Matemática Contemporânea, Vol 34, 195-232

PARABOLIC SUBMANIFOLDS OF RANK TWO

M. Dajczer **P. Morais**

An immersed submanifold $f\colon M^n \to \mathbb{R}^N$, $n \geq 3$, into Euclidean space with the induced metric is called of *rank two* if at any point the kernel of its vector valued second fundamental form has codimension two. Equivalently, we have that the image of the Gauss map in the Grassmannian of non-oriented n-planes G_n^N is a surface. These submanifolds have been the object of a great deal of work in Riemannian Geometry since long time ago. For instance, see [2] and references therein. This interest is in good part motivated by the fact that their curvature tensor is "as flat as possible" without vanishing altogether.

The subspace spanned by the second fundamental form, usually called the first normal space and denoted by N_1, of a rank two submanifold satisfies $\dim N_1 \leq 3$ at any point. It turns out that if in substantial codimension, any rank two submanifold is a hypersurface if $\dim N_1 = 1$ at any point. Then f is either a Euclidean surface or the cone over a spherical surface, up to a Euclidean factor, if dim $N_1 = 3$ everywhere. Submanifolds in the remaining and much more interesting case, namely, when $\dim N_1 = 2$ everywhere, have been divided in three classes: elliptic, hyperbolic and parabolic. A complete parametric description of the elliptic submanifolds was given in [5].

For codimension $N - n = 2$, it was shown in [6] that elliptic and nonruled parabolic submanifolds are *genuinely* rigid. This means that given any other isometric immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+2}$ there is an open dense subset of M^n such that restricted to any connected component $f|_U$ and $\tilde{f}|_U$ are either congruent or there are an isometric embedding $j\colon U \hookrightarrow N^{n+1}$ into a Riemannian manifold N^{n+1} and either flat or isometric noncongruent hypersurfaces $F, \tilde{F}\colon N^{n+1} \to \mathbb{R}^{n+2}$ such that $f|_U = F \circ j$ and $\tilde{f}|_U = \tilde{F} \circ j$. Recently, we proved [8] that nonruled parabolic submanifolds in codimension two are not only genuinely rigid but, in fact, isometrically rigid.

The goal of this paper is to classify parametrically parabolic submanifolds in any codimension. First, we describe the ones that are ruled and show that they are the only parabolic submanifolds that admit an isometric immersion as a hypersurface. Then, we classify the nonruled ones by two different means. In fact, we provide the *polar* and *bipolar* parametrizations, each of which is associated to a parabolic surface and a function on the surface which satisfies a parabolic differential equation. To conclude, we describe the structure of the singular set of the nonruled parabolic submanifolds.

1 Parabolic submanifolds.

In this section, we introduce the concept of parabolic submanifold and study in detail the structure of the normal bundle.

We denote by $f\colon M^n \to \mathbb{Q}_\epsilon^N$, $\epsilon = 0, 1$, a connected n-dimensional submanifold of either Euclidean space $\mathbb{R}^N$ $(\epsilon = 0)$ or unit Euclidean sphere $\mathbb{S}^N$ $(\epsilon = 1)$ with codimension $N - n$. The k^{th}-*normal space* $N_k^f(x)$ of f at $x \in M^n$ is defined as

$$N_k^f(x) = \text{span}\{\alpha_f^{k+1}(X_1, \ldots, X_{k+1})\,;X_1, \ldots, X_{k+1} \in T_xM\}.$$

Here, $\alpha_f^\ell : TM \times \cdots \times TM \to T_f^\perp M$, $\ell \geq 2$, is the symmetric tensor known as the ℓ^{th}-*fundamental form* and given by

$$\alpha_f^\ell(X_1, \ldots, X_\ell) = \pi^{\ell-1}\left(\nabla^\perp_{X_\ell} \ldots \nabla^\perp_{X_3}\alpha_f(X_2, X_1)\right)$$

where π^ℓ stands for the orthogonal projection $\pi^\ell\colon T_f^\perp M \to (N_1^f \oplus \ldots \oplus N_{\ell-1}^f)^\perp$ and $T_f^\perp M$ is endowed with the normal connection $\nabla^\perp$ induced by the metric connection $\tilde\nabla$ in the ambient space. We agree that $\alpha_f^1\colon TM \to TM$ is $\alpha_f^1 = I$ and denote $\alpha_f^2 = \alpha_f$ $(\pi^1 = I)$ as usual.

We always assume that $f\colon M^n \to \mathbb{Q}_\epsilon^N$ is substantial and has rank 2. The later condition is denoted as $\text{rank}_f = 2$, and means that the relative nullity subspaces $\Delta(x) \subset T_xM$ defined as

$$\Delta(x) = \{X \in T_xM : \alpha_f(X, Y) = 0\,; Y \in T_xM\},$$

form a tangent subbundle of codimension two. It is a standard fact that the relative nullity distribution is integrable and that the leaves are totally geodesic submanifolds of the ambient space $\mathbb{Q}_\epsilon^N$.

The cone $Cf\colon M^n \times \mathbb{R}_+ \to \mathbb{R}^{N+1}$ of a submanifold $f\colon M^n \to \mathbb{S}^N$ of rank two has the same rank since the relative nullity leaves of Cf are the cones of the relative nullity leaves of f. Moreover, one has that $N_k^{Cf} = N_k^f$, $k \geq 1$, up to parallel transport in $\mathbb{R}^{N+1}$. Thus, it suffices to consider the Euclidean case since we had restricted ourselves to submanifolds of $\mathbb{R}^N$ and $\mathbb{S}^N$.

The condition $\operatorname{rank}_f = 2$ and the symmetry of the second fundamental form imply that the first normal spaces of f satisfy $\dim N_1^f \leq 3$ at any point. By Theorem 1 in [9] we have that f is a hypersurface in substantial codimension if $\dim N_1^f = 1$ everywhere. On the other hand, it is not difficult to show that a submanifold with $\dim N_1^f = 3$ everywhere is either a Euclidean surface or the cone over a spherical surface up to Euclidean factor. In the remaining case when $\dim N_1^f = 2$ everywhere, either there exists a pair of linearly independent "conjugate directions" $X_1, X_2 \in \Delta^\perp$, i.e.,

$$\alpha_f(X_1, X_1) \pm \alpha_f(X_2, X_2) = 0, \tag{1}$$

or f admits an "asymptotic direction" $0 \neq Z \in \Delta^\perp$, i.e., $\alpha_f(Z, Z) = 0$. In cases (1) the submanifold was called *elliptic* for the plus sign and *hyperbolic* for the minus sign in [5].

Definition 1. A submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$ is called *parabolic* if we have:

(i) $\operatorname{rank}_f = 2$,

(ii) $\dim N_1^f = 2$,

(iii) There is a nonsingular asymptotic vector field $Z \in \Delta^\perp$, i.e., $\alpha_f(Z, Z) = 0$.

Notice that cones of parabolic spherical submanifolds are also parabolic.

Let $f\colon M^n \to \mathbb{R}^N$ be a parabolic submanifold. We always denote by $\{X, Z\}$ an orthonormal frame in $\Delta^\perp$ where Z is an asymptotic vector field. Clearly, we can always take an orthonormal smooth frame $\{\eta_1, \eta_2\}$ in N_1^f such that the shape operators take the form

$$A^f_{\eta_1}|_{\Delta^\perp} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A^f_{\eta_2}|_{\Delta^\perp} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

where the functions b, c never vanish. In particular, we see that the asymptotic field Z is unique up to sign.

An easy argument given in [5] proves the following fact.

Proposition 2. *Assume that* $f\colon M^n \to \mathbb{Q}_\epsilon^N$ *satisfies* $\dim N_1^f = 2$ *at any point. Then, we have that* $\dim N_k^f \leq 2$ *for all* $k \geq 1$.

We always admit that the fibers of any N_k^f have constant dimension and thus form subbundles of the normal bundle. If $\tau = \tau^f$ denotes the index of the "last" of the normal subbundles of f, then $T_f^\perp M = N_1^f \oplus \cdots \oplus N_\tau^f$ since, by assumption, f is substantial.

We denote

$$\xi_1^k = \alpha_f^{k+1}(X, \ldots, X) \quad \text{and} \quad \xi_2^k = \alpha_f^{k+1}(Z, X, \ldots, X).$$

Since $\alpha_f^{k+1}(Z, Z, Y_1, \ldots, Y_{k-1}) = 0$, it is clear that

$$N_k^f = \operatorname{span}\{\xi_1^k, \xi_2^k\} \quad \text{for} \quad 1 \leq k \leq \tau^f.$$

Proposition 3. *For* $1 \leq k \leq \tau^f - 1$ *the following holds:*

(i) $(\tilde\nabla_Z\, \xi_1^k)_{N^f_{k+1}} = (\tilde\nabla_X\, \xi_2^k)_{N^f_{k+1}} = \xi_2^{k+1}$,

(ii) $(\tilde\nabla_X\, \xi_1^k)_{N^f_{k+1}} = \xi_1^{k+1}$,

(iii) $(\tilde\nabla_Z\, \xi_2^k)_{N^f_{k+1}} = 0$.

Proof: From the definition of the k-normal spaces, given $\eta \in N_l^f$ we have

$$\nabla_Y^\perp \eta \in N_{l-1}^f \oplus N_l^f \oplus N_{l+1}^f \tag{3}$$

where $N_0^f = 0 = N_{\tau^f+1}^f$. Then,

$$\xi_2^{k+1} = (\nabla_Z^\perp(\nabla_X^\perp \dots \nabla_X^\perp \alpha_f(X,X))_{N_k^f})_{N_{k+1}^f} = (\tilde\nabla_Z\, \xi_1^k)_{N_{k+1}^f},$$

$$\xi_2^{k+1} = (\nabla_X^\perp(\nabla_Z^\perp \dots \nabla_X^\perp \alpha_f(X,X))_{N_k^f})_{N_{k+1}^f} = (\tilde\nabla_X\, \xi_2^k)_{N_{k+1}^f}$$

and (i) has been proved. The proof of (ii) is similar. For (iii), we have

$$(\tilde\nabla_Z \xi_2^k)_{N_{k+1}^f} = (\nabla_Z^\perp(\nabla_X^\perp \dots \nabla_X^\perp \alpha_f(X,Z))_{N_k^f})_{N_{k+1}^f} = \alpha_f^{k+2}(X,\dots,Z,Z) = 0.$$

□

The following fact was proved in [5].

Proposition 4. *If $f\colon M^n \to \mathbb{R}^N$ is a parabolic submanifold, then the normal subbundles N_k^f, $1 \le k \le \tau^f$, are parallel in $\mathbb{R}^N$ along Δ.*

Let $\mathcal{V}_k \subset N_k^f \times N_k^f$, $0 \le k \le \tau^f$, be the subspace defined as

$$\mathcal{V}_k = \{(\mu_1,\mu_2) \in N_k^f \times N_k^f : \langle \mu_2, \xi_2^k\rangle = 0 \ \text{ and } \ \langle \mu_2, \xi_1^k\rangle = \langle \mu_1, \xi_2^k\rangle\}.$$

It is easy to see that $\mathcal{V}_k$ is independent of the base $\{X, Z\}$ with Z asymptotic. Clearly, $\xi_1^k = 0$ implies that $\mathcal{V}_k = 0$. We also have the following facts.

Lemma 5. *For $1 \le k \le \tau^f$ the following holds:*

(i) $\dim \mathcal{V}_k = 2$ *if and only if* $\dim N_k^f = 2$,

(ii) $\dim \mathcal{V}_k = 1$ *if and only if* $\dim N_k^f = 1$ *and* $\xi_2^k = 0$,

(iii) $\dim \mathcal{V}_k = 0$ *if and only if* $\dim N_k^f = 1$ *and* $\xi_2^k \neq 0$.

