mathcal{E} \longrightarrow M$ and a section s of it and an equation of type

$$\frac{\partial s}{\partial t} = \Delta s + F(s)\,,$$

where F is a bundle-valued function satisfying some properties (see [11] for detailed examples). The key step consists in studying the system of O.D.E.

$$\frac{ds}{dt} = F(s)\,. \tag{3}$$

For the case of the Riemann tensor, in dimension 3, let $\lambda(x,t) \geq \mu(x,t) \geq \nu(x,t)$ be its eigenvalues, depending on $x \in M$ and the time $t \in \mathbf{R}$. Depending on the normalization of the curvature tensor these numbers are sectional curvatures of 2-planes up to a multiplicative constant. We recall that if $\dim M = 3$ then $\dim \Lambda(M) = 3$. Equation (3) now reads

$$\begin{cases} \frac{d\lambda}{dt} & = \lambda^2 + \mu\nu, \\ \frac{d\mu}{dt} & = \mu^2 + \lambda\nu, \\ \frac{d\nu}{dt} & = \nu^2 + \lambda\mu. \end{cases} \tag{4}$$

The squares come from the square of the curvature operator in (1) and the other terms from what we called $\mathrm{Rm}^\sharp$ which we could take here as a definition of this tensor. Now let φ be the reciprocal function to $x \longrightarrow x\log x - x$. Let us assume that the original metric $g(0)$ is normalized so that $\forall x \in M$, $R(x,0) \geq -1$ and $\nu(x,0) \geq -\varphi(R(x,0))$, which is always possible to achieve by an homothety, then

Theorem 6.2 (R. Hamilton, [17]). *If $(M, g(0))$ is normalised as above, then, for all t,* $\nu(x,t) \geq -\varphi(R(x,t))$.

This shows that in dimension three there is no much room so that the scalar curvature completely controls the full Riemann tensor. Indeed, since R is (up to a constant) the sum of the eigenvalues of the curvature tensor, one has the following corollary,

Corollary 6.3. *For all $x \in M$ and t, $R(x,t)+2\varphi(R(x,t)) \geq \lambda(x,t) \geq \nu(x,t) \geq -\varphi(R(x,t))$.*

These are the basic analytic tools that are used in the study of the flow.

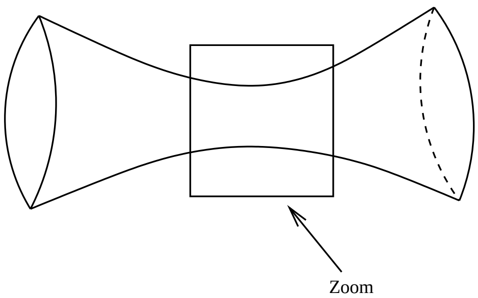
7 Some geometric tools

When the flow is only defined on a finite interval, we need to understand what happens when the metric approaches the maximal time. We know that the curvature blows up but we need to understand more about the geometry near these singular situations. The idea is now to rescale the metric around points of large curvature in order to zoom in the manifold while slowing down the time evolution. More precisely let us define the parabolic rescaling of an evolving metric. We suppose that the flow is defined for $t \in [0, T[$, with $T < +\infty$. Let us assume that (x_k, t_k) satisfies $t_k \to T$, and that

$$\sup\{R(x,t);\, x \in M,\, t \in [0, t_k]\} = R(x_k, t_k) = Q_k \longrightarrow +\infty .$$

We define the *parabolic rescaling* of $g(t)$ at (x_k, t_k) to be

$$\bar{g}_k(t) = Q_k g(t_k + \frac{t}{Q_k}) .$$



We then get a family of metrics $\bar{g}_k(t)$ satisfying, for each k, the Ricci flow equation. They furthermore have the following properties.

- The flow $\bar{g}_k(.)$ is defined for $t \in [-Q_k t_k, 0] \subset [-Q_k t_k, (T - t_k)Q_k)$,

- with obvious notations,
$$\frac{R_{\min}(t_k + t/Q_k)}{Q_k} \leq \bar{R}_k(x,t) \leq \frac{R(x_k,t_k)}{Q_k} = 1\,;$$
hence all sectional curvatures of $\bar{g}_k(t)$ are bounded,
- $\bar{R}_k(x_k, 0) = 1\,.$

If we assume that there is a limit flow, say $\bar{g}_\infty$, then

- it is defined on $(-\infty, 0]$, that is, it is a so-called ancient solution,
- $\bar{R}_\infty \geq 0$ hence by the maximum principle we have $0 \leq \mathrm{Rm} \leq C$,
- it is non flat, since the limit of the x_k's is a point with scalar curvature 1.

