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A SURVEY ON ALEXANDROV-BERNSTEIN-HOPF THEOREMS

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

We give proofs of Alexandrov, Bernstein and Hopf Theorems. Then, we discuss the developments of the theory of constant mean curvature surfaces ensuing from them.

1 Introduction

The followings are very impressive theorems in the theory of constant mean curvature hypersurfaces in Euclidean space.

Alexandrov Theorem. *[1] A compact constant mean curvature hypersurface embedded in $\mathbb{R}^{n+1}$ is a round sphere.*

Bernstein Theorem. *[2], [6], [21], [31], [53], [56] A minimal hypersurface in $\mathbb{R}^{n+1}$, $n < 7$, which is a complete graph over a hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is a hyperplane.*

Do Carmo and Lawson in [23] emphasized that Alexandrov's and Bernstein's Theorems together give a characterization of the (complete) totally umbilical hypersurfaces embedded in $\mathbb{R}^{n+1}$, among those of constant mean curvature (we recall that, at any point of a totally umbilical hypersurface, all the principal curvatures are equal). Actually, the characterization of totally umbilical hypersurfaces with constant mean curvature was known previously: all the principal curvatures of an umbilical hypersurface with constant mean curvature are equal to a constant, in particular the hypersurface is isoparametric. Isoparametric hypersurfaces have been classified in [41], [55], [13] (for the case of $\mathbb{R}^3$, $\mathbb{R}^n$ with $n > 3$, space forms, respectively).

This unified view over Alexandrov's and Bernstein's Theorems, leads Do Carmo and Lawson to prove some results in hyperbolic space $\mathbb{H}^{n+1}$. Among their results, the following natural generalization of the Alexandrov Theorem sticks out.

Theorem 1.1 (Do Carmo-Lawson). *[23] Let S be a complete properly embedded hypersurface in $\mathbb{H}^{n+1}$, with constant mean curvature and exactly one point in the asymptotic boundary. Then S is a horosphere.*

The asymptotic boundary will be defined in Section 2.

Later on, this kind of results were pursued by many authors and this subject is still very fruitful. The aim of this survey is to describe results in the theory of constant mean curvature surfaces from this point of view.

In order to generalize the Alexandrov Theorem to ambient spaces different from $\mathbb{R}^{n+1}$, it is worth to make some remarks about rotationally invariant spheres, geodesic spheres and totally umbilical hypersurfaces. In space forms, rotationally invariant constant mean curvature hypersurfaces are totally umbilical. This is not the case in a general homogeneous manifolds, as for example the simply connected homogeneous 3-manifolds with isometry group of dimension four: $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Heisenberg group Nil_3, the Berger spheres and $\widetilde{PSL_2}(\mathbb{R})$, and with isometry group of dimension three: Sol_3.

Moreover, while in space forms, geodesic spheres have constant mean curvature, this is not the case in a general homogeneous manifold. Hence, in order to obtain Alexandrov type result in homogeneous simply connected 3-manifold, one has to understand first, which may be the surface for which one is looking for uniqueness. As we will see later, when the isometry group of the homogeneous simply connected 3-manifold has dimension four, the desired surface is a rotationally invariant sphere with the same mean curvature, while in Sol_3, where no rotation is avaiable, it is a deformation of the solution of the isoperimetric problem (we will be more precise in Section 4).

It is worth to introduce the Hopf Theorem as an aspect of the discussion.

Hopf Theorem. *[34] A constant mean curvature sphere immersed in $\mathbb{R}^3$ is a round sphere.*

Hopf Theorem can be generalized immediately to $\mathbb{H}^3$ and $\mathbb{S}^3$ with the same proof as in [34]. Furthermore Hopf Theorem still holds for surfaces in space forms of dimension higher than three, provided the mean curvature vector is parallel [65].

In general, a Hopf type theorem is an uniqueness theorem about constant mean curvature immersed topological spheres. In simply connected homogeneous manifold with isometry group of dimension four, the surface that will be unique by a Hopf type theorem is a rotationally invariant sphere. In the case of Sol_3, the surface that will be unique by a Hopf type theorem is a deformation of the solution of the isoperimetric problem (as in the case of Alexandrov type theorem).

The paper is organized as follows. Each section is devoted to one of the Theorems by Alexandrov, Bernstein and Hopf: the section contains a proof of the corresponding theorem and a discussion about its generalizations.

As a general reference for surfaces theory we suggest [24], while for basic notions about the theory of submanifolds we suggest [25].

2 Alexandrov Problem

Alexandrov Theorem. *[1] A compact constant mean curvature hypersurface S embedded in $\mathbb{R}^{n+1}$ is a round sphere.*

We learned the proof of Alexandrov's result in [34], where Hopf proves the Alexandrov Theorem for the case $n = 3$ and S of class C^3. We report a simplified version of that proof. Hopf himself wrote:

...It is my opinion that this proof by Alexandrov, and especially the geometric part, opens important new aspects in differential geometry in the large...

The following developments of the theory of constant curvature surfaces show that Hopf's sentence was very far-sighted. In fact, the geometric part of the Alexandrov's proof is what is now known as Alexandrov reflection method or technique of moving planes and it is a very powerful tool.