Proof: If $\dim \mathcal{V}_k = 2$, we either may choose $(\mu_1,\mu_2) \in \mathcal{V}_k$ such that $\mu_1 \neq 0 \neq \mu_2$ or we are done. It is easy to see that μ_1 and μ_2 must be linearly independent, and thus $\dim N_k^f = 2$. Then, take $0 \neq v \in N_k^f$ such that $\langle v, \xi_2^k\rangle = 0$, and set $u = (\langle v, \xi_1^k\rangle / \|\xi_2^k\|^2)\xi_2^k$. Hence, u, v are a base of N_k^f and $(u,v), (u+v,v) \in \mathcal{V}_k$ are linearly independent. This proves (i). The proofs of (ii) and (iii) follow easily form the definition of $\mathcal{V}_k$.

□

Definition 6. Given a parabolic submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N \subseteq \mathbb{R}^{N+\epsilon}$, we call an element $\beta \in C^\infty(M^n, \mathbb{R}^{N+\epsilon})$ a *k–cross section* to f, $1 \leq k \leq \tau^f$, if at any point

$$\beta_*(TM) \subset N_k^f \oplus \cdots \oplus N_{\tau^f}^f,$$

up to parallel transport in $\mathbb{R}^{N+\epsilon}$.

Lemma 7. *Let $\mathcal{P}_k\colon C^\infty(M^n, \mathbb{R}^{N+\epsilon}) \to N_k^f \times N_k^f$, $1 \leq k \leq \tau^f$, be the tensor*

$$\mathcal{P}_k(\beta) = ((\beta_* X)_{N_k^f}, (\beta_* Z)_{N_k^f}).$$

Then $\mathcal{P}_k(\beta) \in \mathcal{V}_k$ for any k–cross section β to f. Moreover, the tensor

$$\mathcal{P}_k|_{N_{k+1}^f}\colon N_{k+1}^f \to \mathcal{V}_k, \quad 1 \leq k \leq \tau^f - 1,$$

is injective.

Proof: We have,

$$\begin{aligned}
\langle \beta_* X, \xi_2^k \rangle &= \langle \tilde{\nabla}_X \beta, \tilde{\nabla}_Z (\nabla_X^\perp \dots \nabla_X^\perp \alpha_f(X,X)) \rangle \\
&= Z \langle \tilde{\nabla}_X \beta, \nabla_X^\perp \dots \nabla_X^\perp \alpha_f(X,X) \rangle - \langle \tilde{\nabla}_Z \tilde{\nabla}_X \beta, \nabla_X^\perp \dots \nabla_X^\perp \alpha_f(X,X) \rangle \\
&= \langle \tilde{\nabla}_Z \beta, \tilde{\nabla}_X (\nabla_X^\perp \dots \nabla_X^\perp \alpha_f(X,X)) \rangle \\
&= \langle \beta_* Z, \xi_1^k \rangle.
\end{aligned}$$

A similarly argument gives

$$\langle \beta_* Z, \xi_2^k \rangle = \langle \beta_* X, \alpha_f^{k+1}(Z, Z, X, \dots, X) \rangle = 0.$$

To conclude, observe that if $\eta \in N_{k+1}^f$ satisfies $\mathcal{P}_k(\eta) = 0$, then

$$0 = \langle \eta_* X, \xi_j^k \rangle = \langle \tilde{\nabla}_X \eta, \xi_j^k \rangle = -\langle \eta, \tilde{\nabla}_X \xi_j^k \rangle = -\langle \eta, \xi_j^{k+1} \rangle, \;\; j = 1, 2.$$

Hence, $\eta = 0$.

$\square$

Proposition 8. *Let $f\colon M^n \to \mathbb{R}^N$ be a parabolic submanifold. Then, we have:*

(i) $\xi_1^k \neq 0$ *for any* $1 \leq k \leq \tau^f - 1$,

(ii) $\xi_2^k = 0$ *if and only if* $\dim N_k^f = 1$,

(iii) If $\xi_2^k = 0$, *then* $\xi_2^j = 0$ *for* $j \geq k$.

Proof: To prove (i) suppose that $\xi_1^k = 0$. Thus, $\mathcal{V}_k = 0$. Then Lemma 7 gives $N_{k+1}^f = 0$, which is not possible. For (ii) suppose that $\dim N_k^f = 1$ and $\xi_2^k \neq 0$. We have that $\mathcal{V}_k = 0$ from Lemma 5, and by Lemma 7 this is a contradiction. Finally, to prove (iii) assume $\xi_2^k = 0$. Using (3) we have

$$\begin{aligned} \xi_2^{k+1} &= \pi^{k+1}(\nabla_X^\perp \nabla_Z^\perp \nabla_X^\perp \ldots \nabla_X^\perp \alpha_f(X,X)) \\ &= \pi^{k+1}(\nabla_X^\perp(\pi^k(\nabla_Z^\perp \nabla_X^\perp \ldots \nabla_X^\perp \alpha_f(X,X))) \\ &= \pi^{k+1}(\nabla_X^\perp \xi_2^k) = 0. \end{aligned}$$

$\square$

Definition 9. We say that a parabolic submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$ has *critical index* $\tau_0^f \in \{1,\ldots,\tau^f - 1\}$ if $\xi_2^{\tau_0^f} \neq 0$ and $\xi_2^k = 0$ for any $k \geq \tau_0^f + 1$.

Corollary 10. *Assume that* f *possesses critical index. Then:*

(i) $\dim N_k^f = 2,\ 1 \leq k \leq \tau_0^f$,

(ii) $\dim N_k^f = 1,\ \tau_0^f + 1 \leq k \leq \tau^f$,

(iii) The tensor, $\mathcal{P}_k|_{N_{k+1}^f}\colon N_{k+1}^f \to \mathcal{V}_k$ *is an isomorphism for* $k \leq \tau_0^f - 1$.

2 Intrinsic proprieties

In this section we analyze the metric structure of the parabolic submanifolds.

Proposition 11. *Let* $f\colon M^n \to \mathbb{R}^N$ *be a parabolic submanifold. Then,*

$$\mathcal{F} = span\{Z\} \oplus \Delta$$

is an integrable distribution and the leaves are flat hypersurfaces.

Proof: We first show that the line bundle $L = \mathrm{span}\{\xi_2^1\}$ is parallel along the leaves of relative nullity. The unit vector field $\eta \in N_1^f$ orthogonal to ξ_2^1 is the only one, up to sign, such that $A_\eta^f Z = 0$. Thus A_η^f has rank 1. In view of Proposition 4 it is sufficient to show that η is parallel along Δ.

Recall that the *splitting tensor* C associates to $T \in \Delta$ the endomorphism C_T of $\Delta^\perp$ defined as

$$C_T X = -\left(\nabla_X T\right)_{\Delta^\perp}.$$

It is well-known [7] that the differential equation

$$\nabla_T A_\xi^f = A_\xi^f \circ C_T \tag{4}$$

is satisfied along $\Delta^\perp$ if $\xi \in T_f^\perp M$ is parallel along Δ.

Let $x \in M^n$ and γ a geodesic with $\gamma(0) = x$ contained in the corresponding leaf of Δ. If δ_t is the parallel transport of η_x along γ, we have

$$\nabla_{\gamma'} A_{\delta_t}^f = A_{\delta_t}^f \circ C_{\gamma'}.$$

Hence, $A_{\delta_t}^f = A_{\eta_x}^f e^{\int_0^t C_{\gamma'} d\tau}$. Thus $A_{\delta_t}^f$ has rank 1 and, therefore $\eta = \delta_t$ is parallel.

Since the left hand side of

$$\nabla_T A_\eta^f = A_\eta^f \circ C_T$$

is symmetric, we obtain that

$$A_\eta^f C_T Z = C_T^t A_\eta^f Z = 0.$$

Thus $C_T Z \in \mathrm{span}\{Z\}$, that is, $\langle \nabla_Z T, X\rangle = 0$. Then the Codazzi equation yields

$$\nabla_T^\perp \alpha_f(Z,X) - \langle \nabla_T Z, X\rangle \alpha_f(X,X) + \langle \nabla_Z T, Z\rangle \alpha_f(Z,X) = 0.$$

Using that L is parallel along Δ, we obtain that $\langle \nabla_T Z, X\rangle = 0$. Hence $\mathcal{F}$ is integrable. Moreover, the second fundamental form of a leaf U is

$$A_X^U = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

where $\lambda = \langle \nabla_Z Z, X\rangle$. Thus the leaves of $\mathcal{F}$ are flat.

$\square$

Recall that a submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$ is called *ruled* when M^n admits a hypersurface foliation of totally geodesic submanifolds of $\mathbb{Q}_\epsilon^N$.

Example 12. Ruled Euclidean submanifolds of rank 2 without flat points and substantial codimension at least 2 are basic examples of parabolic submanifolds. In fact, it follows from Corollary 4.7 in [3] that $\dim N_1^f = 2$.

From the proof of Proposition 11 we have the following fact.

Corollary 13. *Let $f\colon M^n \to \mathbb{R}^N$ be a ruled parabolic submanifold. Then the leaves of $\mathcal{F}$ are totally geodesic in M^n.*

3 Regularity

A key ingredient in the parametric description of the elliptic submanifolds given in [5] was the regularity of the k-normal spaces. In fact, any elliptic submanifold f satisfies $\dim N_k^f = 2$, $1 \le k \le \tau^f - 1$, whereas the dimension of $N_{\tau^f}^f$ is determined by the codimension. In this paper, that a parabolic submanifold is regular roughly means that the N_k^f's behave as in the elliptic case. The main result in this section is that nonregular parabolic submanifolds are necessarily ruled.

Definition 14. We say that a parabolic submanifold $f\colon M^n \to \mathbb{R}^N$ is *regular* if $\dim N_k^f = 2$ for any $1 \le k \le \tau^f - 1$.

By Corollary 10, the following holds:

$$f \text{ is regular if and only if } \begin{cases} \dim N_{\tau^f}^f = 2 \iff \xi_2^{\tau^f} \neq 0, \text{ if } N-n \text{ is even} \\ \dim N_{\tau^f-1}^f = 2 \iff \xi_2^{\tau^f-1} \neq 0, \text{ if } N-n \text{ is odd.} \end{cases}$$

Observe that ruled surfaces with $\dim N_1 = 2$ are parabolic. We give next an example of such a surface that is nonregular.

Example 15. Let $c\colon I\subset\mathbb{R}\to\mathbb{R}^6$ be a smooth curve parametrized by arc length with Frenet frame $E_1,\ldots,E_6$ and constant Frenet curvatures $k_j\neq 0, 1\leq j\leq 5$. The map $X\colon\mathbb{R}^2\to\mathbb{R}^6$ given by

$$X(s,t)=c(s)+tE_2(s)$$

parametrizes a substantial complete surface that is parabolic for $t\neq 0$. An easy calculation gives $\xi_2^2=0$, that is, $\tau_0^X=1$. Hence, $\dim N_2^X=1$ and therefore X is nonregular.

By a parabolic submanifold being *nonruled* we understand that none of the leaves of $\mathcal{F}$ is totally geodesic in M^n or, equivalently, in $\mathbb{R}^N$.

Theorem 16. *Nonruled parabolic submanifolds $f\colon M^n\to\mathbb{R}^N$ are regular.*

The proof of Theorem 16 will follow from two results. First, we give a sufficient condition for a parabolic submanifold in odd codimension to be ruled.

Proposition 17. *Let $f\colon M^n\to\mathbb{R}^N$ be a regular parabolic submanifold satisfying that $\xi_2^{\tau^f}=0$ at any point. Then f is ruled.*

Proof: We claim that f is ruled if and only if $L=\operatorname{span}\{\xi_2^1\}$ is parallel along $\mathcal{F}$. From the proof of Proposition 11, we know that L is parallel along Δ. Clearly, that f is ruled is equivalent to $\nabla_Z Z=0$. Take an orthonormal frame $\{\eta_1,\eta_2\}$ in N_1^f as in (2). Since $\eta_1\in L$, we have to show that

$$\nabla_Z Z=0 \text{ if and only if } (\nabla^\perp_Z\eta_1)_{N_1^f}=0. \tag{5}$$

From the Codazzi equation

$$\langle(\nabla_X A^f_{\eta_2})Z-(\nabla_Z A^f_{\eta_2})X,Z\rangle=0,$$

we get

$$c\langle\nabla_Z Z,X\rangle=b\langle\nabla^\perp_Z\eta_1,\eta_2\rangle.$$

Being f parabolic we obtain $b\neq 0\neq c$, and the claim follows.