We thus need to prove that this sequence of flow converges and this is the purpose of the following theorem proved by R. Hamilton.

Theorem 7.1 (R. Hamilton, [20])**.** *Let $(M_k, \bar{g}_k(t), x_k)$ be a sequence of pointed Ricci flow defined on (A, Ω), with $A < 0 < \Omega$. If*

i) $\forall r > 0, \quad |\mathrm{Rm}_{\bar{g}_k(t)}| \leq C(r,t)$ *on* $B_{\bar{g}_k(t)}(x_k, r)\,,$

ii) $\mathrm{inj}(x_k, \bar{g}_k(0)) \geq C > 0,$

then the sequence subconverges to a flow $(M_\infty, \bar{g}_\infty(t), x_\infty)$.

This convergence is in a very strong sense. Indeed, the flow behaves like a heat equation and hence for a positive time we can get control on the curvature and its derivatives of an arbitrary order. Consequently the above convergence is in the C^k-topology for k large. We see that the sequence subconverges, if we can get a uniform lower bound on the injectivity radius at the points x_k. This is exactly where G. Perelman's works come into the picture. A first breakthrough, proved in the text [28], is this estimate. More precisely for $\kappa > 0$, let us define, for an n-dimensional Riemannian manifold

Definition 7.2. *(M^n, g) is said to be κ-noncollapsed at scale ρ if, $\forall r < \rho$, $\forall x \in M$*

$$|\mathrm{Rm}| \leq r^{-2} \text{ on } B(x,r) \Rightarrow \mathrm{vol}(B(x,r)) \geq \kappa r^n \,.$$

This is a scale invariant property. After rescaling so that one has $r = 1$, this non-collapsing property reads

$$|\mathrm{Rm}| \leq 1 \text{ on } B(x,1) \Rightarrow \mathrm{vol}(B) \geq \kappa \,.$$

We then get the key result

Theorem 7.3 (G. Perelman, [28])**.** *Let $(M^n, g(t))$ be a Ricci flow defined on $[0,T)$ for $T < +\infty$, there exists $\kappa := \kappa(g(0),T) > 0$, such that $g(t)$ is κ-noncollapsed at scale $T^{1/2}$.*

This amazing result is true in any dimension. There are several proofs each using quantities which are monotonic along the Ricci flow, for example a logarithmic Sobolev constant. Now in our situation, in dimension 3, this non-collapsing applied to the family $\bar{g}_k$ which has bounded sectional curvature amounts to a uniform lower bound on the injectivity radius at x_k. This is exactly what is needed in order to apply Hamilton's compactness theorem and get a limiting Ricci flow.

8 Models for singularities

If the family $\bar{g}_k$ subconverges towards a flow $(M_\infty, \bar{g}_\infty(t), x_\infty)$, then it satisfies the following properties:

- It is an ancient solution that is defined on $(-\infty, 0]$,
- it is complete and has non negative curvature operator and bounded sectional curvature; furthermore it is non flat,
- it is κ-non collapsed at any scale.

These are infinitesimal models for the points where the curvature is high, in fact maximal, if we follow our construction of $\bar{g}_k$. Now, a classification of those will give a precise description of a neighbourhood of these points. This classification was initiated by R. Hamilton without the non collapsing property; with it one can rule out one case which is not of the type described below, this is the so-called cigar-solution (see [21]). Now, there may exist points whose scalar curvature goes to infinity at a rate which is slightly lower than at the maximum and we should know more about these.

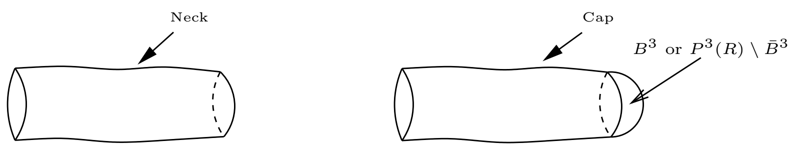
G. Perelman went much further and gave a classification of the geometry of the neighbourhoods of points with high scalar curvature without being maximal. This is the second breakthrough that we meet in this text. Let us be more precise. We refine the notion of normalized initial metric by asking $g(0)$ to satisfy: $\forall x \in M$

$$R(x,0) \geq -1,\, \nu(x,0) \geq \varphi(R(x,0)),\, \mathrm{vol}(B(x,1)) \geq \frac{1}{2}\omega_3\,.$$