We use the following characterization of the sphere in $\mathbb{R}^3$.

Lemma 2.1 *A compact embedded surface in $\mathbb{R}^3$ that has a plane of symmetry in every direction, is a round sphere* (Lemma 2.2. Chapter VII in [34]).

Then we need some results from PDE's theory. Let p be a point of a surface S of constant mean curvature H. Locally around the point p, one can write the surface S as a graph of a C^2 function $u(x,y)$ over the tangent plane to S at p. Then, the function u satisfies

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} - 2H(1+u_x^2+u_y^2)^{\frac{3}{2}} = 0 \qquad (1)$$

Denote by $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{yy}$, $t = u_{xy}$, then equation (1) writes as

$$\Phi(p,q,r,s,t) = (1+q^2)r - 2pqs + (1+p^2)t - 2H(1+p^2+q^2)^{\frac{3}{2}} = 0$$

As the quadratic form $\Delta = \Phi_r\lambda^2 + \Phi_s\lambda\mu + \Phi_t\mu^2 = \lambda^2 + \mu^2 + (q\lambda - p\mu)^2$ is positive definite, $\Phi = 0$ is an elliptic partial differential equation of second order. Then, we are able to prove the following result.

Theorem 2.1 *Let $\Phi = 0$ be a partial differential equation of second order elliptic in a neighborhood of $(0,0)$. Let u_1 and u_2 be two C^2 solutions of $\Phi = 0$ such that at $(0,0)$*

$$u_1(0,0) = u_2(0,0), \quad p_1(0,0) = p_2(0,0), \quad q_1(0,0) = q_2(0,0) \qquad (2)$$

but $u_1 \neq u_2$ in any neighborhood of $(0,0)$. Then, $w = u_2 - u_1$ does not have a sign in any neighborhood of $(0,0)$.

The geometric version of the previous theorem is known as the Maximum Principle and it claims as follows.

Maximum Principle. *Let S_1 and S_2 be two surfaces with the same constant mean curvature H, that are tangent at a point $p \in int(S_1) \cap int(S_2)$. Assume that the mean curvature vectors of S_1 and S_2 at p coincide and that, around p, S_1 lies on one side of S_2. Then $S_1 \equiv S_2$. When the intersection point p belongs to the boundary of the surfaces, the result holds as well, provided further that the two boundary are tangent and both are local graphs over a common neighborhood in $T_pS_1 = T_pS_2$.*

Proof of Theorem 2.1. By the proof of Theorem 10.1 in [32], w satisfies a linear elliptic partial differential equation of second order whose highest order terms are of the form $\Delta + a_{ij}D_{ij}$, where each a_{ij} is C^∞ and $O(|x|^2)$, $x = (x_1, x_2)$ (one must do the computation in the proof of Theorem 10.1 in [32], keeping in mind that $u_1(0,0) = u_2(0,0)$ and $\nabla u_1(0,0) = \nabla u_2(0,0)$). This allows us to use Theorem 1.1 in [43], to conclude that

$$w(x) = h(x) + O(|x|^{n+1})$$

for some $n \geq 2$ and h is a homogeneous harmonic polynomial of degree n.

Let $z = x_1 + ix_2$. It follows that $h = h(z)$ is the real part of a holomorphic function. Since h is homogeneous of degree n, we have $h(z) = Re(cz^n)$, for some nonzero complex constant c. By rotating the coordinates, if necessary, we may assume that c is real. Hence

$$h(z) = h(re^{i\theta}) = cr^n \cos n\theta$$

As h changes sign in any neighborhood of (0,0), so does w.

$\square$

We are now ready to prove the Alexandrov Theorem.

Proof of the Alexandrov Theorem. The proof relies on Alexandrov reflection method. Let P_t, $t \in \mathbb{R}$, be the family of parallel planes in $\mathbb{R}^3$ orthogonal to a given direction of $\mathbb{R}^3$. Denote by P_t^- the halfspace containing $P_{t'}$ with $t' \leq t$

and let $P_t^+ = (\mathbb{R}^3 \setminus P_t^-) \cup P_t$. Denote by $S(t)^- = S \cap P_t^-$, by $S(t)^+ = S \cap P_t^+$ and by $S(t)_*$ the reflection of $S(t)^+$ across P_t.

Let $\mathcal{W}$ be the compact region of $\mathbb{R}^3$ with boundary S. As $\mathcal{W}$ is compact, we can choose a P_t disjoint from $\mathcal{W}$. Move P_t parallel to itself (decreasing t, say) until t_0 such that P_{t_0} touches S at a first point q. Then continue to decrease t. At the beginning $S(t)^+$ is a graph of bounded slope over a part of P_t and $int(S(t)_*)$ is contained in $\mathcal{W}$. Furthermore the mean curvature vector at any point of $S(t)_*$ is the reflection of the mean curvature vector at the corresponding point of $S(t)^+$. Now continue to decrease t till the first τ where one of the following conditions fails to hold:

(a) $int(S(\tau)_*) \subset \mathcal{W}$.