We first consider the case $N-n=3$. We have, $\dim N_1^f=2,\ \dim N_2^f=1$ and $\xi_2^2=0$. It suffices to show that η_1 is parallel along Z. By Proposition 3, the subbundles N_1^f, N_2^f are parallel along Z. Thus, the Codazzi equation gives

$$A^f_{\nabla^\perp_X\delta}Z=A^f_{\nabla^\perp_Z\delta}X=0$$

where $\delta\in N_2^f$ has unit length. Using (2) we obtain

$$\left(\nabla^\perp_X\delta\right)_{N_1^f}\in\operatorname{span}\{\eta_2\}. \tag{6}$$

From $X\langle\eta_1,\delta\rangle=0$ and (6) we have

$$\left(\nabla^\perp_X\eta_1\right)_{N_2^f}=0. \tag{7}$$

The Ricci equation, using (6), (7) and the parallelism of N_1^f along Z gives

$$\begin{aligned}0=\langle R^\perp(X,Z)\eta_1,\delta\rangle &= \langle\nabla^\perp_X\nabla^\perp_Z\eta_1-\nabla^\perp_Z\nabla^\perp_X\eta_1-\nabla^\perp_{[X,Z]}\eta_1,\delta\rangle\\ &= \langle\nabla^\perp_X\nabla^\perp_Z\eta_1,\delta\rangle=-\langle\nabla^\perp_Z\eta_1,\nabla^\perp_X\delta\rangle\\ &= \langle\nabla^\perp_Z\eta_1,\eta_2\rangle\langle\nabla^\perp_X\eta_2,\delta\rangle.\end{aligned}$$

But $\langle\nabla^\perp_X\eta_2,\delta\rangle\neq 0$ since N_1^f is not parallel. Thus, $(\nabla^\perp_Z\eta_1)_{N_1^f}=0$.

We now consider the general case $N-n\geq 5$. Take an orthonormal basis $\{\eta_1^k,\eta_2^k\}$ of N_k^f for any $1\leq k\leq\tau^f-1$ such that

$$\xi_1^k=a_k\eta_1^k+c_k\eta_2^k\quad\text{and}\quad\xi_2^k=b_k\eta_1^k.$$

Proposition 3 gives

$$\left(\nabla^\perp_Z\eta_1^k\right)_{N_{k+1}^f}=0\quad\text{and}\quad c_k\left(\nabla^\perp_Z\eta_2^k\right)_{N_{k+1}^f}=b_k\left(\nabla^\perp_X\eta_1^k\right)_{N_{k+1}^f}. \tag{8}$$

Since $\dim N_k^f=2$, $1\leq k\leq\tau^f-1$, it follows from (8) that

$$N_k^f=\operatorname{span}\left\{\left(\nabla^\perp_X\eta_1^{k-1}\right)_{N_k^f},\left(\nabla^\perp_X\eta_2^{k-1}\right)_{N_k^f}\right\}. \tag{9}$$

From (8) and $\xi_2^{\tau^f}=0$, we have

$$(\nabla^\perp_Z\eta_1^{\tau^f-1})_{N_{\tau^f}^f}=(\nabla^\perp_X\eta_1^{\tau^f-1})_{N_{\tau^f}^f}=(\nabla^\perp_Z\eta_2^{\tau^f-1})_{N_{\tau^f}^f}=0. \tag{10}$$

Thus $N_1^f \oplus \ldots \oplus N_{\tau^f-1}^f$ and $N_{\tau^f}^f$ are both parallel along Z. The Ricci equation for $\delta \in N_{\tau^f}^f$ and (10) give

$$\begin{aligned} 0 = \langle R^\perp(X,Z)\eta_1^{\tau^f-1}, \delta\rangle &= \langle \nabla_X^\perp \nabla_Z^\perp \eta_1^{\tau^f-1}, \delta\rangle = -\langle \nabla_Z^\perp \eta_1^{\tau^f-1}, \nabla_X^\perp \delta\rangle \\ &= \langle \nabla_Z^\perp \eta_1^{\tau^f-1}, \eta_2^{\tau^f-1}\rangle \langle \nabla_X^\perp \eta_2^{\tau^f-1}, \delta\rangle. \end{aligned}$$

But $\langle \nabla_X^\perp \eta_2^{\tau^f-1}, \delta\rangle \neq 0$ since f is substantial. Therefore,

$$(\nabla_Z^\perp \eta_1^{\tau^f-1})_{N_{\tau^f-1}^f} = 0.$$

To conclude again that $\langle \nabla_Z^\perp \eta_1^1, \eta_2^1\rangle = 0$, it suffices to show that if

$$(\nabla_Z^\perp \eta_1^{\ell+1})_{N_{\ell+1}^f} = 0, \quad 1 \leq \ell \leq \tau^f - 2, \tag{11}$$

then

$$(\nabla_Z^\perp \eta_1^{\ell})_{N_\ell^f} = 0. \tag{12}$$

Being η_1^ℓ collinear with ξ_2^ℓ and $\eta_1^{\ell+1}$ with $\xi_2^{\ell+1}$, then $\eta_1^{\ell+1}$ and $(\nabla_X^\perp \eta_1^\ell)_{N_{\ell+1}^f}$ are also collinear. From (11), we have

$$\langle \nabla_Z^\perp (\nabla_X^\perp \eta_1^\ell)_{N_{\ell+1}^f}, \eta_2^{\ell+1}\rangle = 0. \tag{13}$$

The Ricci equation using (8) and (13) yields

$$\begin{aligned} 0 &= \langle R^\perp(X,Z)\eta_1^\ell, \eta_2^{\ell+1}\rangle = \langle \nabla_X^\perp \nabla_Z^\perp \eta_1^\ell - \nabla_Z^\perp \nabla_X^\perp \eta_1^\ell - \nabla_{[X,Z]}^\perp \eta_1^\ell, \eta_2^{\ell+1}\rangle \\ &= \langle \nabla_X^\perp \langle \nabla_Z^\perp \eta_1^\ell, \eta_2^\ell\rangle \eta_2^\ell, \eta_2^{\ell+1}\rangle - \langle \nabla_Z^\perp (\nabla_X^\perp \eta_1^\ell)_{N_\ell^f}, \eta_2^{\ell+1}\rangle - \langle \nabla_X Z, X\rangle \langle \nabla_X^\perp \eta_1^\ell, \eta_2^{\ell+1}\rangle \\ &= \langle \langle \nabla_Z^\perp \eta_1^\ell, \eta_2^\ell\rangle \nabla_X^\perp \eta_2^\ell - \langle \nabla_X^\perp \eta_1^\ell \eta_2^\ell\rangle \nabla_Z^\perp \eta_2^\ell, -\langle \nabla_X Z, X\rangle \nabla_X^\perp \eta_1^\ell, \eta_2^{\ell+1}\rangle. \end{aligned}$$

Thus,

$$\big(\langle \nabla_Z^\perp \eta_1^\ell, \eta_2^\ell\rangle \nabla_X^\perp \eta_2^\ell - \langle \nabla_X^\perp \eta_1^\ell, \eta_2^\ell\rangle \nabla_Z^\perp \eta_2^\ell - \langle \nabla_X Z, X\rangle \nabla_X^\perp \eta_1^\ell\big)_{N_{\ell+1}^f} \in \operatorname{span}\{\eta_1^{\ell+1}\},$$

and we obtain (12) from (8) and (9).

□

To conclude that f is ruled, from (11) and (12) in the proof of the preceding result it is sufficient to show that there exists an index $1 \leq \ell \leq \tau^f - 2$ such that $(\nabla_Z^\perp \eta_1^{\ell+1})_{N_{\ell+1}^f} = 0$. Thus, this gives the following fact.

Corollary 18. *Let $f\colon M^n \to \mathbb{R}^N$ be a regular parabolic submanifold. If there is an index $1 \le s \le \tau^f - 1$ such that $\eta_1^s = \xi_2^s/\|\xi_2^s\| \in N_s^f$ satisfies $(\nabla_Z^\perp \eta_1^s)_{N_s^f} = 0$, then f is ruled.*

Our next result deals with nonregular parabolic submanifolds.

Proposition 19. *Let $f\colon M^n \to \mathbb{R}^N$ be a simply connected parabolic submanifold. Assume that $\dim N_{k_0-1}^f = 2$ and $\dim N_{k_0}^f = 1$ for some index $2 \le k_0 \le \tau^f - 1$. Then, there exists a parabolic regular isometric immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+2k_0-1}$ such that the subbundles $N_s^{\tilde{f}}$ and N_s^f, $1 \le s \le k_0$, endowed with the induced connection, correspond by a parallel isometry.*

Proof: Consider the normal subbundle $\mathcal{T} = N_1^f \oplus \ldots \oplus N_{k_0}^f$ with the induced connection $\hat{\nabla}_Y^\perp \eta = (\nabla_Y^\perp \eta)_{\mathcal{T}}$. We have to show that α_f still satisfies the Gauss, Codazzi and Ricci equations. In fact, the Gauss and Codazzi equations are trivially satisfied. By Propositions 3 and 8, the subbundles $\mathcal{T}$ and $\mathcal{T}^\perp$ are parallel in the normal connection along Z. Given $\eta \in \mathcal{T}$, a simple calculation yields

$$\hat{R}^\perp(X,Z)\eta - R^\perp(X,Z)\eta = -\left(\nabla_X^\perp \nabla_Z^\perp \eta\right)_{\mathcal{T}^\perp} + \nabla_Z^\perp \left(\nabla_X^\perp \eta\right)_{\mathcal{T}^\perp} + \left(\nabla_{[X,Z]}^\perp \eta\right)_{\mathcal{T}^\perp}.$$

Since $R^\perp(X,Z)\eta \in \mathcal{T}$ by the Ricci equation, the left hand side vanishes and thus

$$\hat{R}^\perp(X,Z)\eta = R^\perp(X,Z)\eta.$$

Now using Proposition 4 we conclude that the Ricci equation is satisfied. Since M^n is simply connected, the result follows from the Fundamental theorem of submanifolds.

□

Finally, we are in condition to prove Theorem 16.

Proof: Assume that f is nonregular. By Proposition 8 there exists $k_0 \le \tau^f - 1$ such that $\xi_2^{k_0} = 0$. By Proposition 19, there is a regular parabolic submanifold

$\tilde{f}\colon M^n \to \mathbb{R}^{n+2k_0-1}$ with $\xi_2^{\tau^{\tilde{f}}} = 0$. It follows from Proposition 17 that f is ruled.

□

4 Ruled parabolic

The simple structure of ruled parabolic submanifolds allows us to give a parametric description of these submanifolds. Using this description, we conclude that this submanifolds are generically regular. Then, we show that ruled parabolic submanifolds are the only parabolic submanifolds that admit isometric immersions as hypersurfaces.