Here ω_3 is the volume of the unit ball in the 3-dimensional Euclidean space. Then one finds in [28], the following canonical neighbourhood theorem

Theorem 8.1 (G. Perelman, [28])**.** *There exists $r = r(\kappa, \varphi) > 0$ such that for any normalized Ricci flow $(M, g(t))$, if $Q := R(x,t) \geq r^{-2}$ then a space-time neighbourhood of (x,t) is, after rescaling by Q, close (in smooth topology) to a neighbourhood of a point in a model solution.*

A more precise statement can be found in the original paper. This is much deeper than what was explained before on points of maximal curvature. The classification of κ-solutions is then possible and we get, for the above neighbourhood, essentially two situations.



$$\text{or } S^3/\Gamma$$

The last case, S^3/Γ, corresponds to the situation where the manifold is so small that it is contained in one single canonical neighbourhood. Besides this case the neighbourhood of high curvature points are then almost cylindrical or cylinders closed by caps which are topological 3-balls or complement of a ball in a projective space.

We may now make two simplification, first we may assume that M is irreducible, that is,

Definition 8.2. *M is irreducible if every embedded S^2 is the boundary of an embedded B^3.*

We may also assume that M does not contain any embedded $\mathbf{R}P^2$. Indeed the only closed, connected, orientable and irreducible 3-manifold which contains $\mathbf{R}P^2$ is $\mathbf{R}P^3$.

8.1 The first singular time

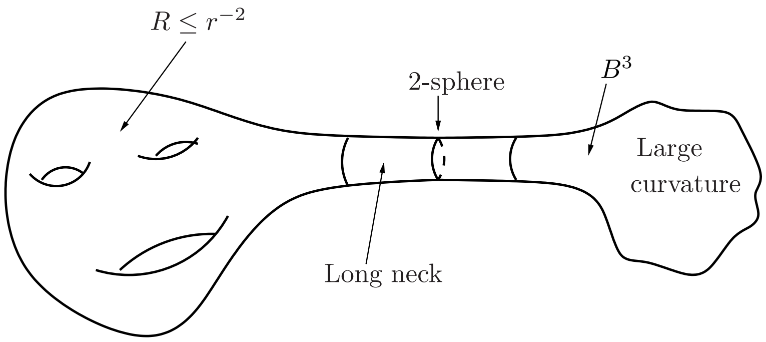
When the scalar curvature reaches the threshold given by the canonical neighbourhood of Theorem 8.1 then two possibilities occur

1st case: for some t, $R(x,t) \geq r^{-2}$ for all $x \in M$. Then M is covered by canonical neighbourhoods and we get the following classification

Proposition 8.3. *M is diffeomorphic to S^3/Γ.*

Notice that a union of cylinders and caps could give a manifold diffeomorphic to $S^2 \times S^1$ or $\mathbf{R}P^3 \sharp \mathbf{R}P^3$. However, with our hypothesis, these cases are ruled out, since the first one is not irreducible and the second contains an embedded $\mathbf{R}P^2$. The short cut for the proof of the Poincaré conjecture shows that this happens in finite time (see, [12]).

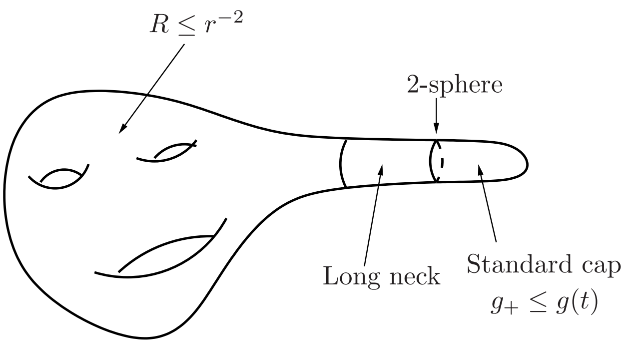
2nd case: The curvature is not big everywhere. The situation could be summarized by the following picture,



The connected components (one in the picture) of the part where the scalar curvature is large are linked to the part of the manifold where the curvature is small by (at least) a neck. The middle sphere of the neck is a boundary of a ball by the irreducibility hypothesis, hence the curvature is large in a topological ball. This is schematic and the reader is referred to [4] for the details.

