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# CONFORMAL INVARIANTS INTERPRETED IN DE SITTER SPACE

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**Abstract**

We give a new interpretation of the conformally invariant differential 1-forms along the curvature lines of hypersurfaces in $I\!R^{n+1}$, obtained in [10], in terms of the infinitesimal arc-length of conveniently chosen curves in the $(n+2)$-dimensional de Sitter space.

## Introduction

In [3], R. L. Bryant gave an interpretation of the conformally invariant differential 2-form $(K_1 - K_2)^2\, dx_1 \wedge dx_2$ on a surface $M$ in $I\!R^3$ as the area of the surface $M'$, in the 5-dimensional Minkowski space, determined by the family of tangent spheres to $M$ whose curvature coincides with the mean curvature of $M$ at each point. More recently in [5], R. Langevin and J. O'Hara use the natural association between the points of the de Sitter $(n+2)$-space and the hyperspheres of $I\!R^{n+1}$ (conveniently immersed in the $(n+3)$-dimensional Minkowski space) in order to attach to the one-parameter family of osculating circles of a curve $\alpha$ in $I\!R^2$ or $S^2$ a null curve $\gamma$ in the de Sitter 4-space. As a consequence they obtain a new interpretation of the conformal arc-lenght of $\alpha$ in terms of a $\frac{1}{2}$-dimensional measure on $\gamma$.

In this work we use an analogous idea in order to provide a geometrical interpretation of the following conformally invariant differential 1-forms along to the $i$-th curvature lines $\varphi_i$, $i = 1, \cdots n$ that were defined in [10] for hypersurfaces in $I\!R^{n+1}$:

$$\left\{ \sqrt{\mid K_i{}'(t) \mid \, dt} \right\}_{i=1}^{n} \quad \text{and} \quad \{\, (K_i - K_j)\, dt\, \}_{1 \leq i \neq j \leq n}$$

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We characterize the first family of 1-forms in terms of the $\frac{1}{2}$-dimensional length element of a lightlike curve in the $(n+2)$-dimensional de Sitter space. Such a curve is determined by the $i$-th focal hyperspheres along the corresponding $i$-th curvature line on the considered hypersurface. We obtain a characterization for the ridges of hypersurfaces in $I\!R^{n+1}$ as points for which the tangent of one of these lightlike curves is a lightlike vector. On the other hand, we interpret the second family of 1-forms in terms of the infinitesimal arc-length of curves determined in the $(n+2)$-dimensional de Sitter space by the set of $i$-th focal hyperspheres along the $j$-th curvature lines. We then characterize the umbilic points as points at which the tangent of one of these curves is a lightlike vector.

We also apply this method to the study of the conformal invariants

$$\frac{\sqrt{\|c'_n(t)\|^2 - {r'_n}^2(t)}}{r_n(t)}\, dt,$$

of curves in $I\!R^{n+1}$, $n \geq 2$, related to the osculating hyperspheres obtained in [9]. We obtain a characterization for the vertices of curves in $I\!R^{n+1}$ as points for which the induced curve in de Sitter $(n+2)$-space has a lightlike tangent.

# 1 Basic concepts in Minkowski space

Let $E^{n+3} = \{(x_0, \cdots, x_{n+2}) : x_i \in I\!R, i = 0, 1, \cdots, n+2\}$ be the $(n+3)$-dimensional vector space with the pseudo-metric

$$\langle x, y\rangle_1 = -x_0 y_0 + \sum_{i=1}^{n+2} x_i y_i$$

where $x = (x_0, \cdots, x_{n+2})$, $y = (y_0, \cdots, y_{n+2}) \in E^{n+3}$. The space $(E^{n+3}, \langle , \rangle_1)$ is called Minkowski $(n+3)$-space and written by $E_1^{n+3}$.

A non zero vector $x \in E_1^{n+3}$ is said to be spacelike, lightlike or timelike according to $\langle x, y\rangle_1 > 0, = 0$ or $< 0$. The de Sitter $(n+2)$-space is defined as

$$S_1^{n+2} = \{x \in E_1^{n+3} : \langle x, x\rangle_1 = 1\}$$

and the quadratic lightcone is given by $Q = \{x \in E_1^{n+3} : \langle x, x\rangle_1 = 0\}$.