Let $v\colon I \subset \mathbb{R} \to \mathbb{R}^N$ be a smooth curve parametrized by arc length in some interval. Set $e_1 = dv/ds$ and let $e_2, \ldots, e_{n-1}$ be orthonormal normal vector fields along $v = v(s)$ parallel in the normal connection of v in $\mathbb{R}^N$. Thus,

$$\frac{de_j}{ds} = b_j e_1, \quad 2 \leq j \leq n-1, \tag{14}$$

where $b_j \in C^\infty(I)$. Set $\Delta = \operatorname{span}\{e_2, \ldots, e_{n-1}\}$ and let $\Delta^\perp$ be the orthogonal complement in the normal bundle. Take $e_0 \in \Delta^\perp$ along v such that

$$P = \{e_0, (de_1/ds)_{\Delta^\perp}\} \subset \Delta^\perp$$

satisfy that

$$\dim P = 2 \tag{15}$$

and that P is nowhere parallel in $\Delta^\perp$ along v, that is,

$$\operatorname{span}\{(de_0/ds)_{\Delta^\perp}, (d^2 e_1/ds^2)_{\Delta^\perp}\} \not\subset P. \tag{16}$$

We parametrize a ruled submanifold M^n by

$$f(s, t_1, \ldots, t_{n-1}) = c(s) + \sum_{j=1}^{n-1} t_j e_j(s) \tag{17}$$

where $(t_1,\dots,t_{n-1}) \in \mathbb{R}^{n-1}$ and $c(s)$ satisfies $dc/ds = e_0$. To see that f is parabolic, first observe that

$$TM = \operatorname{span}\{f_s\} \oplus \operatorname{span}\{e_1\} \oplus \Delta$$

where $f_s = e_0 + t_1 de_1/ds + \sum_{j\geq 2} t_j b_j e_1$. Consider the orthogonal decomposition

$$\left(\frac{de_1}{ds}\right)_{\Delta^\perp} = a_1 e_0 + \eta. \tag{18}$$

Thus $\eta(s) \neq 0$ for all $s \in I$ from (15). Hence,

$$TM = \operatorname{span}\{e_0 + t_1(a_1 e_0 + \eta)\} \oplus \operatorname{span}\{e_1\} \oplus \Delta. \tag{19}$$

Since $f_{st_j} = b_j e_1 \in TM,\ 2 \leq j \leq n-1$, we have that $\Delta \subset \Delta_f$. It follows easily from (18), (19) and $\eta(s) \neq 0$ that

$$f_{st_1} = \frac{de_1}{ds} \notin TM.$$

It is easy to see that $f_{ss} \notin \operatorname{span}\{f_{st_1}\} \oplus TM$, i.e., $\dim N_1^f = 2$, is equivalent to

$$\left(\frac{de_0}{ds}\right)_{\Delta^\perp} + t_1 \left(\frac{d^2 e_1}{ds^2}\right)_{\Delta^\perp} \notin P.$$

It follows that $\Delta = \Delta_f$. Therefore f is parabolic in, at least, an open dense subset of M^n.

Let $f\colon M^n \to \mathbb{R}^N$ be a ruled parabolic submanifold and $\{e_2,\dots,e_{n-1}\}$ an orthonormal frame for Δ_f along an integral curve $c = c(s),\ s \in I$, of the unit vector field X orthogonal to the rulings. Without loss of generality (see Lemma 2.2 in [1]) we may assume that

$$\frac{de_j}{ds} \perp \Delta_f,\ \ 2 \leq j \leq n-1.$$

Now parametrize f by (17), where $e_0 = X$ and $e_1 = Z$. That $f_{st_j} \in TM$ implies

$$\frac{de_j}{ds} \in \operatorname{span}\{e_1, f_s\}, \quad 2 \leq j \leq n-1. \tag{20}$$

Taking $t_1 = 0$, we obtain that

$$\frac{de_j}{ds} = a_j e_0 + b_j e_1, \quad 2 \leq j \leq n-1, \tag{21}$$

where $a_j, b_j \in C^\infty(I)$. Since $\dim N_1^f = 2$, we have

$$\frac{de_1}{ds} = a_1 e_0 + (de_1/ds)_\Delta + \eta \tag{22}$$

where $\eta \perp \operatorname{span}\{e_0, e_1\} \oplus \Delta$ satisfies $\eta(s) \neq 0$. Thus (20) reduces to

$$a_j e_0 \in \operatorname{span}\{(1 + t_1 a_1 + \ldots + t_{n-1} a_{n-1}) e_0 + t_1 \eta\}, \quad 2 \leq j \leq n-1.$$

Therefore $a_j = 0$. From (21) we have $de_j/ds = b_j e_1$ for $2 \leq j \leq n-1$.

We have proved the following result.

Proposition 20. *Let $c\colon I \subset \mathbb{R} \to \mathbb{R}^N$, $N - n \geq 2$, be a smooth curve. Let $\{e_0 = dc/ds, e_1(s), \ldots, e_{n-1}(s)\}$ be orthonormal fields satisfying (14), (15) and (16) at any point. Then, the submanifold parametrized by*

$$f(s, t_1, \ldots, t_{n-1}) = c(s) + \sum_{j \geq 1} t_j e_j(s) \tag{23}$$

where $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$, defines a ruled submanifold, that is parabolic in an open dense subset of M^n. Conversely, any ruled parabolic submanifold can be parametrized as in (23).

Let f be a ruled parabolic submanifold parametrized by (23). Assume that f has critical index $k - 1 = \tau_0^f$. The condition $\dim N_k^f = 1$ is equivalent to

$$\frac{d^k e_1}{ds^k} \in TM \oplus \operatorname{span}\left\{\frac{d^{\ell-1} e_1}{ds^{\ell-1}}, \frac{d^{\ell-1} e_0}{ds^{\ell-1}} + t_1 \frac{d^\ell e_1}{ds^\ell}, \ \ 2 \leq \ell \leq k\right\} \tag{24}$$

where TM was given by (19). In particular, for $t_1 = 0$ and using (22) we have

$$\frac{d^{k-1}(a_1 e_0 + \eta)}{ds^{k-1}} \in TM \oplus \operatorname{span}\left\{\frac{d^{\ell-2}(a_1 e_0 + \eta)}{ds^{\ell-2}}, \frac{d^{\ell-1} e_0}{ds^{\ell-1}}, \ \ 2 \leq \ell \leq k\right\} \tag{25}$$

where now $TM = \operatorname{span}\{e_0, e_1\} \oplus \Delta$.

It is easy to see that (24) and (25) are equivalent. In fact, in (25) taking $\ell = 2$ we obtain that η belongs to the subspace. If (25) is satisfied, it follows that the subspace in (24) is independent of the parameter t_1. In particular, this shows again that $\dim N_k^f = 1$ is equivalent to $\xi_2^k = 0$. Finally, we have that (25) is equivalent to

$$\frac{d^{k-1}\eta}{ds^{k-1}} \in \operatorname{span}\left\{e_0, \frac{de_0}{ds}, \ldots, \frac{d^{k-1}e_0}{ds^{k-1}}, \eta, \ldots, \frac{d^{k-2}\eta}{ds^{k-2}}\right\} \oplus \Delta.$$

It is now clear that (24) will not be satisfied in general. In that sense and recalling Theorem 16, we can say that the parabolic submanifolds are *generically* regular.

Remark 21. A condition for a ruled regular parabolic submanifold in odd codimension to satisfies $\xi_2^{\tau^f} = 0$ is the following:

$$\frac{d^{\tau^f-1}\eta}{ds^{\tau^f-1}} \in \operatorname{span}\left\{e_0, \frac{de_0}{ds}, \ldots, \frac{d^{\tau^f-2}e_0}{ds^{\tau^f-2}}, \eta, \ldots, \frac{d^{\tau^f-2}\eta}{ds^{\tau^f-2}}\right\} \oplus \Delta.$$

Next we extend the characterization of ruled parabolic submanifolds in codimension two given in [6] to arbitrary codimension.

Definition 22. We say that a submanifold $f\colon M^n \to \mathbb{R}^N$ is of *surface type* if either $f(M) \subset L^2 \times \mathbb{R}^{n-2}$ where $L^2 \subset \mathbb{R}^{N-n+2}$ or $f(M) \subset CL^2 \times \mathbb{R}^{n-3}$ where $CL^2 \subset \mathbb{R}^{N-n+3}$ is a cone over a spherical surface $L^2 \subset \mathbb{S}^{N-n+2}$.

Theorem 23. *Let $f\colon M^n \to \mathbb{R}^N$ be a ruled parabolic submanifold. If M^n is simply connected then it admits an isometric immersion as a ruled hypersurface in $\mathbb{R}^{n+1}$ with the same rulings. Conversely, if M^n admits an isometric immersion as a hypersurface in $\mathbb{R}^{n+1}$ and f is not of surface type in any open subset, then f is ruled.*

Proof: To prove the converse, assume that there exists an isometric immersion $g\colon M^n \to \mathbb{R}^{n+1}$ with Gauss map N. We first show that

$$\Delta_g = \Delta_f. \tag{26}$$

Let $\beta\colon T_xM \times T_xM \to \mathbb{R}\langle\eta_1\rangle \oplus \mathbb{R}\langle N\rangle = \mathbb{R}^2$ be the symmetric bilinear form

$$\beta(Y,V) = (\langle A^f_{\eta_1}Y, V\rangle, \langle A^g_N Y, V\rangle)$$

where $\{\eta_1, \eta_2\}$ is as in (2). By the Gauss equation, β is flat with respect to the Lorentzian metric in $\mathbb{R}^2$ defined as $\|\eta_1\|^2 = 1 = -\|N\|^2$ and $\langle \eta_1, N\rangle = 0$, that is,

$$\langle \beta(X,Y), \beta(V,W)\rangle - \langle \beta(X,W), \beta(V,Y)\rangle = 0.$$

If (26) is not satisfied, and since $\dim \Delta_g \le n-2$, it follows easily that

$$S(\beta) = \mathrm{span}\{\beta(Y,V) : \; Y, V \in T_xM\}$$

satisfies $S(\beta) = \mathbb{R}^2$. From Corollary 1 in [11] we have $\dim N(\beta) = n-2$ where

$$N(\beta) = \{Y \in T_xM : \beta(Y,V) = 0, \;\; V \in T_xM\}.$$

But since $N(\beta) = \Delta_g \cap \Delta_f$, it follows that (26) holds.

Let

$$A^g_N|_{\Delta^\perp} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{bmatrix}.$$

From (4) we have

$$C_T = \begin{bmatrix} m & 0 \\ n & m \end{bmatrix}$$

for any $T \in \Delta$. On the other hand,

$$A^g_N \circ C_T = \begin{bmatrix} \bar{a}m + bn & \bar{b}m \\ \bar{b}m + \bar{c}n & \bar{c}m \end{bmatrix}.$$

The symmetry of $A^g_N \circ C_T$ allows to conclude that $\bar{c}n = 0$. Since f is nowhere of surface type, it follows from Lemma 6 in [4] that $n \neq 0$ for some $T \in \Delta$ in an open dense subset of M^n. Thus $\bar{c} = 0$ and therefore, by the Gauss equation, we may assume that $\bar{b} = b$.

The Codazzi equation for $A^f_{\eta_1}$ gives

$$\nabla_X bX - \langle \nabla_X Z, X\rangle(aX + bZ) - \nabla_Z(aX + bZ) + \langle \nabla_Z X, Z\rangle bX + \langle \nabla^\perp_Z \eta_1, \eta_2\rangle cX = 0.$$

Taking the Z-component yields

$$2b\langle \nabla_X X, Z\rangle - a\langle \nabla_Z X, Z\rangle - Z(b) = 0. \tag{27}$$

The Codazzi equation for A^g_N, that $\bar{c} = 0$ and $\bar{b} = b$ give

$$\nabla_X bX - \langle \nabla_X Z, X \rangle (\bar{a}X + bZ) - \nabla_Z (\bar{a}X + bZ) + \langle \nabla_Z X, Z \rangle bX = 0.$$

Taking the Z-component yields

$$2b\langle \nabla_X X, Z \rangle - \bar{a}\langle \nabla_Z X, Z \rangle - Z(b) = 0. \tag{28}$$

Subtracting (27) from (28), gives $(a - \bar{a})\langle \nabla_Z Z, X \rangle = 0$. If $\langle \nabla_Z Z, X \rangle = 0$, then f is ruled. Thus, we may assume that $a = \bar{a}$. Now taking the X-component in both Codazzi equations yields

$$X(b) - a\langle \nabla_X Z, X \rangle - Z(a) + 2b\langle \nabla_Z X, Z \rangle + c\langle \nabla^{\perp}_Z \eta_1, \eta_2 \rangle = 0$$

and

$$X(b) - a\langle \nabla_X Z, X \rangle - Z(a) + 2b\langle \nabla_Z X, Z \rangle = 0.$$

It follows from the last two equations that

$$\langle \nabla^{\perp}_Z \eta_1, \eta_2 \rangle = 0, \tag{29}$$

and we conclude from (5) that f is ruled.