8.2 The surgery

Now, the idea of surgery introduced in [18] and developed in [30] could be adapted to our situation. It amounts *grosso modo* in removing the part where the curvature is large and replacing it by a (almost) standard metric on a topological ball, much less curved, called a standard cap. The picture then becomes



In our situation, since we replace a topological ball by another there is no (topological) surgery, strictly speaking. In fact we just suitably modify the metric in the topological ball; this can even be done so that the identity map between the pre-surgery and post-surgery Riemannian manifolds decreases the metric. If t is a singular time, that is a time when we perform this "surgery", the pre-surgery metric $g(t)$ is replaced by $g_+(t)$ which differs from $g(t)$ on a topological ball and such that $g(t) \geq g_+(t)$.

This construction (or destruction) should be carefully perform and several key parameters are used in order to describe the situation. Here is the list,

- The scalar curvature of the middle sphere of the neck is a number h^{-2} for $h << r$. More precisely there is a point on this sphere which has this scalar curvature and the sphere is almost round.
- In fact, after rescaling by h^{-2}, the neck is very close to $S^2 \times] - 1/\delta, 1/\delta[$, for some number δ very small.
- The threshold, which is used to decide that the scalar curvature is large, is $Dh^{-2} >> r^{-2}$.

We can summarize a key theorem of [4] in

Theorem 8.4 ([4])**.** *For each $r > 0$ and $\delta > 0$ small enough, there exist $h(r, \delta)$ and $D(r, \delta)$ as above.*

In fact these quantities, h and D are universal, they do not depend on the particular Ricci flow as long as the initial metric is suitably normalized. What is the role of δ? In order to answer this question we should notice that the post-surgery metric should satisfy several properties:

- The maximum principle with the function φ,
- The κ-non collapsing,
- The canonical neighbourhood property with the threshold parameter r.

These are produced by the flow (after a certain time) and are satisfied by $g(t)$, however g_+ is a brand new metric which may not satisfy them. One can show that if δ is chosen small enough then they persist after the "surgery". The above scheme is a slight modification of Perelman construction (see [30]) which has the advantage of simplifying some technical aspects.

The prove of these results is a painstaking work and is done by contradiction using Hamilton's compactness theorem for flows.

We then restart the flow with the new metric and we reach a new singular time. The starting point is now not normalized any more so the canonical neighbourhood scale r may decrease as well as δ, h, κ. One possibility that we have to rule out is that the surgeries may accumulate. The keystone of Perelman's work is the

Theorem 8.5 (G. Perelman)**.** *There exist $r(t) > 0$, $\delta(t) > 0$, $\kappa(t) > 0$ such that the flow-with-surgery exists for all t.*

We adapted this result to our situation but it is essentially the proof presented in [30]. We can then summarized the situation. We construct a flow-with-surgery on a manifold M which is irreducible and does not contain any $\mathbf{R}P^2$ so that during the process there are no changes in the topology. At each surgery the metric decreases and is discontinuous. These surgeries occur at times which form a discrete subset of $\mathbf{R}$.

9 The long-term behaviour of the flow with surgery

The goal is to obtain a thick-thin decomposition of the manifold for large time. The thick part ought to become hyperbolic (after a suitable rescaling) and the difficulty, mentioned in the introduction, is to describe the thin part. Before doing this, let us define these two classical notions.

9.1 The thick-thin decomposition

For a Riemannian (M, g), let us chose $\epsilon > 0$,

Definition 9.1. *The ϵ-thin part of (M, g), denoted $\mathrm{Thin}_\epsilon(M, g)$, is the set of $x \in M$ such that $\exists \rho = \rho(x, \epsilon) \in (0, 1)$ with the following property*

$$\mathrm{Rm} \geq -\rho^{-2} \text{ on } B(x, \rho) \text{ and } \mathrm{vol}(B(x, \rho)) < \epsilon \rho^3 \,.$$