(b) $S(\tau)^+$ is a graph of bounded slope over a part of P_τ.

If (a) fails first, one applies the Maximum Principle to $S(\tau)^-$ and $S(\tau)_*$ at the point where they touch to conclude that P_τ is a plane of symmetry of S. If (b) fails first, then the point p where the tangent space of $S(\tau)^+$ becomes orthogonal to P_τ belongs to $\partial S(\tau)^+ = \partial S(\tau)^- \subset P_\tau$ and one apply the boundary Maximum Principle to $S(\tau)_*$ and $S(\tau)^-$ to conclude that P_τ is a plane of symmetry of S.

Thus, for any direction, one finds a plane of symmetry of S orthogonal to that direction. Hence S has a plane of symmetry in any direction and one concludes that it is a sphere, by Lemma 2.1.

$\square$

As it is clear from the proof, the key properties for the Alexandrov Theorem holding are:

- The surface must satisfy an elliptic equation.
- The ambient space must have "many" totally geodesic surfaces that are symmetry submanifolds by an ambient isometry.

It is easy to see that the analogous of the Alexandrov Theorem holds in $\mathbb{H}^{n+1}$ and in a hemisphere of $\mathbb{S}^{n+1}$ with the same proof as in [1]. In the case of

$\mathbb{S}^{n+1}$, the hypersurface S must be contained in a hemisphere, in order to start Alexandrov reflection method with a totally geodesic hypersurface $\mathbb{S}^n$ disjoint from S.

Furthermore, one can extend the Alexandrov Theorem to embedded hypersurfaces of $\mathbb{R}^{n+1}$ having positive constant scalar curvature, or such that any other symmetric function of the principal curvatures is a positive constant [52]. Alexandrov type results where obtained for Weingarten surfaces [11], [58] and for constant mean curvature surfaces bounded by convex curves in space forms [12], [59], [60].

The following Theorem is a generalization of Alexandrov's result to $\mathbb{H}^2 \times \mathbb{R}$ and to a hemisphere of $\mathbb{S}^2$ times $\mathbb{R}$.

Theorem 2.2 (Hsiang-Hsiang) *[36] A compact embedded constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ or in a hemisphere of $\mathbb{S}^2$ times $\mathbb{R}$ is a rotational sphere.*

The proof of Theorem 2.2 is similar to the proof of Alexandrov Theorem. In fact, any reflection about a vertical plane (i.e. a horizontal geodesic times $\mathbb{R}$) is an isometry of $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. In $\mathbb{H}^2 \times \mathbb{R}$ one applies Alexandrov reflection method with vertical planes in order to prove that for any horizontal direction, there is a vertical plane of symmetry of the surface, orthogonal to that direction. This means that the surface is invariant by rotation about a vertical axis i.e. it is a rotational sphere. The proof is analogous in $\mathbb{S}^2 \times \mathbb{R}$. We have only to notice that in order to start Alexandrov reflection method with vertical planes, one need to find, for any horizontal direction, a vertical plane orthogonal to that direction, non intersecting the surface. In $\mathbb{S}^2 \times \mathbb{R}$, this fact is guaranteed by the hypothesis that the surface is contained in a hemisphere times $\mathbb{R}$.

Alexandrov's problem in Nil_3, $\widetilde{PSL_2}(\mathbb{R})$ and in the Berger spheres is still open, since no reflections are available in these spaces.

In [29], Espinar, Galvez, Rosenberg remarked that in Sol_3 there are two orthogonal foliations by totally geodesic surfaces such that each leaf of the two orthogonal foliations is a symmetry submanifold by an ambient isometry and thus the Alexandrov reflection method can be used to prove that a compact embedded surface with constant mean curvature is a topological sphere. Then, once a Hopf type theorem is proved in Sol_3, an Alexandrov type result is proved too (see Theorem 4.2 and the discussion there).

Do Carmo-Lawson extension of Alexandrov Theorem (Theorem 1.1 in the Introduction) was suggested by the fact that in hyperbolic space, umbilical hypersurfaces are somewhat more interesting than in Euclidean space. We recall that that $\mathbb{H}^{n+1}$ has a natural compactification $\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \cup \mathbb{S}^n(\infty)$ where the points of $\mathbb{S}^n(\infty)$ can be viewed as classes of geodesic rays in $\mathbb{H}^{n+1}$ (two rays are identified if their distance tends to zero at infinity). If Σ is a submanifold of $\mathbb{H}^{n+1}$, the asymptotic boundary of Σ is defined as $\overline{\Sigma} \cap \mathbb{S}^n(\infty)$ and it is denoted by $\partial_\infty \Sigma$. Umbilical hypersurfaces in $\mathbb{H}^{n+1}$ are of three types, according to the value of the mean curvature (and their asymptotic boundary):

- $1 < H < \infty$: spheres (compact case).
- $H = 1$: horospheres (asymptotic boundary: one point).
- $0 \leq H < 1$: equidistant spheres, in particular hyperplanes when $H = 0$ (asymptotic boundary: a codimension two sphere).