We now prove the direct statement. In view of (2), we consider the tensor $A : TM \to TM$ where $\operatorname{Ker} A = \Delta$ and

$$A|_{\Delta^\perp} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}.$$

Since (29) holds by assumption, it is easy to see that the tensor A satisfies the Gauss and Codazzi equations as a hypersurface, and this concludes the proof.

$\square$

Corollary 24. *Let $f\colon M^n \to \mathbb{R}^N$ be a simply connected parabolic submanifold. Assume that there is $2 \leq k_0 \leq \tau^f - 1$ such that $\dim N^f_{k_0} = 1$. Then f is ruled and M^n admits an isometric immersion as a ruled hypersurface.*

Proof: We know from Proposition 19 that there exists a regular parabolic isometric immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+2k_0-1}$ such that $\xi_2^{k_0} = 0$. It follows from Theorem 17 that f is ruled. The result follows from Theorem 23.

$\square$

5 Nonruled parabolic submanifolds

In this section we study parabolic surfaces. First we show that they are associated to parabolic differential equations. Then we give a complete characterization of their s-cross sections.

Let L^2 be a Riemannian manifold endowed with a global system of coordinates. Then, let $f\colon L^2 \to \mathbb{Q}_\epsilon^N \subset \mathbb{R}^{N+\epsilon}$ where $\epsilon = 0, 1$ and $N \geq 4$, be a surface of the sphere or the Euclidean space whose coordinate functions are linearly independent solutions (of length 1 if $\epsilon = 1$) of the parabolic equation

$$\frac{\partial^2 u}{\partial z^2} + W(u) + \epsilon\lambda u = 0 \tag{30}$$

where $W \in TL$ and $\lambda \in C^\infty(L^2)$. If $\epsilon = 0$, then (30) is equivalent to

$$\tilde{\nabla}_Z f_* Z + f_* W = 0$$

where $Z = \partial/\partial z$. Thus $\alpha_f(Z, Z) = 0$. If $\epsilon = 1$, we have

$$\tilde{\nabla}_Z f_* Z + f_* W + \lambda f = 0$$

and again $\alpha_f(Z, Z) = 0$. In both situations f is parabolic with Z asymptotic.

Conversely, let $f\colon L^2 \to \mathbb{Q}_\epsilon^N$ be parabolic endowed with the induced metric and coordinates (x, z) such that $\partial/\partial z = Z$ is asymptotic. The latter means that the coordinate functions of f satisfy (30) with $W = -\nabla_Z Z$ and $\lambda = \|Z\|^2$.

Let $g\colon L^2 \to \mathbb{Q}_\epsilon^N$ be a parabolic surface and Σ the vector space of classes of functions $u \in C^\infty(L)$ that satisfy (30), where for $\epsilon = 0$ we identify two functions when they differ by a constant. Consider L^2 with the induced metric by g. Then (30) takes the form

$$\mathrm{Hess}_u(Z, Z) + \epsilon u = 0 \tag{31}$$

where $Z \in TL$ is an unit asymptotic field.

Given a parabolic submanifold $f : M^n \to \mathbb{Q}_\epsilon^N$, we denote

$$\tau_*^f = \begin{cases} \tau & \text{if } N-n \text{ is even} \\ \tau - 1 & \text{if } N-n \text{ is odd.} \end{cases}$$

Let Γ_r, $1 \leq r \leq \tau_*^g$, be the vector space of classes of r–cross sections of L^2 where we identify two sections when, up to a constant, they differ by a section of $N_{r+1}^g \oplus \ldots \oplus N_{\tau^g}^g$. Take $[\,h\,] \in \Gamma_r$ with $r < \tau_*^g$ and $1 \leq r < s \leq \tau_*^g$. Then, set $\mathcal{P}_r(h) = (\mu_1, \mu_2) \in \mathcal{V}_r$. By Corollary 10, there exists an unique section $\gamma_{r+1} \in N_{r+1}^g$ such that

$$\mathcal{P}_r(h) = \mathcal{P}_r(-\gamma_{r+1}).$$

Thus $\bar{h}_{r+1} = h + \gamma_{r+1}$ satisfies that $\bar{h}_{r+1} = h + \gamma_{r+1} \in \Gamma_{r+1}$. Using the above argument, it follows easily that there exist unique sections $\gamma_j \in N_j^g$, $r+1 \leq j \leq s$, such that

$$\bar{h} = h + \gamma_{r+1} + \ldots + \gamma_s \tag{32}$$

satisfies $[\,\bar{h}\,] \in \Gamma_s$.

We show next that all the Γ_r's are isomorphic to Σ. Given $[\,h\,] \in \Gamma_r$, set

$$h = \epsilon\varphi g + W + \delta$$

where $W \in TL$, $\delta \in T^\perp L$ and $\varphi \in C^\infty(L)$ if $\epsilon = 1$. Given $Y \in TL$, we have

$$h_*(Y) = \epsilon((Y(\varphi) - \langle Y, W\rangle)g + \varphi Y) + \nabla_Y W + \alpha_g(Y, W) - A_\delta^g(Y) + \nabla_Y^\perp \delta.$$

Since the TL-component of $h_*(Y)$ vanishes, we obtain

$$\epsilon\varphi Y + \nabla_Y W = A_\delta^g Y. \tag{33}$$

In particular, the map $(Y, U) \mapsto \langle \nabla_Y W, U\rangle$ is symmetric. Thus, if $\epsilon = 0$ and setting $\Theta(U) = \langle W, U\rangle$, we have $d\Theta(Y, U) = 0$. Thus $W = \nabla\varphi$, for $\varphi \in C^\infty(L^2)$. If $\epsilon = 1$, that the span$\{g\}$-component of $h_*(Y)$ vanishes gives $Y(\varphi) = \langle Y, W\rangle$, and again $W = \nabla\varphi$. In both cases, we obtain from (33) we that

$$\mathrm{Hess}_\varphi + \epsilon\varphi I = A_\delta^g. \tag{34}$$

Consider the linear map $\Upsilon\colon \Gamma_r \to \Sigma$ defined by $\Upsilon([h]) = [\varphi]$. Assume that $\Upsilon([h]) = 0$. Then $(h)_{T_gL} = \nabla\varphi = 0$. From (34) we obtain $A^g_\delta = 0$, which means $(h)_{N^g_1} = 0$. Using (iii) in Corollary 10 we obtain $h \in N^g_{r+1} \oplus \ldots \oplus N^g_{\tau^g}$. We conclude from the definition of Γ_r that Υ is injective.

Take $\varphi \in \Sigma$ and set

$$\mathcal{S} = \{\psi \in L_{sim}(TL, TL) : \langle \psi Z, Z\rangle = 0\}.$$

Let $\Phi\colon N^g_1 \to \mathcal{S}$ be the injective linear map defined by $\Phi(v) = A^g_v$. From (31) and $\dim N^g_1 = 2$, we have that Φ is an isomorphism. It follows that there exists a unique $\gamma_1 \in N^g_1$ such that

$$A^g_{\gamma_1} = \mathrm{Hess}_\varphi + \epsilon\varphi I.$$

We define $\hat{h} = \epsilon\varphi g + \nabla\varphi + \gamma_1$. Then,

$$\hat{h}_* X = \epsilon X(\varphi) g + \epsilon\varphi X + \tilde{\nabla}_X \nabla\varphi + \tilde{\nabla}_X \gamma_1 = \alpha_g(X, \nabla\varphi) + \nabla^\perp_X \gamma_1,$$

and thus $[\,\hat{h}\,] \in \Gamma_1$. We conclude from (32) that Υ is an isomorphism. In this way, we obtain the following recursive procedure for the construction of the r–cross sections for the parabolic surfaces.

Proposition 25. *Let $g\colon L^2 \to \mathbb{Q}^N_\epsilon$ be a regular parabolic surface. Then, any r–cross section, $1 \leq r \leq \tau^g_*$ can be written as*

$$h_\varphi = \epsilon\varphi g + g_*\nabla\varphi + \gamma_0 + \gamma_1 + \cdots + \gamma_r, \tag{35}$$

where φ satisfies (30) and is unique (up to a constant if $\epsilon = 0$), γ_0 is any section of $N^g_{r+1} \oplus \ldots \oplus N^g_{\tau^g}$, $\gamma_1 \in N^g_1$ is the unique solution of $A^g_{\gamma_1} = \mathrm{Hess}_\varphi + \epsilon\varphi I$ and γ_j , $2 \leq j \leq r$, are the unique sections given by (32). Conversely, any function h_φ with the form (35) is a r–cross section to g.

6 The parametrizations

In this section, we provide a parametrically description of all regular parabolic Euclidean submanifolds. There are two alternative representation, the polar and bipolar parametrizations, each of which is determined by a parabolic surface and a solution of a differential equation.

Our starting point, is to show how to construct parabolic submanifolds using parabolic surface with non vanishing normal vector ξ_2^τ, in particular, any nonruled parabolic surface.

Let g: $L^2 \to \mathbb{Q}_\epsilon^N$ a parabolic surface with $Z \in TL$ asymptotic and whose normal vector field $\xi_2^{\tau^g}$ does not vanish at any point. Let h be a s–cross section to g and $\Lambda_s = N_{s+1}^g \oplus \ldots \oplus N_{\tau^g}^g$ for $1 \leq s \leq \tau_*^g$. Let Ψ: $\Lambda_s \to \mathbb{R}^{N+\epsilon}$ be the map

$$\Psi(\delta) = h(x) + \delta$$

where $\delta \in \Lambda_s(x)$.

Proposition 26. *At regular points, $M^n = \Psi(\Lambda_s)$ is a regular parabolic submanifold. Moreover, M^n is nonruled if g is nonruled.*

For the proof we use the following general results.

Lemma 27. *Let f: $M^n \to \mathbb{R}^N$ be a parabolic submanifold. Then, we have:*

(i) If $\dim N_{k+1}^f = 2$, then there exists $\eta \in N_{k+1}^f$ such that the components of $\mathcal{P}_k(\eta)$ form a base of N_k^f.

(ii) Suppose that $N - n$ is odd, $\dim N_{\tau^f-1}^f = 2$ and that $\xi_2^{\tau^f}$ never vanishes. Then $\mathcal{P}_{\tau^f-1}(\xi_2^{\tau^f})$ is a base of $N_{\tau^f-1}^f$.

Proof: We prove (i). From Corollary 10 we have that $\mathcal{P}_k|_{N_{k+1}^f}$ is an isomorphism and from Lemma 5 that $\dim N_k^f = 2$. Since N_k^f has dimension 2, there exists at least one vector $(\mu_1, \mu_2) \in \mathcal{V}_k$ with $\mu_2 \neq 0$. Thus μ_1 and μ_2 are linearly independent and form a base of N_k^f.

For the proof of (ii) it is sufficient to show that $(\nabla_Z^\perp \xi_2^{\tau^f})_{N_{\tau^f-1}^f} \neq 0$. If the vector field vanishes, from the definition of $\mathcal{V}_{\tau^f-1}$ we have $\langle \nabla_X^\perp \xi_2^{\tau^f}, \xi_2^{\tau^f-1} \rangle = 0$. Thus $\xi_2^{\tau^f} = 0$ from Proposition 3, and this is a contradiction.

$\square$

Lemma 28. *Let β: $M^n \to \mathbb{R}^{N+\epsilon}$ a s–cross section to f, $1 \leq s \leq \tau^f$. Then,*

$$(\tilde{\nabla}_Z \beta_*(Z))_{N^f_{s-1}} = 0.$$

Proof: For $s \geq 2$, we have that $\langle \beta_*(Z), \xi_2^{s-1} \rangle = 0$. Then,

$$\begin{aligned} 0 &= Z\langle \beta_*(Z), \xi_2^{s-1} \rangle = \langle \tilde{\nabla}_Z \beta_*(Z), \xi_2^{s-1} \rangle + \langle \beta_*(Z), \alpha^{s+1}(Z, Z, X, \ldots, X) \rangle \\ &= \langle \tilde{\nabla}_Z \beta_*(Z), \xi_2^{s-1} \rangle. \end{aligned}$$

Using Lemma 7, is easy to prove by a similar argument that

$$\langle \tilde{\nabla}_Z \beta_*(Z), \xi_1^{s-1} \rangle = 0.$$

For $s = 1$, since $N^f_0 = \Delta^\perp$, $\xi_1^0 = X$ and $\xi_2^0 = Z$, the proof follows easily.