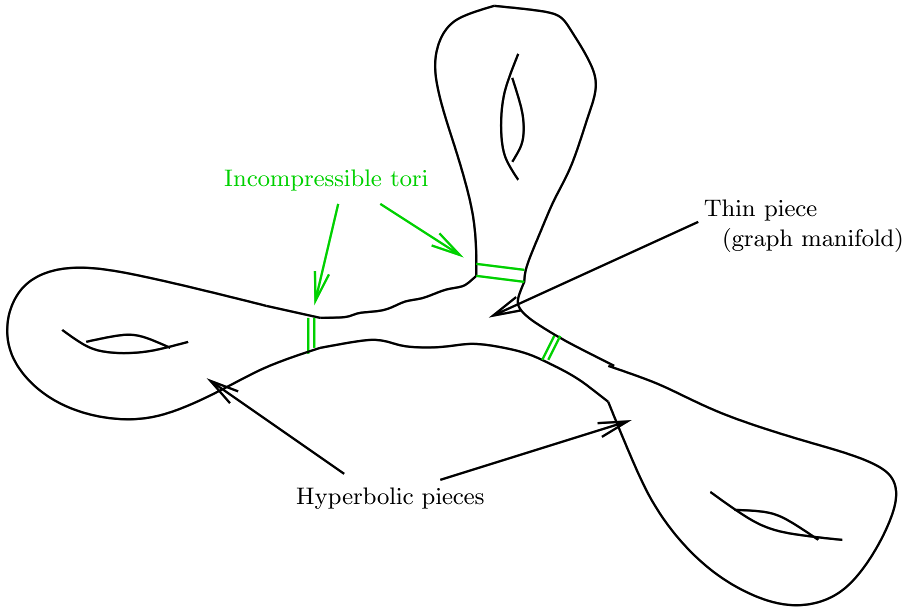
The thick part, $\mathrm{Thick}_\epsilon(M, g)$ *is* $M \setminus \mathrm{Thin}_\epsilon(M, g)$.

We shall say that M is ϵ-thin if $M = \mathrm{Thin}_\epsilon(M, g)$. An important result can be summarized as

Theorem 9.2 (G. Perelman, [30])**.** *If $x \in \mathrm{Thick}_\epsilon(M, g(t))$ for all large t then $(M, x, \frac{1}{4t} g(t))$ converges to a pointed complete hyperbolic manifold.*

Notice that the only instance in which we know the limit metric of a general Ricci flow, possibly after rescaling, is when it converges towards a constant negative curvature metric, that is the content of the above theorem. Other than that, it is only for very special initial metrics that we know the long-term limit of the flow.

We can then summarize the situation by the following picture,



We must now describe the topology of the thin part. Precisely the goal is to prove that it is a graphed manifold.

9.2 Geometric decomposition *à la* Thurston

Let us recall, in a non precise way, some basic definitions

Definition 9.3. *M^3 closed, connected and orientable,*

- *is Seifert if it fibers over some* 2*-orbifold.*
- *is a graphed manifold if it is a union of Seifert bundles glued along their boundaries.*
- *is aspherical if its universal cover is contractible, which in our case is equivalent to having an infinite fundamental group.*
- *is Haken if it is irreducible and contains an incompressible surface or has a non empty boundary.*

Thurston's conjecture states that every irreducible 3-manifold admits a (geometric) decomposition as shown on the picture, that is, can be cut open along incompressible tori so that the pieces left are either hyperbolic or graphed. More precise statements can be found in the literature and for example in [7]. We then have the important result

Theorem 9.4 (W. Thurston, 1980). *Haken manifolds have a geometric decomposition.*

We shall use this theorem for irreducible 3-manifolds with boundary, which are Haken.

9.3 The thin part is graphed

In [5] we describe the structure of the thin part relying on Thurston's theorem. A simplified statement is the following. Let M^3 be closed, connected, orientable and aspherical.

Theorem 9.5 ([5]). *Let g_n be a sequence of metrics on M such that $M_n = (M, g_n)$ is ϵ_n-thin for ϵ_n going to 0 as n goes to infinity, then M is a graphed manifold.*

This sequence of metrics could be produced by the Ricci flow with surgery. Indeed, if we suppose that we have a Ricci flow with surgery $(M, g(t))$ defined on $[0, +\infty)$, we may consider $g_n = \frac{1}{4n} g(n)$. Note that in the proof of the geometrization conjecture one has to deal with simultaneous presence of thick and thin parts. The necessary modifications to the proof that will be sketched below can be read in [5].

Sketch of proof. The idea starts in the same way as the other approaches, that is describing local geometric models for such a manifold. Then one has to glue them together in order to get a global picture or alternatively have global arguments; this second point of view is the one used below.

The following lemma gives a local picture, that is a description of the geometric structure of the metric g_n for large n.