Notice that properly embedded hypersurfaces with compact connected asymptotic boundary and constant mean curvature $H \geq 1$, do not exist in $\mathbb{H}^{n+1}$. Furthermore, Anderson [3], proved that any closed submanifold N^{p-1} immersed in $\mathbb{S}^n(\infty)$ is the asymptotic boundary of a minimal submanifold M^p of $\mathbb{H}^{n+1}$.

Hardt-Lin [37], [40] discussed the regularity at infinity in the case of hypersurfaces of $\mathbb{H}^{n+1}$. Tonegawa extended their results to submanifolds of different codimension [64].

Hence, it is clear that the asymptotic boundary has a crucial role in the discussion about constant mean curvature hypersurfaces in hyperbolic space.

Before stating next result we need to recall the following notion. Let Σ be a hypersurface in $\mathbb{H}^{n+1}$ such that $\partial_\infty\Sigma$ is a codimension two sphere. Then $\partial_\infty\Sigma$ can be assumed to be an equator of $\mathbb{S}^n(\infty)$. We say that Σ separates poles if the north pole and the south pole with respect to such equator are in different component of $\overline{\mathbb{H}^{n+1}} \setminus \overline{\Sigma}$ (see [23]).

Theorem 2.3 (Do Carmo-Lawson) *[23] Let Σ be a complete constant mean curvature hypersurface properly emebedded in $\mathbb{H}^{n+1}$ and let $\partial_\infty\Sigma$ be the asymptotic boundary of Σ.*

1. *If $\partial_\infty\Sigma$ is one point, then Σ is a horosphere.*
2. *If $\partial_\infty\Sigma$ is codimension two sphere and Σ separates poles, then Σ is a equidistant sphere.*

The first part of Theorem 2.3 is Theorem 1.1 stated in the Introduction. Notice that it is false without the assumption of embeddedness [42].

Proof of Theorem 2.3 Part 1.

The proof is a smart variation of the Alexandrov reflection method. Consider the half-space model for $\mathbb{H}^{n+1}$, that is

$$\mathbb{R}^{n+1}_+ = \mathbb{R}^n_0 \times \mathbb{R} = \{(\overline{x}, x_{n+1}) \in \mathbb{R}^{n+1} \ \ | \ \ x_{n+1} > 0\}$$

Assume that the asymptotic boundary $\partial_\infty\Sigma = \infty$ i.e. the point at infinity of $\mathbb{R}^{n+1}_+$. Then, Σ separates $\mathbb{H}^{n+1}$ into two connected components, $\mathcal{W}^+$ and $\mathcal{W}^-$ such that $\partial_\infty\mathcal{W}^+ = \infty$ and $\partial_\infty\mathcal{W}^- = \mathbb{R}^n_0$. In the half-space model, the horospheres whose asymptotic boundary is ∞ are $x_{n+1} = const$. The geodesics orthogonal to this family of horospheres are vertical half-lines, parameterized as follows: $\gamma_{\overline{x}} = (\overline{x}, t)$, $t > 0$, $\overline{x} \in \mathbb{R}^n_0$. Each geodesic $\gamma_{\overline{x}}$ determines a family of hyperbolic hyperplanes $\mathcal{H}_{\gamma_{\overline{x}}}(t)$ orthogonal to $\gamma_{\overline{x}}$. Geometrically, each $\mathcal{H}_{\gamma_{\overline{x}}}(t)$ is a half-sphere centered at $\overline{x}$, that is

$$\mathcal{H}_{\gamma_{\overline{x}}}(t) = \{x \in \mathbb{R}^{n+1}_+ \mid \|x - (\overline{x}, 0)\| = t\}.$$

As $\partial_\infty \Sigma = \infty$, then, for any geodesic $\gamma_{\overline{x}}$ and t small

$$\mathcal{H}_{\gamma_{\overline{x}}}(t) \cap \ \Sigma = \emptyset.$$

Recall that any inversion with respect to a halfsphere $\mathcal{H}_{\gamma_{\overline{x}}}(t)$ orthogonal to $\mathbb{S}^n(\infty)$ is a hyperbolic isometry, called the reflection with respect to $\mathcal{H}_{\gamma_{\overline{x}}}(t)$.

We apply Alexandrov reflection method with the family of hyperplanes $\mathcal{H}_{\gamma_{\overline{x}}}(t)$.