$\square$

We now prove Proposition 26.

Proof: Take a coordinate system (x, z) of L^2 such that $Z = \partial/\partial z$ is asymptotic and let $\{\eta_1, \ldots, \eta_k\}$ be an orthonormal frame of Λ_s. We parametrize M^n by

$$\Psi(x, z, t_1, \ldots, t_k) = h(x, z) + \sum_{j=1}^{k} t_j \eta_j(x, z)$$

where $k = N - 2s$ and $(t_1, \ldots, t_k) \in \mathbb{R}^k$. From Lemma 27, we have $TM = \Lambda_{s-1}$ and $\Delta_\delta = \Lambda_s$. We claim that $\Psi_*(Z)$ is asymptotic, that is, $\tilde{\nabla}_Z \Psi_*(Z) \in TM$. In view of (3) it is sufficient to show for $\upsilon \in N^g_{s-1}$ that $\langle \tilde{\nabla}_Z \Psi_*(Z), \upsilon \rangle = 0$. Let $X = \partial/\partial x \in TL$. We have that

$$\begin{aligned} \langle \tilde{\nabla}_Z \Psi_*(Z), \xi_1^{s-1} \rangle &= \langle \tilde{\nabla}_Z h_*(Z), \xi_1^{s-1} \rangle + \sum_{j=1}^{k} t_j \langle \tilde{\nabla}_Z \tilde{\nabla}_Z \eta_j, \alpha^s_\Psi(X, \ldots, X) \rangle \\ &= \langle \tilde{\nabla}_Z h_*(Z), \xi_1^{s-1} \rangle - \sum_{j=1}^{k} t_j \langle \eta_j, \tilde{\nabla}_Z \alpha^{s+1}_\Psi(Z, X, \ldots, X) \rangle \\ &= \langle \tilde{\nabla}_Z h_*(Z), \xi_1^{s-1} \rangle. \end{aligned}$$

By a similar argument, we obtain

$$\langle \tilde{\nabla}_Z \Psi_*(Z), \xi_2^{s-1} \rangle = \langle \tilde{\nabla}_Z h_*(Z), \xi_2^{s-1} \rangle.$$

Now Lemma 28 and $N^g_{s-1} = \operatorname{span}\{\xi_1^{s-1}, \xi_2^{s-1}\}$ give the claim. Observe that it follows from Lemma 27 that $N^\Psi_k = N^g_{s-k}$. This concludes the first part of the proof.

Assume that g is nonruled. From Lemma 7 we have that ξ_2^s and $\Psi_*(Z)$ are orthogonal. Being $\eta_s \in \Delta_\Psi^\perp = N_s^g$ a unit asymptotic vector field to Ψ, we obtain that Ψ is ruled if and only if $(\tilde\nabla_Z \eta_s)_{N_s^g} = 0$. Now the proof follows from Corollary 18.

$\square$

Our goal now is to show that any parabolic submanifolds with non vanishing normal vector field ξ_2^τ, in particular, all nonruled regular parabolic submanifolds, can be locally parametrized by a parabolic surface using Proposition 26.

Given a parabolic submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$, due to the local nature of our work, we may assume that f is the saturation of a fixed cross section $L^2 \subset M^n$ to the relative nullity foliation. From Proposition 4, each N_k^f can be viewed as a plane bundle along L^2.

Definition 29. Let $f\colon M^n \to \mathbb{Q}_\epsilon^{N-\epsilon}$ be a regular parabolic submanifold. A *polar surface* to f is an immersion of a cross section L^2 as above, defined as follows:

(i) If $N - n - \epsilon$ is odd, then $g\colon L^2 \to \mathbb{S}^{N-1}$ is defined by

$$\operatorname{span}\{g(x)\} = N^f_{\tau^f}(x).$$

(ii) If $N - n - \epsilon$ is even, then $g\colon L^2 \to \mathbb{R}^N$ is any surface such that

$$T_{g(x)}L = N^f_{\tau^f}(x),$$

up to parallel identification in $\mathbb{R}^n$.

Proposition 30. *Any regular parabolic submanifold $f\colon M^n \to \mathbb{Q}_\epsilon^N$ with non vanishing normal vector field $\xi_2^{\tau^f}$ admits a polar surface g locally. Moreover, g is parabolic and nonruled if f is nonruled and has no Euclidean factor.*

We will use the following fact.

Lemma 31. *Assume that f has even codimension. Let $\eta \in N^f_{\tau^f}$ and*

$$\mu_1 = (\tilde\nabla_X \eta)_{N^f_{\tau^f-1}}, \quad \mu_2 = (\tilde\nabla_Z \eta)_{N^f_{\tau^f-1}}$$

be such that $\mu_2 \neq 0$. Then,

$$\mathcal{V}_{\tau^f-1} = \{(a\mu_1 + b\mu_2, a\mu_2) : \ a, b \in C^\infty(M)\}.$$

Proof: Since $\langle (\tilde\nabla_Z \eta)_{N^f_{\tau^f-1}}, \xi_2^{\tau^f-1}\rangle = \langle \eta, \tilde\nabla_Z \xi_2^{\tau^f-1}\rangle = 0$, the definition of $\mathcal{V}_{\tau^f-1}$ and Lemma 3 yield $(\mu_2, 0) \in \mathcal{V}_{\tau^f-1}$. Since $\dim N^f_{\tau^f-1} = 2$, we easily conclude that $N^f_{\tau^f-1} = \operatorname{span}\{(\mu_1, \mu_2), (\mu_2, 0)\}$, and the proof follows.

$\square$

Remark 32. Notice that $\eta = \xi_2^{\tau^f}/\|\xi_2^{\tau^f}\| \in N^f_{\tau^f}$ satisfies $(\tilde\nabla_Z \eta)_{N^f_{\tau^f-1}} \neq 0$. In fact, from Proposition 3 it is easy to see that $\langle \tilde\nabla_Z \eta, \xi_1^{\tau^f-1}\rangle \neq 0$.

We now prove Proposition 30.

Proof: In the case of odd codimension, the existence of a polar surface follows from (ii) of Lemma 27. Assume that $\dim N^f_{\tau^f} = 2$. Let $\{\eta_1, \eta_2\}$ be a base of $N^f_{\tau^f}$ constant along Δ. We show that there exist linearly independent $1-$forms, θ_1, θ_2 so that the differential equation

$$dg = \theta_1 \eta_1 + \theta_2 \eta_2 \tag{36}$$

has solution.

Take a non vanishing asymptotic vector field $Z \in TM$ and consider the isomorphism P: $\Delta^\perp \to TL$. Let $U = P(Z) \in TL$ and (u, w) a coordinate system on L^2 such that $U = \partial/\partial u$. Set $W = \partial/\partial w \in TL$ and $X = P^{-1}(W) \in \Delta^\perp$. Endow L^2 with the metric which makes the base $\{U, W\}$ orthonormal and positively oriented. Let $\eta_1, \eta_2 \in N_{\tau^f}$ be linearly independents vector fields constant along Δ. Without loss of generality, we my assume $\mu_2 = (\tilde\nabla_Z \eta_1)_{N^f_{\tau^f-1}} \neq 0$. According to Lemma 31, there are $a, b \in C^\infty(M)$ with $b \neq 0$ such that

$$\mathcal{P}_{\tau^f-1}(\eta_1) = (\mu_1, \mu_2) \quad \text{and} \quad \mathcal{P}_{\tau^f-1}(\eta_2) = (a\mu_1 + b\mu_2, a\mu_2). \tag{37}$$

Consider 1-forms

$$\theta_1 = a^1 du + a^2 dw \quad \text{e} \quad \theta_2 = b^1 du + b^2 dw, \tag{38}$$

where $a^1, a^2, b^1, b^2 \in C^\infty(L^2)$. We show that we can choose $a^1, a^2, b^1, b^2 \in C^\infty(L)$ such that (36) has solution . The integrability condition for (36) is

$$\begin{aligned} 0 &= d\theta_1 \eta_1 + d\theta_2 \eta_2 + \theta_1 \wedge d\eta_1 + \theta_2 \wedge d\eta_2 \\ &= d\theta_1 \eta_1 + d\theta_2 \eta_2 + (a^1 \frac{\partial \eta_1}{\partial w} - a^2 \frac{\partial \eta_1}{\partial du}) dV + (b^1 \frac{\partial \eta_2}{\partial v} - b^2 \frac{\partial \eta_2}{\partial u}) dV \\ &= d\theta_1 \eta_1 + d\theta_2 \eta_2 + (\tilde{\nabla}_{a^1 W - a^2 U}\, \eta_1 + \tilde{\nabla}_{b^1 W - b^2 U}\, \eta_2) dV \end{aligned}$$

where dV stands for the volume element of L^2. Then, we must have

$$(\tilde{\nabla}_{a^1 W - a^2 U}\, \eta_1 + \tilde{\nabla}_{b^1 W - b^2 U}\, \eta_2)_{N_{\tau f-1}} = 0.$$

From (37) we may rewrite the above equation as

$$\begin{cases} a^1 + ab^1 = 0 \\ a^2 - bb^1 + ab^2 = 0. \end{cases} \tag{39}$$

Then, let $e, \ell \in C^\infty(L)$ be such that

$$\tilde{\nabla}_{a^1 W - a^2 U}\, \eta_1 + \tilde{\nabla}_{b^1 W - b^2 U}\, \eta_2 = e\eta_1 + \ell \eta_2.$$

We claim that there exist $a^1, a^2, b^1, b^2 \in C^\infty(L)$ such that θ_1, θ_2 satisfy(39) and

$$\begin{cases} d\theta_1 &= e\, dV \\ d\theta_2 &= \ell\, dV, \end{cases}$$

or equivalently,

$$\begin{cases} a^2_u - a^1_w &= e \\ b^2_u - b^1_w &= \ell. \end{cases} \tag{40}$$

From (39) and (40) we have

$$\begin{cases} a^1 = -ab^1 \\ a^2 = bb^1 - ab^2 \\ b_u b^1 + bb^1_u - a_u b^2 - a(b^2_u - b^1_w) + a_w b^1 = e \\ b^2_u - b^1_w = \ell. \end{cases}$$

The two last equations give

$$\begin{cases} b_u b^1 + b b_u^1 - a_u b^2 + a_w b^1 = e + a\ell \\ b_u^2 - b_w^1 = \ell. \end{cases} \tag{41}$$

We assume $a_u \neq 0$ without loss of generality. The first equation of (41) yields

$$b^2 = -\frac{1}{a_u}(e + a\ell - (b_u + a_w)b^1 + b b_u^1).$$

We take b^1 to be a solutions of the above linear parabolic equation (see p. 367 of [10]), and now the claim follows easily.

If f has a Euclidean factor, take T a parallel subbundle of the relative nullity subbundle of f. It is easy to see that under these conditions the subbundle $T \oplus \nabla^{\perp} \oplus N_1^g$ is a normal parallel subbundle of g. Thus, the codimension of g can be reduced. The converse is similar.

We claim that g has an asymptotic vector. First observe that $N_1^g = N^f_{\tau_*^f - 1}$. Thus, in odd codimension, we have from (36) and (39) that

$$g_* \partial/\partial u = a^1 \eta_1 + b^1 \eta_2 = -ab^1 \eta_1 + b^1 \eta_2. \tag{42}$$

Therefore, in view of (37) we obtain

$$(\tilde{\nabla}_Z g_* \partial/\partial u)_{N^f_{\tau^f - 1}} = -ab^1 \mu_2 + ab^1 \mu_2 = 0.$$

For even codimension, the claim follow from Lemma 28. Hence g is parabolic.