Lemma 9.6. *$\forall D > 0$, $\exists n_0 = n_0(D)$ such that for $n \geq n_0$:*

- *either M_n is $\frac{1}{D}$-close to a compact manifold X_n with $\mathrm{Rm} \geq 0$, hence the differentiable manifold M has a Euclidean structure.*

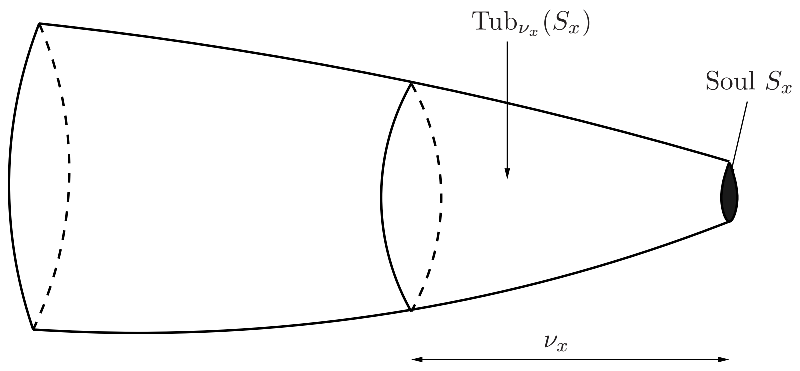
- *or $\forall x \in M_n$, $\exists \nu_{x,n} \in (0, \rho_n(x))$, $X_{x,n}$ non compact with $\mathrm{Rm} \geq 0$ such that:*

  i) $B(x, \nu_{x,n})$ *is* $\frac{1}{D}$*-close to* $\mathrm{Tub}_{\nu_{x,n}}(S_{x,n}) \subset X_{x,n}$*, where* $S_{x,n}$ *is the soul of* $X_{x,n}$*,*

  ii) $\mathrm{diam}(S_{x,n}) \geq \nu_{x,n}/D$*,*

  iii) $\mathrm{vol}(B(x, \nu_{x,n})) < \frac{1}{D}\nu_x^3$ *and on* $B(x, \nu_x)$, $\mathrm{Rm} \geq -\frac{1}{\nu_{x,n}^2}$.

The first part of the theorem tells us that the manifold is close in a smooth topology to a compact manifold of non negative curvature operator which, by Hamilton's result (see [18]), is covered by S^3, $S^1 \times S^2$ or a torus. The asphericity hypothesis leaves us with the last possibility and in that case M is obviously a graph manifold. The second part says that if we are not in the previous situation, then around any point of M_n there is a ball which looks like the tubular neighbourhood of the soul of a non negatively curved non compact manifold. Furthermore the soul is small and we have control on the volume of the ball and the sectional curvature therein. For simplicity we shall drop the subscript n but we have to keep in mind that the metric varies. The situation in the second case is better summarized by the following picture.



Since the manifold X_x is non compact and the soul is, the only possibilities are $S_x = \star, S^1, S^2, T^2, K^2$, where $\star$ denotes a point, T^2 the 2-torus and K^2 the Klein bottle. Notice that $\mathbf{R}P^2$ is excluded by our assumptions. Then the ball around x has to be

$$B(x, \nu_x) = B^3, S^1 \times D^2, S^2 \times I, T^2 \times I, K^2 \tilde{\times} I\,,$$

where I is an interval and D^2 is a 2-disc. Here $K^2 \tilde{\times} I$ denotes a twisted bundle over the klein bottle (in order to have an oriented manifold). We see the local fibred structure. The question is how these local foliations match. This is addressed in the reference [26] and in the forthcoming article by B. Kleiner and J. Lott. The proof of the above lemma is done by contradiction which is a classical approach for this kind of results.

Now, we wish to find one of these local models, say $\mathcal{V}$ so that $M \setminus \mathcal{V}$ is Haken and hence has a geometric decomposition. Then we just have to show that in its geometric decomposition there are no hyperbolic pieces i.e. only graphed parts are left. A manifold with non empty boundary is Haken if it is irreducible. We then have to find a local model $\mathcal{V}$, one of the balls $B(x, \nu_x)$, so that $M \setminus \mathcal{V}$ is irreducible. This assumption on M, which is minor and is natural when one considers the Kneser-Milnor decomposition, becomes crucial now.

Let us take a local model $\mathcal{V}$ and consider a 2-sphere embedded in $M \setminus \mathcal{V}$.

This sphere is the boundary of a 3-(topological) ball in M. The irreducibility assumption shows, by an easy argument, that the 2-sphere separates, hence the 3-ball could be on one side or the other, up to isotopy. If it is in $M \setminus \mathcal{V}$ then the sphere is a boundary in this manifold. If it is on the other side then it has to contain $\mathcal{V}$. This shows that if every local model is contained in a topological ball then we cannot pursue the proof. We thus have to find a local model $\mathcal{V}$ not included in a 3-(topological) ball. In fact we can prove the following proposition.