Let $\mathcal{H}_{\gamma_{\overline{x}}}(t)^+$ and $\mathcal{H}_{\gamma_{\overline{x}}}(t)^-$ be the two halfspaces determined by $\mathcal{H}_{\gamma_{\overline{x}}}(t)$

$$\mathcal{H}_{\gamma_{\overline{x}}}(t)^+ = \cup\{\mathcal{H}_{\gamma_{\overline{x}}}(s) \mid s \geq t\}, \quad \mathcal{H}_{\gamma_{\overline{x}}}(t)^- = \cup\{\mathcal{H}_{\gamma_{\overline{x}}}(s) \mid s \leq t\}$$

Let $\Sigma(t)^+ = \Sigma \cap \mathcal{H}_{\gamma_{\overline{x}}}(t)^+$ and $\Sigma(t)^- = \Sigma \cap \mathcal{H}_{\gamma_{\overline{x}}}(t)^-$. As the asymptotic boundary of Σ is ∞, for t small, $\Sigma(t)^- = \emptyset$. Let $\Sigma(t)_*$ be the hyperbolic reflection of $\Sigma(t)^-$ across $\mathcal{H}_{\gamma_{\overline{x}}}(t)$. Let $t_0 > 0$ be the smallest t such that $\mathcal{H}_{\gamma_{\overline{x}}}(t) \cap \Sigma \neq \emptyset$. Then, increase t. At the beginning $int(\Sigma(t)_*) \subset \mathcal{W}^+$ and $\Sigma(t)^-$ is not orthogonal to $\mathcal{H}_{\gamma_{\overline{x}}}(t)$ along the boundary.

Now continue to increase t till the first τ where one of the following conditions fails to hold:

(a) $int(\Sigma(\tau)_*) \subset \mathcal{W}^+$.

(b) $\Sigma(\tau)^-$ is not orthogonal to $\mathcal{H}_{\gamma_{\overline{x}}}(\tau)$ along the boundary.

This yields that $\Sigma(\tau)_*$ and $\Sigma(\tau)^+$ are tangent at an interior or boundary point p. Furthermore, in a neighborhood of p, $\Sigma(\tau)_*$ lies on one side of $\Sigma(\tau)^+$ and the mean curvature vectors coincide. Then, by the maximum principle, $\Sigma(\tau)^*$ and $\Sigma(\tau)^+$ must coincide. This yields that Σ is compact. Contradiction.

As (a) and (b) hold for every vertical geodesic $\gamma_{\overline{x}}$ and every t, one can conclude that the tangent plane to Σ is horizontal at any point, and Σ is a horosphere.

$\square$

Remark 2.1 In $\mathbb{H}^3$, horospheres are unique also in another sense. A properly embedded, constant mean curvature one, simply connected surface is a horosphere, while the only annulus in such hypothesis is a rotational surface with two point at infinity, known as the catenoid cousin [16]. The deep reason of this fact is that, in $\mathbb{H}^3$, a properly embedded annular end of a surface with constant mean curvature one has finite total curvature. On the contrary, the analogous assertion for a minimal surface in $\mathbb{R}^3$ is false: the end of the helicoid is annular, properly embedded but does not have finite total curvature (see [16] for details).

Remark 2.2 In $\mathbb{H}^2 \times \mathbb{R}$, the constant mean curvature $H = \frac{1}{2}$ plays a role analogous to that of $H = 0$ in $\mathbb{R}^3$ and $H = 1$ in $\mathbb{H}^3$. It is worth to notice that in $\mathbb{H}^2 \times \mathbb{R}$ there exists a complete, non umbilic, rotationally invariant, vertical graph with constant mean curvature $H = \frac{1}{2}$ [44], [48], [54], [57]. Discussions and conjectures about the uniqueness of complete, $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ can be found in [33], [45]. Recently Berard and Sa Earp described rotational hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and discussed about their classification [10].

3 Bernstein Problem

Bernstein Theorem. *[2], [6], [21], [31], [53] A minimal hypersurface M in $\mathbb{R}^{n+1}$, $n < 7$, which is a complete graph over a hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is a hyperplane.*

There are different proofs of Bernstein Theorem, according to the dimension. In $\mathbb{R}^3$, Bernstein himself [6], Heinz [35] and Osserman [49] proved Bernstein's result. The methods they used are strictly two dimensional. Simons [53] settled the result for $n \leq 6$. Bombieri-De Giorgi-Giusti [9] proved the existence of complete minimal graphs over $\mathbb{R}^n$, provided $n \geq 7$, hence a Bernstein type result is false for $n \geq 7$.

Let us explain the relations between minimal graphs, area minimizing hypersurfaces and stable hypersurfaces. A minimal hypersurface ($H = 0$) is a critical point of the volume with respect to deformations with compact support. A minimal hypersurface is called stable if the second variation of the volume is nonnegative for all compactly supported deformations. An area minimizing hypersurface is a minimum for the volume and of course an area minimizing hypersurface is stable. Furthermore a minimal graph M over a domain of $\mathbb{R}^n$ is stable. This is a well known fact, but we think it is worth to give a proof of it. Let $|A|$ be the second fundamental form of M. A hypersurface M is stable if and only if the first eigenvalue λ of the Jacobi operator $L = \Delta - |A|^2$, acting on $C_0^\infty(M)$ is non negative (see [7]). By contradiction, let D be a domain in M with compact closure, assume that $\lambda < 0$ and let f be the first eigenfunction. Then, $Lf = -\lambda f$, $f_{|\partial D} = 0$ and one can assume $f_{|D} > 0$. Let Φ_t be the variation of D such that $< \frac{d\Phi_t}{dt}_{|t=0}, N >= f$, where N is the unit normal vector field to M pointing upward. The first variation of the mean curvature for the normal variation fN is given by

$$\dot{H}(0)f = Lf = -\lambda f > 0.$$

Hence, for positive small t, at any interior point of the variation $\Phi_t(D)$, the mean curvature is greater than zero. Now, translate D upward, such that $D \cap \Phi_t(D) = \emptyset$. Then, translate D downward: at the first contact point between the translation of D and $\Phi_t(D)$, the mean curvature of D (zero) is smaller than the mean curvature of $\Phi_t(D)$, but D is above $\Phi_t(D)$. This is a contradiction by the maximum principle. Hence a minimal graph is stable.