To complete the proof suppose that f is nonruled. We show that g is also nonruled. If the codimension of f is odd, since $\xi_2^{\tau^f} \neq 0$, then TL is spanned by $\{(\tilde{\nabla}_X \xi_2^{\tau^f})_{N^f_{\tau_*^f}}, (\tilde{\nabla}_Z \xi_2^{\tau^f})_{N^f_{\tau_*^f}}\}$, being $(\tilde{\nabla}_Z \xi_2^{\tau^f})_{N^f_{\tau_*^f}}$ an asymptotic field.

The definition of $\mathcal{V}_{\tau_*^f}$ allows us to conclude that the unit asymptotic field γ is normal to $\xi_2^{\tau_*^f}$ Then, g is ruled if and only if $(\tilde{\nabla}_Z \gamma)_{N^f_{\tau_*^f}} = 0$. Thus g is nonruled by Corollary 18. In the even codimension case, we have

$$N_1^g = \text{span}\{(\tilde{\nabla}_Z \eta_1)_{N^f_{\tau^f - 1}}, (\tilde{\nabla}_X \eta_1)_{N^f_{\tau^f - 1}}\}.$$

From (37) and (42) it is easy to conclude that

$$\xi_2^{1\,g} = b\mu_2 = b(\tilde{\nabla}_Z \eta_1)_{N_{\tau^f - 1}}. \tag{43}$$

Let $\lambda = \|b(\tilde\nabla_Z\eta_1)_{N_{\tau^f-1}}\|^{-1}$. It follows from (5) that g is ruled if and only if

$$(\tilde\nabla_U\lambda(\tilde\nabla_Z\eta_1)_{N_{\tau^f-1}})_{N^f_{\tau^f-1}} = 0.$$

From our assumption that η_1 is constant along Δ_f, it follows that

$$0 = (\tilde\nabla_U\lambda(\tilde\nabla_Z\eta_1)_{N^f_{\tau^f-1}})_{N^f_{\tau^f-1}} = U(\lambda)(\tilde\nabla_Z\eta_1)_{N^f_{\tau^f-1}} + \lambda(\tilde\nabla_Z(\tilde\nabla_Z\eta_1)_{N^f_{\tau^f-1}})_{N^f_{\tau^f-1}}.$$

Thus,

$$(\tilde\nabla_Z(\tilde\nabla_Z\eta_1)^f_{N_{\tau^f-1}})_{N^f_{\tau^f-1}} \in (\tilde\nabla_Z\eta_1)_{N^f_{\tau^f-1}}.$$

Since $(\tilde\nabla_Z\eta_1)_{N_{\tau^f-1}}$ is normal to $\xi_2^{\tau^f-1\,f}$, we obtain

$$(\tilde\nabla_Z\,\xi_2^{\tau^f-1\,f}/\|\xi_2^{\tau^f-1\,f}\|)_{N^f_{\tau^f-1}} = 0,$$

and conclude from Corollary 18 that f is ruled. This is a contradiction.

$\square$

The following is the *polar parametrization.*

Theorem 33. *Given a parabolic surface $g\colon L^2 \to \mathbb{Q}^N_\epsilon$ with non vanishing normal vector $\xi_2^{\tau^g}$ and $1 \le s \le \tau^g_*$, consider the smooth map $\Psi\colon \Lambda_s \to \mathbb{R}^N$ defined by*

$$\Psi(\delta) = h(x) + \delta \tag{44}$$

where $\delta \in \Lambda_s = N^g_{s+1} \oplus \ldots \oplus N^g_{\tau^g}$ and h is any s–cross section to g. Then, at regular points, $M^n = \Psi(\Lambda_s)$ is a regular parabolic submanifold with polar surface g. Moreover, if g is nonruled, then $M^n = \Psi(\Lambda_s)$ is nonruled.

Conversely, any parabolic submanifold $f\colon M^n \to \mathbb{R}^N$ without local Euclidean factor and with non vanishing normal vector $\xi_2^{\tau^f}$ admits a local parametrization (44), where g is a polar surface to f.

Proof: The direct statement follows from Proposition 26. For the converse, take a polar surface $g\colon L^2 \to \mathbb{Q}^N_\epsilon$ to f. It is easy to see that under these conditions that $\Delta_f = \Lambda_{\tau^f_*}$ and $TM = \Lambda_{\tau^f_*-1}$ along L^2. Thus, the section $h = f_{|_{L^2}}$ is a τ^f_*–cross section to g. $\square$

Observe that picking a different γ_0 in (35) only results in a reparametrization of $\Psi(\Lambda_s)$. Hence, it is convenient to take $\gamma_0 = 0$ when using the recursive procedure to generate s–cross sections.

The polar parametrization is very effective for submanifolds in low codimension since the recursive procedure has few iterations. For instance, in codimension two it suffices to take a 1–cross section of the form $h_\varphi = \nabla\varphi + \gamma_1$, where $\gamma_1 \in N_1^f$ is unique satisfying $A_{\gamma_1} = \mathrm{Hess}_\varphi$ for a given solution φ of (30).

Definition 34. We define the *bipolar surface* to a parabolic submanifold f to be any polar surface to a polar surface to f.

Proposition 35. *Any nonruled parabolic submanifolds admits locally a bipolar surface.*

Proof: From Proposition 30, f admits locally a nonruled polar surface g. Then, Proposition 17 gives $\xi_2^{\tau^g} \neq 0$. The proof now follows from Proposition 30

$\square$

Definition 36. Let $g\colon L^2 \to \mathbb{Q}_\epsilon^N$ be a parabolic surface and $0 \leq s \leq \tau_*^g - 1$. We call *dual s–cross section* to g any element $h \in C^\infty(L^2, \mathbb{R}^{N+\epsilon})$ satisfying

$$h_*(TL) \subset \epsilon\,\mathrm{span}\{g\} \oplus N_0^g \oplus \ldots \oplus N_s^g$$

at any point.

Notice that a dual 0-section to a parabolic surface in Euclidean space is just a bipolar surface.

Proposition 37. *Let $g\colon L^2 \to \mathbb{Q}_\epsilon^N$ be a regular parabolic surface with polar surface $\hat{g}$. Any dual s-section to g is a $([N/2] - s - 1)$-section to $\hat{g}$.*

Proof: We have $\tau^g_* = \tau^{\hat g}_* = [N/2]-1$ and $N^g_s = N^{\hat g}_{\tau^{\hat g}_*-s}$. The proof follows easily.

□

The following is the *bipolar parametrization.*

Theorem 38. *Given a parabolic surface $g\colon L^2 \to \mathbb{Q}^N_\epsilon$ with non vanishing normal vector $\xi_2^{\tau^g}$ and $0 \le s \le \tau^g_* - 1$, consider the smooth map $\tilde\Psi\colon \tilde\Lambda_s \to \mathbb{R}^N$ defined by*

$$\tilde\Psi(\tilde\delta) = \tilde h(x) + \tilde\delta \tag{45}$$

where $\tilde\delta \in \tilde\Lambda_s = \epsilon\, span\{g\} \oplus N^g_0 \oplus \ldots \oplus N^g_{s-1}$ and $\tilde h$ is any dual s–cross section to g. Then, at regular points, $M^n = \tilde\Psi(\tilde\Lambda_s)$ is a nonruled parabolic submanifold with bipolar surface g.

Conversely, any nonruled parabolic submanifold $f\colon M^n \to \mathbb{R}^N$ without local Euclidean factor admits a local parametrization (45), where g is a bipolar surface to f.

Proof: The result follows from Theorem 33 and Propositions 35 and 37.

□

Next, we give a simple way to parametrize parabolic submanifolds.

Let $g\colon L^2 \to \mathbb{Q}^N_\epsilon$ be a simply connected nonruled parabolic surface endowed with the metric induced by g and $\{X, Z\}$ an orthonormal tangent frame with Z asymptotic. Let $J \in End\,(TL)$ be defined by

$$J(X) = Z \;\text{ and }\; J(Z) = 0$$

and let $R \in End\,(TL)$ the reflection defined by

$$R(X) = X \;\text{ and }\; R(Z) = -Z.$$

Now consider the linear second order parabolic operator

$$L(\varphi) = ZZ(\varphi) + \Gamma_2 X(\varphi) - \Gamma_1 Z(\varphi) + (X(\Gamma_2) - Z(\Gamma_1) + (\Gamma_1)^2 - (\Gamma_2)^2 - \epsilon)\varphi$$

where $Y = [X, Z] = \Gamma_2 Z - \Gamma_1 X$. Let $\varphi \in C^\infty(L)$ satisfy $L(\varphi) = 0$ and let ψ be the 1-form such that $d\psi(X, Z) = -\varphi$.

Lemma 39. *The differential equation*

$$d\theta = d\varphi \circ J + \varphi Y^* \circ R + \epsilon\psi \tag{46}$$

is integrable.

Proof: From our assumptions, we easily obtain $d^2\theta(X,Z) = -L(\varphi)$, and this concludes the proof.

□

Lemma 40. *The differential equation*

$$dh = \epsilon\psi g + dg \circ (\theta I + \varphi J) \tag{47}$$

is integrable, where θ is a solution of (46).

Proof: An easy computation yields

$$\begin{aligned} d^2h(X,Z) &= \epsilon(d\psi(X,Z) + \varphi)g + (d\theta(X) + \varphi\Gamma_1 - Z(\varphi) - \epsilon\psi(X))Z \\ &- (d\theta(Z) + \varphi\Gamma_2 - \epsilon\psi(Z))X. \end{aligned}$$

Thus, we conclude that $d^2h = 0$.

□

Theorem 41. *Let g: $L^2 \to \mathbb{Q}_\epsilon^{N-\epsilon}$ a simply connected nonruled parabolic surface, $\varphi \in C^\infty(L)$ so that $L(\varphi) = 0$ and h: $L^2 \to \mathbb{R}^N$ a solution of (47). Then, the map Ψ: $L^2 \times \mathbb{R}^{2s-\epsilon} \to \mathbb{R}^N$ defined by,*

$$\Psi(x,t) = h(x) + \epsilon\, t_0 g(x) + \sum_{j=1}^{s} \left(t_{2j-1} \frac{\partial^j g}{\partial v \partial u^{j-1}} + t_{2j} \frac{\partial^j g}{\partial u^j} \right)(x)$$

where $0 \le s \le [(N-\epsilon)/2] - 2$ and (u,v) is a coordinate system of L^2 such that $\partial/\partial v$ is asymptotic, parametrizes, at regular points, a parabolic submanifold.

Conversely, any nonruled parabolic submanifold without local Euclidean factor can be locally parametrized in this way.

Proof: It is clear for $0 \le j \le \tau_*^g$ that

$$N_j^g = \operatorname{span}\left\{ \left(\frac{\partial^{j+1} g}{\partial u^j \partial v}\right)_{N_j^g}, \left(\frac{\partial^{j+1} g}{\partial u^{j+1}}\right)_{N_j^g} \right\}.$$

In (45) we take $\tilde{h}$ to be a dual 0–cross section to g without loss of generality. It remains to show that any dual 0-section to g can be written as a solution of (47).