Proposition 9.7. *There exists a local model $\mathcal{V} = B(x, \nu_x)$ such that the subgroup $\mathrm{Im}(\pi_1(\mathcal{V}) \longrightarrow \pi_1(M))$ is not trivial.*

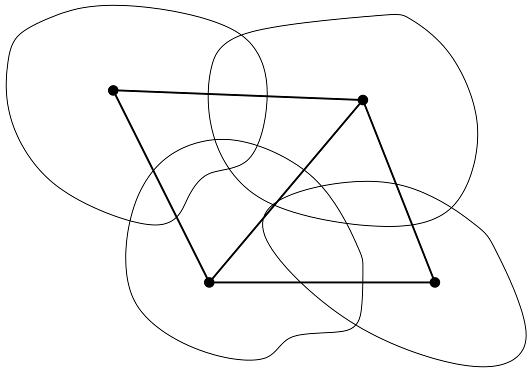
This is sufficient to ensure that $\mathcal{V}$ is not included in a contractible set. We again proceed by contradiction and assume that for all $x \in M$, $B(x, \nu_x)$ is homotopically trivial, that is $\mathrm{Im}(\pi_1(B(x, \nu_x)) \longrightarrow \pi_1(M))$ is trivial. We shall extract from the covering by these balls a finite sub-covering and show that the sets can be shrunk so as to form a covering of dimension at most 2. The contradiction then comes from the following result due to C. MacMullen (see [24]). This approach borrows ideas from M. Gromov (see [16])

Theorem 9.8 (C. McMullen)**.** *Let N^d be a closed, orientable and aspherical d-manifold with $d \geq 1$. Every locally finite covering of N^d by homotopically trivial open sets has dimension at least d.*

Let us recall the definition of the dimension of a covering.

Definition 9.9. *The dimension of a covering $\{U_i\}_{i \in I}$ is the smallest integer q such that for all $x \in M$, x belongs to at most $q + 1$ sets U_i.*

Let us construct the sub-covering and explain how we can reduce the dimension. We choose a finite collection of the local models $B(x_1, \frac{1}{4}r_1)$, $B(x_2, \frac{1}{4}r_2), \ldots$ which is a maximal and pairwise disjoint family, where for the sake of simplicity we have set $r_i = \nu_{x_i}$. By maximality the family $\{B(x_1, r_1), \ldots, B(x_q, r_q)\}$ is a covering of M_n whose nerve is denoted K (we recall that the nerve is a subset of the $q-1$-dimensional standard simplex). The dimension of K, that is the maximal dimension of a simplex in K, is the dimension of the covering.



By an easy argument relying on Bishop-Gromov's comparison theorem we can show that there exists an *a priori* uniform bound on the number of overlaps at each point. We remember that we are working with a family of metrics g_n and by uniform we mean that the bound does not depend on n. We can now use the characteristic map from M_n to K defined by

$$f = \frac{1}{\sum \phi_i}(\phi_1, \ldots, \phi_q) : M_n \longrightarrow K\,,$$

where $\phi_i : B(x_i, r_i) \longrightarrow [0, 1]$ is a cut-off function. We choose the ϕ_i's satisfying the following properties.

- $\phi_i|_{B(x_i, \frac{2}{3}r_i)} \simeq 1$, $\phi_i|_{\partial B(x_i, r_i)} = 0$,
- $|\nabla f| \leq \frac{C}{r_i}$ on $B(x_i, r_i)$ for a uniform constant C (we here need the *a priori* bound on the number of overlaps).

Again, by uniform we mean independent of n. Notice that f^{-1}(open star of the i-th vertex) $\subset B(x_i, r_i)$. The idea is now to retract f to $f^{(2)} : M_n \longrightarrow K^{(2)}$, the 2-keleton of K, by a suitable projection of K onto $K^{(2)}$ preserving the 2-skeleton. We then define a new covering, $V_i = (f^{(2)})^{-1}$(open star of the i-th vertex) $\subset B(x_i, r_i)$. Clearly we still have that $\mathrm{Im}(\pi_1(V_i) \longrightarrow \pi_1(M))$ is trivial and the covering by the V_i's has dimension at most 2 by construction. This yields the contradiction.