The key point in Simons' proof of the Bernstein Theorem is Simons inequality.

Let M be a minimal hypersurface in $\mathbb{R}^{n+1}$, Δ the laplacian on M and $|A|$ the norm of the second fundamental form of M. Simons' inequality is the following:

$$\Delta|A|^2 \geq -2|A|^4 + 2(1 + \frac{2}{n})|\nabla|A||^2$$

Simons was able to prove that no non trivial n-dimensional stable minimal cones exist in $\mathbb{R}^{n+1}$ for $n \leq 6$. By a result of Fleming [31], the non existence of non trivial stable minimal cones in $\mathbb{R}^{n+1}$ implies that the only area minimizing hypersurfaces in $\mathbb{R}^{n+1}$ are hyperplanes. Hence any area minimizing hypersurface in $\mathbb{R}^{n+1}$, $n \leq 6$, is a hyperplane.

By adding some hyphotesis on the growth of the minimal graph, one obtains Bernstein type result in any dimension. Caffarelli-Nirenberg-Spruck proved that there are no complete minimal graphs of a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $|Du| = o(|x|^{\frac{1}{2}})$ [17]. Ecker-Huisken proved that there are no complete minimal graphs of a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that u is at most linear [28]. Schoen-Simon-Yau [56] generalized Simons inequality to different ambient manifolds. Then, they obtain an L^p inequality for the norm of the second fundamental form of a stable minimal hypersurface. Let us recall the following theorem from [56].

Theorem 3.1 (Schoen-Simon-Yau)*[56] Let M^n be a stable minimal immersion in a manifold $\mathcal{N}^{n+1}$. Let K_1 and K_2 be the lower and the upper bound of the sectional curvatures of $\mathcal{N}^{n+1}$ respectively and let c be a bound for the norm of the gradient of the curvature tensor. Then, for each $p \in [4, 4 + \sqrt{\frac{8}{n}})$ and for each non-negative smooth function f with compact support in M^n, one has*

$$\int_M |A|^p f^p \leq \beta \int_M [|\nabla f|^p + (c^{\frac{2}{3}} + K_1 - K_2 + \max\{-K_2, 0\})^{\frac{p}{2}} f^p] \tag{3}$$

where β is a constant depending only on n and p.

In [56], using Theorem 3.1, Schoen-Simon-Yau gave a simplified proof of Bernstein Theorem in the case $n < 6$. In fact, their proof works for area minimizing hypersurfaces. We give a sketch of the proof in [56].

Proof of Bernstein Theorem. As we remarked before, if M is a complete minimal graph over $\mathbb{R}^n$, then M is area-minimizing, in particular it is stable.

Define $B_R = \{x \in \mathbb{R}^{n+1} \cap M \ \mid \ |x| \leq R\}$. As M is area-minimizing, $vol(B_R) \leq \frac{1}{2}vol(S_R)$ where S_R is the Euclidean sphere of radius R in $\mathbb{R}^{n+1}$. Hence

$$vol(B_R) \leq \frac{(n+1)\omega_{n+1}}{2} R^n, \tag{4}$$

where ω_{n+1} is the volume of S_1. Let $|A|$ be the norm of the second fundamental form of M. Schoen-Simon-Yau deduced from (3) (choosing f as a distance function) that, for any $\theta \in (0,1)$ and $p \in (0, 4+\sqrt{\frac{8}{n}})$, there exists a constant β depending only on n and p (see [56]) so that

$$\int_{B_{\theta R}} |A|^p \leq \frac{\beta}{(1-\theta)^p} R^{-p} vol(B_R). \tag{5}$$

Replacing (4) in (5), one has

$$\int_{B_{\theta R}} |A|^p \leq \frac{\beta(n+1)\omega_{n+1}}{2(1-\theta)^p} R^{n-p}. \tag{6}$$

If $n < 6$, there is a $p > n$ satisfying (6). For such p, letting $R \longrightarrow \infty$ in (6), one has $|A| \equiv 0$ at any point, that is, M is a hyperplane.

$\square$

As the previous proof depends on inequality (3), it is not hard to see that one can obtain a Bernstein type result in ambient manifolds whose sectional curvatures satisfy some conditions [56]. Furthermore, as Theorem 3.1 holds for stable hypersurfaces, many authors try to extend the Bernstein Theorem to minimal stable hypersurfaces (parametric Bernstein Problem).

Conjecture. *[46] [47] The only complete minimal stable hypersurfaces in* $\mathbb{R}^{n+1}$, $3 \leq n \leq 7$, *are hyperplanes.*

Some partial results are contained in [14], [18], [19], [26], [46], [47], [50], [61].