Given a dual 0-section $\tilde{h}$ to g, we need a 1-form Ψ and $S \in End(TL)$ such that

$$d\tilde{h} = \epsilon \Psi g + dg \circ S.$$

An easy computation yields

$$\begin{aligned} d^2\tilde{h}(X,Z) &= \epsilon(d\psi(X,Z) - \langle X, SZ\rangle + \langle Z, SX\rangle)g + (\nabla_X S)Z + \alpha_g(X, SZ) \\ &\quad -(\nabla_Z S)X - \alpha_g(Z, SX) + \epsilon(\psi(Z)X - \psi(X)Z). \end{aligned}$$

Thus, the integrability conditions reduces to the equations

$$\alpha_g(X, SZ) = \alpha_g(Z, SX), \tag{48}$$

$$(\nabla_X S)Z - (\nabla_Z S)X = \epsilon(\psi(X)Z - \psi(Z)X), \tag{49}$$

and for $\epsilon = 1$ the additional equation

$$d\psi(X,Z) = \langle SZ, X\rangle - \langle SX, Z\rangle. \tag{50}$$

From (48) and since $\alpha_g(X,X)$ and $\alpha_g(X,Z)$ are linearly independent, we have

$$S = \theta I + \varphi J$$

where $\theta, \varphi \in C^\infty(L)$. The left side of (49) gives us

$$\nabla_X \theta Z - \nabla_Z(\theta X + \varphi Z) + \Gamma_1 SX - \Gamma_2 SZ = (d\theta(X) + \varphi\Gamma_1 - d\varphi(Z))Z - (d\theta(Z) + \varphi\Gamma_2)X.$$

Thus (49) is equivalent to

$$\left\{ \begin{array}{rcl} d\theta(X) &=& -\Gamma_1\varphi + d\varphi(Z) + \epsilon\psi(X) \\ d\theta(Z) &=& \langle Y, -Z\rangle\varphi + \epsilon\psi(Z). \end{array} \right.$$

Hence,

$$d\theta = d\varphi \circ J + \varphi Y^* \circ R + \epsilon\psi,$$

and from (50) we easily get $d\psi(X, Z) = -\varphi$. The result follows from Theorem 38 and Lemma 40.

□

7 The singularities

In this section we show that the nowhere nonruled complete parabolic submanifolds are surface-like, that is, they are isometric to $L^2 \times \mathbb{R}^{n-2}$. We also describe the singular set of nonruled parabolic submanifolds of dimension at least four.

The complete submanifolds $f\colon M^n \to \mathbb{R}^N$ with rank $\rho \leq 2$, had been studied in [7]. If M^n does not contain an open set $L^3 \times \mathbb{R}^{n-3}$ with L^3 unbounded, then the following holds in the open set $M^* \subset M^n$ where $\rho = 2$.

(i) M^* is an union of smoothly ruled strips.

(ii) If f is completely ruled on M^*, then it is completely ruled everywhere and a cylinder on each component of the complement of the closure of M^*.

A ruled submanifold is called *completely ruled* if each leaf is a complete affine space. The leaves in each connected component of M^n, called a *ruled strip*, form an affine vector bundle over a curve with or without end point [7].

Given a ruled parabolic submanifold $f\colon M^n \to \mathbb{R}^N$, let $\tilde{M}^n$ be the extension of $f(M^n)$ (with possible singularities) obtained by extending each leaf to a complete affine Euclidean space $\mathbb{R}^{n-1}$. We have the following result.

Proposition 42. *Let $f\colon M^n \to \mathbb{R}^N$ a ruled parabolic submanifold. Then $\tilde{M}^n$ is a ruled strip. Moreover, if c is complete and the function a_1 defined in (18) satisfy $|a_1(s)| \leq K < +\infty$, then $\tilde{M}^n$ is complete.*

Proof: Using (23) we parametrize M^n by

$$f(s,t_1,\ldots,t_{n-1}) = c(s) + \sum_{j\geq 1} t_j e_j(s) w$$

where

$$\frac{de_1}{ds} = a_1 e_0 + \delta + \eta \quad \text{and} \quad \frac{de_j}{ds} = b_j e_1, \ \ 2 \leq j \leq n-1,$$

$\delta = (de_1/ds)_\Delta$ and $\eta \perp \operatorname{span}\{e_0, e_1\} \oplus \Delta$ is nonsingular for every $s \in I$. We have,

$$TM = \operatorname{span}\{(1+t_1 a_1)e_0 + t_1\eta\} \oplus \operatorname{span}\{e_1,\ldots,e_{n-1}\},$$

and is now easy to conclude that f is nonsingular. Thus $\tilde{M}^n$ is a ruled strip.

Next, suppose that c is complete. Notice that

$$\|f_s\|^2 \geq (1+t_1 a_1(s))^2 + t_1^2 \|\eta(s)\|^2.$$

We claim that $\tilde{M}^n$ is complete. If $|t_1| \leq M < \infty$, from our assumption that $|a_1(s)| \leq K < \infty$ we obtain $\|f_s\|^2 \geq L > 0$. On the other hand, it is easy to see that any divergent curve $\gamma(u) = f(s(u), t_1(u), ..., t_{n-1}(u)), u \in [0,+\infty)$, in $\tilde{M}^n$ with at least one $t_i, 1 \leq i \leq n-1$, unbounded has infinity length. Thus, any divergent curves in $\tilde{M}^n$ has infinity length, and the proof follows.

□

Observe that any ruled parabolic submanifold parametrized by (23) with $b_j = 0,\ 2 \leq j \leq n-1$, everywhere is a product $L^2 \times \mathbb{R}^{n-2}$. On the other hand, if there exist $j \in \{2,\ldots,n-1\}$ such that $b_j \neq 0$ everywhere then the submanifold does not contain an open set $L^2 \times \mathbb{R}^{n-2}$.

Theorem 43. *Let $f\colon M^n \to \mathbb{R}^N$, $n \geq 3$, be a complete submanifold which is nonruled in any open set and parabolic in an open dense set $\mathcal{O}$. Then, any connected component of $\mathcal{O}$ is isometric to $L^2 \times \mathbb{R}^{n-2}$ and f splits accordingly.*

Proof: From Lemma 6 in [7] it is easy to see that either $C = 0$ or

$$C_T = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} \tag{51}$$

where $T \perp \operatorname{Ker} C$. We have a disjoint decomposition $\mathcal{O} = M_0 \cup M_1$, where M_0 is the closet set where $C = 0$. We now argue that the open set M_1 is empty. It follows from Lemma 1.8 in [7] that M_0 and M_1 are saturated, i.e. they are unions of complete leaves of Δ. We have from Lemma 1.5 in [7] and (51) that

$$0 = (\nabla_X C_T)Z - (\nabla_Z C_T)X = n\langle \nabla_X Z, X\rangle Z - Z(n)Z - n\langle \nabla_Z Z, X\rangle X$$

where $T \perp \ker C$ is an unit field. Therefore $\langle \nabla_Z Z, X\rangle = 0$, i.e., M_1 is ruled. We conclude that $M_1 = \emptyset$ and the result follows from Lemma 1.1 in [7].

$\square$

Observe that if $f\colon M^n \to \mathbb{R}^N$ is a complete, simply connected parabolic submanifold, then M^n is diffeomorphic to $\mathbb{R}^n$ since its sectional curvature satisfies $K_M \leq 0$. In the ruled case, we have from Theorem 23 that M^n admits an isometric immersion as a ruled hypersurface with the same rulings. There are many examples of complete ruled hypersurfaces [7]. A simple example goes as follows: take $c\colon I \subset \mathbb{R} \to \mathbb{R}^{n+1}$ any unit speed curve, and let $E_0 = dc/ds, E_1, \ldots, E_n$ a Frenet frame. It is easy to see that the hypersurface

$$(s, t_1, \ldots, t_{n-1}) \mapsto c(s) + \sum_{j=1}^{n-1} t_j E_{j+1}$$

is complete.

Given a nonruled parabolic submanifold $f\colon M^n \to \mathbb{R}^N$ without Euclidean factor, let $\tilde{M}^n$ be the extension of $f(M^n)$ in $\mathbb{R}^N$ obtained by extending each leaf of relative nullity of f to a complete affine Euclidean space in $\mathbb{R}^{n-2}$. Our next and last result, describes the singular set of nonruled parabolic submanifolds without Euclidean factor and dimension $n \geq 4$.

Proposition 44. *Let $f\colon M^n \to \mathbb{R}^N$, $n \geq 4$, be a nonruled parabolic submanifold without Euclidean factor. Then the hypersurface given by*

$$\{\lambda \in \tilde{M}^n : \langle \lambda, \xi_2^{s+1}\rangle = 0\}$$

is the singular set of $\tilde{M}$.

Proof: Let $\Psi(\delta) = h(x)+\delta$, $\delta \in \Lambda_s(x)$, be the parametrization in Theorem 33, where h is any s–cross section of a polar surface g to f. Without loss of generality, we assume that h is a τ_*^g-section. Being (x,z) a coordinate system of g with $Z = \partial/\partial z$ asymptotic and $\{\eta_1, \dots, \eta_k\}$ an orthonormal frame of Λ_s, we can write

$$\Psi(x, z, t_1, \dots, t_k) = h(x, z) + \sum_{j=1}^{k} t_j \eta_j(x, z)$$

where $k = N - 2s$ and $(t_1, \dots, t_k) \in \mathbb{R}^k$. Recall that $TM = \Lambda_{s-1}$ and $\Delta = \Lambda_s$. Thus, with $X = \partial/\partial x$, we have that $\Psi(x, z, t_1, \dots, t_k)$ is a singular point if and only if

$$t_1(\nabla_X^\perp \eta_1)_{N_s} + t_2(\nabla_X^\perp \eta_2)_{N_s} \quad \text{and} \quad t_1(\nabla_Z^\perp \eta_1)_{N_s} + t_2(\nabla_Z^\perp \eta_2)_{N_s}$$

are linearly independents. By the definition of $\mathcal{V}_s$, we have

$$\langle \nabla_Z^\perp \eta_1, \xi_2^s \rangle = \langle \nabla_Z^\perp \eta_2, \xi_2^s \rangle = 0.$$

Thus $t_1(\nabla_Z^\perp \eta_1)_{N_s} + t_2(\nabla_Z^\perp \eta_2)_{N_s}$ and ξ_2^s are normal fields. The above condition is now equivalent to

$$\langle t_1(\nabla_X^\perp \eta_1)_{N_s} + t_2(\nabla_X^\perp \eta_2)_{N_s}, \xi_2^s \rangle = 0$$

and, from Proposition 3, equivalent to

$$\langle t_1\eta_1 + t_2\eta_2, \xi_2^{s+1} \rangle = 0.$$

It follows that $\lambda \in \tilde{M}^n$ is a singular point if and only if $\langle \lambda, \xi_2^{s+1} \rangle = 0$.

□

References

[1] Barbosa, J. L., Dajczer, M. and Jorge, L., *Minimal Ruled Submanifolds in Spaces of Constant Curvature*. Indiana Math. J. 33 (1984), 531–547.

[2] Boeckx, E., Kowalski, O. and Vanhecke, L., Riemannian Manifolds of conullity two, World Scientic, Singapore, 1996.

[3] Dajczer, M. et al., Submanifolds and isometric immersions, Mathematics. Lecture Ser. 13, Publish or Perish, Houston, 1990.

[4] Dajczer, M., Florit, L. and Tojeiro, R., *On deformable hypersurfaces in space forms*. Ann. Mat. Pura Appl. 147 (1998) 361–390.

[5] Dajczer, M. and Florit, L., *A Class of austere submanifolds*. Illinois J. Math. 45 (2001) 735–755.

[6] Dajczer, M. and Florit, L., *Genuine rigidity of Euclidean submanifolds in codimension two*. Geom. Dedicata 106 (2004) 195–210.

[7] Dajczer, M. and Gromoll, D., *Rigidity of complete Euclidean hypersurfaces*. J. Diff. Geometry 31 (1990) 401–416.

[8] Dajczer, M. and Morais, P., *Isometric rigidity in codimension two*. To appear in Mich. Math. J.

[9] Dajczer, M. and Tojeiro, R., *Submanifolds with nonparallel first normal bundle*. Canad. Math. Bull. 37 (1994), 330–337.

[10] Evans, L., Partial differential equations, Graduate Studies in Mathematics 19, A.M.S.

[11] Moore, J. D., *Submanifolds of Constant positive curvature I*. Duke Math. J. 44 (1977) 449–484.

IMPA – Estrada Dona Castorina, 110
22460-320 – Rio de Janeiro – Brazil
E-mail: marcos@impa.br

Universidade da Beira Interior
6201-001 – Covilhã – Portugal
E-mail: pmorais@mat.ubi.pt