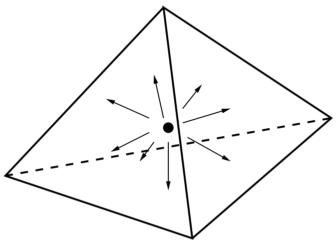
Let us now explain how to reduce the dimension of the covering. It is done in two steps of different nature. First we deform f to a map into the 3-skeleton and then reduce to the 2-skeleton. Let us assume that we already have a map $f : M_n \longrightarrow K^{(3)}$ and explain the last argument. It relies on the following lemma.

Lemma 9.10. *No* 3*-simplex* $\Delta^3 \subset K^{(3)}$ *is filled by* f *for large* D*.*

The proof of this fact reduces to a volume estimate using the metric properties of the local models described in 9.6

$$\begin{aligned} \mathrm{vol}(\mathrm{Im}(f) \cap \Delta^3) \quad &\leq \mathrm{vol}(f(B(x_i, r_i))) \leq |\nabla f|^3 \, \mathrm{vol}(B(x_i, r_i)) \\ &\leq (\tfrac{C}{r_i})^3 \tfrac{r_i^3}{D} \leq \tfrac{C^3}{D} < \mathrm{vol}(\Delta^3)\,. \end{aligned}$$

It suffices then to compose f with the radial projection $\Delta^3 \setminus \{\star\} \longrightarrow \partial\Delta^3$ and get the desired map $f^{(2)}$. This finishes the proof of Proposition 9.7.



$\square$

We then know that there exists a non trivial local model $\mathcal{V}$, such that $M \setminus \mathcal{V}$ is irreducible hence Haken. Being non trivial reduces the list to

$$\mathcal{V} \simeq S^1 \times D^2, T^2 \times I \,\mathrm{or}\, K \widetilde{\times} I\,.$$

By Thurston's geometrization of Haken manifolds, $M \setminus \mathcal{V}$ has a geometric decomposition. Furthermore, the boundary is a torus or a union of two tori. As before we can construct a finite sub-covering of $M \setminus \mathcal{V}$ using the local models and reduce its dimension to be at most 2. There are some issues about the boundary which we omit. The reader is referred to [5]. Looking at the list of possibilities for theses local models we see that the image in $\pi_1(M \setminus \mathcal{V})$ of their fundamental group is virtually abelian, i.e. it has a subgroup of finite index which is abelian. In this situation, a theorem of M. Gromov (see [16]) asserts that the simplicial volume of $M \setminus \mathcal{V}$ vanishes. Let us recall the definition of the simplicial volume of a manifold.

Definition 9.11. *N^k orientable manifold,*

$$||N|| = \inf\{\sum |t_\sigma|;\ [\sum t_\sigma \sigma] = [N] \in H_k(N; \boldsymbol{R})\}\,.$$

The main properties in our context are:

- $||N^3|| = 0$ if N is Seifert fibred,
- $||N^3|| = v_3 \operatorname{vol}(N^3)$ if N is hyperbolic (Gromov-Thurston),
- if N^3 has a geometric decomposition with hyperbolic pieces N_i^3 then $||N^3|| = v_3 \sum \operatorname{vol}(N_i^3)$ (T. Soma, see [35]).

Here v_3 is a universal constant. Hence we get the following characterization of graph manifolds.

Corollary 9.12. *If N^3 has a geometric decomposition and if $||N^3|| = 0$ then N^3 is a graphed manifold.*

We are working with a manifold with boundary and this approach requires to deal with this issue which is left to the reader who is referred to [5]. This shows that $M \setminus \mathcal{V}$ is graphed. It is now not difficult to check that gluing back $\mathcal{V}$, which has a local fibred structure, along tori shows that M is graphed too.

10 Extension and conclusion

The proof sketched above works even in the case when M is not aspherical as long as its fundamental group is not trivial. It requires just to replace McMullen's theorem by the following nice result.

Theorem 10.1 (J.C. Gómez-Larrañaga – F. González-Acuña, see [15]). *Let N be a closed, connected, orientable, irreducible* 3*-manifold. If N has a covering of dimension at most* 2 *by open subsets which are homotopically trivial in N, then N is simply connected.*

It is a pity that we only miss the simply connected case, that is the Poincaré conjecture. In fact we could include it in our framework if we could conclude in the previous theorem that N is a sphere. It is not clear whether this would be as difficult as the Poincaré conjecture itself or of a lesser level of difficulty. The author of this small note is not a topologist and leaves this question to the reader.

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Institut Fourier and C.N.R.S.
Université de Grenoble I
E-mail: g.besson@ujf-grenoble.fr