Let us quote some results that we believe especially interesting. In the following theorem the condition of being a graph is replaced by stability added to a condition on the L^2 norm of the second fundamental form.

Theorem 3.2 (Do Carmo-Peng) *[26] Let M be a stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ such that*

$$\lim_{R \longrightarrow \infty} \frac{\int_{\mathcal{B}_R} |A|^2}{R^{2+2q}} = 0, \quad q < \sqrt{\frac{2}{n}}, \tag{7}$$

where $\mathcal{B}_R$ is the geodesic ball of radius R in M. Then M is a hyperplane.

The proof of Theorem 3.2 is based on a refinement of the technique in [56].

In the following theorem the condition of being a graph is replaced by stability and the topological condition of having at least two ends.

Theorem 3.3 (Cao-Shen-Zhu) [18] *For any $n \geq 3$, if M^n is a complete, stable minimal hypersurface in $\mathbb{R}^{n+1}$, then M^n has only one end.*

The technique of the proof of Theorem 3.3 is different in nature from those using the generalized Simons' inequality. It relies on the existence of a non-trivial bounded harmonic function with finite energy on a stable minimal hypersurface, provided the hypersurface has at least two ends. Then, one uses a Liouville type theorem [63], to prove that such function does not exist.
Many authors try to obtain a Bernstein type theorem for (strongly) stable constant mean curvature hypersurfaces [15], [22], [27], [30], [38], [62]. For the definition of (strong) stability for constant mean curvature hypersurfaces, see [8] and [30].

The following result yields the non existence of complete (strongly) stable hypersurfaces of constant mean curvature different from zero in $\mathbb{R}^{n+1}$, provided $n \leq 4$.

Theorem 3.4 (Elbert-Nelli-Rosenberg) *[30] Let $\mathcal{N}^{n+1}$ be a Riemannian manifold with sectional curvatures uniformly bounded from below. When $n = 3, 4$, $\mathcal{N}^{n+1}$ has no complete (strongly) stable hypersurfaces of constant mean curvature H, without boundary, provided $|H|$ is large enough (with respect to the absolute value of the bound on the sectional curvatures of $\mathcal{N}^{n+1}$). In particular there are no complete (strongly) stable H-hypersurfaces in $\mathbb{R}^{n+1}$ without boundary, $H \neq 0$.*

The proof of Theorem 3.4 relies on the following fact. For $|H|$ large enough, every point of a complete (strongly) stable hypersurface with constant mean curvature H, must have bounded distance from the boundary of the hypersurface. Then, fix a point p on the hypersurface in the hypothesis of Theorem 3.4 and choose balls in the hypersurface centered at the point p, with increasing radius. The distance from p to the boundary of the ball can be made arbitrarily large. Contradiction.

We notice that, the technique of the proof of Theorem 3.4, can be applied to many other cases in order to prove non-existence results [44], [51].

4 Hopf Problem

Hopf Theorem. *[34] A constant mean curvature topological sphere immersed in $\mathbb{R}^3$ is a round sphere.*

Let us recall the definition of Hopf differential on a surface S. Let E, F, G and e, f, g be the coefficients of the first and the second fundamental form of S, respectively. Denote by κ_1 and κ_2 the eigenvalues of the second fundamental form (i.e. the principal curvatures of the surface S). Let (u, v) isothermal parameters on S, that is $ds^2 = E(du^2 + dv^2)$. Then, the Gauss curvature and the mean curvature of S are

$$K = \kappa_1 \kappa_2 = \frac{eg - f^2}{E^2}, \qquad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{e + g}{2E}, \tag{8}$$

while Codazzi equations can be written as

$$\left(\frac{e - g}{2}\right)_u + f_v = EH_u, \qquad \left(\frac{e - g}{2}\right)_v - f_u = -EH_v, \tag{9}$$

and the lines of curvature of S are given by $-f du^2 + (e - g) du dv + f dv^2 = 0$.

Introduce the complex parameters $w = u + iv$, $\overline{w} = u - iv$ and let

$$\Phi(w, \overline{w}) = \frac{e - g}{2} - if. \tag{10}$$

Φ is known as the *Hopf fuction* of S and $\Phi(dw)^2$ as the *Hopf differential* of S.

From (8) and (10), it follows that

$$\frac{|\Phi|}{E} = \frac{|\kappa_1 - \kappa_2|}{2}.$$

As we wrote in the Introduction, if $\kappa_1 = \kappa_2$ at a point $p \in S$, the point p is called umbilic. Hence the umbilic points of S are the zeros of Φ.

Isothermal parameters give to S the structure of a Riemann surface such that the Hopf differential is a complex quadratic differential on S. In terms of Φ, Codazzi equations write as

$$\Phi_{\overline{w}} = EH_w.$$

If the surface S has constant mean curvature H then $H_v = H_u = 0$, and Codazzi equations are equivalent to the real and the imaginary part of Φ satisfying the Cauchy-Riemann equations. Hence Φ is an analytic function of w.

We report the proof of the Hopf Theorem as it is in [34]. We use the following characterization of the sphere in $\mathbb{R}^3$.

Lemma 4.1 *The spheres are the only close surfaces in* $\mathbb{R}^3$ *for which all points are umbilics* (Lemma 1.2. Chapter V in [34]).

Proof of the Hopf Theorem. The zeros of the Hopf differential are the umbilic points, hence one should prove that $\Phi \equiv 0$ on a compact Riemann surface of genus zero. We cover S by two coordinate neighborhoods: $w \in \mathbb{C}$ and $z = \frac{1}{w}$, $w \neq 0$. Then $\Phi(dw)^2 = \Psi(dz)^2$ are related by

$$\Phi(w) = \Psi(z)\left(\frac{dz}{dw}\right)^2 = \Psi(z)w^{-4} = \Psi(z)z^4 \tag{11}$$

Being Φ an entire function of w and being Ψ regular for $z = 0$, then $\Phi = 0$ for $w = \infty$. Hence $\Phi \equiv 0$ by Liouville's Theorem.

$\square$

As we wrote in the Introduction, the Hopf Theorem can be extended easily to space forms $\mathbb{H}^3$ and $\mathbb{S}^3$, because of the fact that Hopf differential is defined as in $\mathbb{R}^3$ and because rotational surfaces are umbilic. On the contrary, in other homogeneous 3-manifolds one has much more work to do. Abresch and Rosenberg extended Hopf's result to simply connected homogeneous 3-manifolds with isometry group of dimension four.

Theorem 4.1 (Abresch-Rosenberg) *[4] [5] A constant mean curvature topological sphere immersed in $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil_3, $\widetilde{PSL_2}(\mathbb{R})$, and in a Berger sphere is a rotationally invariant constant mean curvature sphere.*

As we have seen, the Hopf Theorem's proof relies on the existence on the surface of a holomorphic quadratic differential vanishing at umbilic points. In the ambient spaces of Theorem 4.1, Abresch and Rosenberg succeeded in defining a quadratic differential as a linear combination of the Hopf differential and a term coming from a Killing field of the ambient space. Such quadratic differential is holomorphic on constant mean curvature surfaces and vanishes on rotationally invariant surfaces. Hence they proved that any constant mean curvature topological sphere is rotationally invariant, in such ambient spaces.
We now discuss the extension of Alexandrov and Hopf type result to Sol_3, the only simply connected homogeneous 3-manifold with isometry group of dimension three.

Theorem 4.2 (Daniel-Mira) *[20] Let $H > \frac{1}{\sqrt{3}}$. Then:*

(a) There exists an embedded sphere S_H with constant mean curvature H in Sol_3.
(b) Any immersed sphere with constant mean curvature H in Sol_3 differs from S_H at most by a left translation.
(c) Any compact embedded surface with constant mean curvature H in Sol_3 differs from S_H at most by a left translation.

Moreover, these canonical spheres S_H constitute a real analytic family, they *all have index one and two reflection planes, and their Gauss maps are global diffeomorphisms into* $\mathbb{S}^2$.

It is clear that (b) and (c) in Theorem 4.2 are the Hopf type theorem and the Alexandrov type theorem in Sol_3, respectively.

The proof of Theorem 4.2 is quite articulated. We sketch the main ideas of it.

As we told in Section 2, (c) follows immediately from (b) and the existence part (a), by using the standard Alexandrov reflection technique with respect to the two canonical foliations of Sol_3 by totally geodesic surfaces.

In Sol_3, there are no known explicit constant mean curvature topological spheres. Hence Daniel and Mira had to find, first, the sphere for which they were looking for uniqueness.

As there are no rotations in Sol_3, one can not reduce the problem of finding constant mean curvature spheres, to the problem of solving an ordinary differential equation.

Notice that the solutions of the isoperimetric problem in Sol_3 are embedded spheres. Daniel and Mira proved that the Gauss map of an isoperimetric sphere (and more generally of an index one constant mean curvature sphere) is a diffeomorsphim. They also proved that a constant mean curvature sphere, whose Gauss map is a diffeomorphism, is embedded.

Then, they proved that one can deform (by implicit function theorem) index one constant mean curvature spheres, and that the property of having index one is preserved by this deformation. In this way they proved that there exists an index one sphere S_H with constant mean curvature H for all $H > \frac{1}{\sqrt{3}}$. This last condition comes from the fact that in order to deform, they need a bound on the diameter of the spheres. Such diameter estimate is a consequence of a theorem of Rosenberg [51] holding only for $H > \frac{1}{\sqrt{3}}$.

Then, they succeed in proving the existence of a quadratic differential satisfying the Cauchy-Riemann inequality (see [39]) on constant mean curvature spheres whose Gauss map is a diffeomorphism of $\mathbb{S}^2$. This quadratic differential is used to prove uniqueness in (b).

It is worth to notice that Daniel and Mira were able to prove the Hopf Theorem without knowing *a-priori* explicitly the sphere for which they where looking for uniqueness. This approach seems to be suitable for Hopf type theorems in many other cases.

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