## 1.1 Conformal group

Let $F$ be $I\!R^2$ with the inner product $(\epsilon|\epsilon) = 0$, $(\eta|\eta) = 0$, $(\epsilon|\eta) = 1/2$, and $\{\epsilon, \eta\}$ is a basis of $F$. Take now the orthogonal sum $H = I\!R^{n+1} \perp F$, and consider in $I\!R^{n+1}$ the usual product $g$, such that $H$ is a Minkowski space whose pseudo-metric $h$ is given by

$$h(a\epsilon + x + b\eta, c\epsilon + y + d\eta) = \frac{1}{2}(ad + bc) + g(x, y).$$

We denote by $Q \subset H$ the quadratic lightcone defined by the zeros of

$$q(a\epsilon + x + b\eta) = ab + g(x, x).$$

Consider now the affine hyperplane of $H$ given by $J = \epsilon + (I\!R^{n+1} \oplus I\!R\eta)$ and define an injective map

$$\begin{array}{rlcl} j: & I\!R^{n+1} & \longrightarrow & J \\ & x & \longmapsto & \epsilon + x - g(x, x)\eta. \end{array}$$

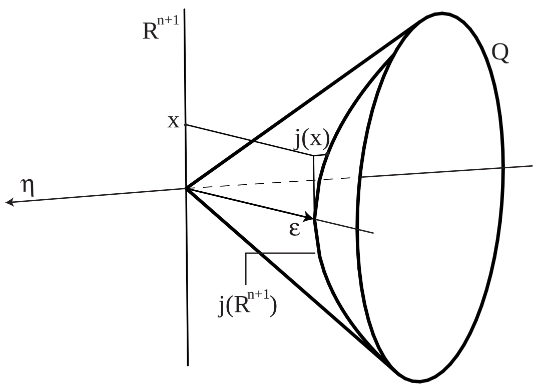


Figure 1: Quadratic light cone

Let $P : H \to P(H)$ be the natural projection of $H$ into its projective space $P(H)$ and put $\tilde{Q} = P(Q)$. The orthogonal group $O(H)$ preserves $Q$ and the group $PO(H) = O(H)/\{-I, I\}$ acts effectively on $P(H)$ and leaves $\tilde{Q}$ invariant.

In the following proposition, proved in [6], we identify $\tilde{Q}$ with the one point compactification of $\mathbb{R}^{n+1}$, and $PO(H)$ with the **Conformal Group** generated by translations, homotheties, rotations and inversions of $\mathbb{R}^{n+1}$ ([4]).

**Proposition 1** *With the notation above, we have that*

$$j(\mathbb{R}^{n+1}) = Q \cap J, \quad \tilde{Q} - P(j(\mathbb{R}^{n+1})) = P(\eta).$$

The next proposition and corollary ([6]), tells us that spheres and planes in $\mathbb{R}^{n+1}$ are in a one-to-one correspondence with Minkowskian linear subspaces in $H$.

**Proposition 2** *By considering the map $j$, we find the following equivalence between subspaces:*

*a) Let $S$ be an $m$-sphere of radius $r$ with center $c$, that lies in the affine subspace $c + W$, where $W$ is an $(m+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$. If the linear map $j_c : \mathbb{R}^{n+1} \to H$ is given by $j_c(x) = x - 2g(c,x)\eta$ and $d = g(c,c)^{\frac{1}{2}}$, then $j(S)$ is the intersection of $j(\mathbb{R}^{n+1})$ with the linear subspace*

$$W_{c,r} = \mathbb{R}(\epsilon + c - (d^2 + r^2)\eta) + j_c(W).$$

*b) Let $c + W$ be an affine subspace of $\mathbb{R}^{n+1}$, where $W$ is an $m$-dimensional linear subspace. Then $j(c+W)$ is the intersection with $j(\mathbb{R}^{n+1})$ of the linear subspace of $H$, $W_{c,\infty} = \mathbb{R}(\epsilon + c) + W + \mathbb{R}\eta$.*

**Corollary 1** *If $w_1, \ldots, w_{m+1}$ is an orthonormal basis of $W$, then the vectors*

$$\begin{aligned} u_0 &= \tfrac{1}{r}(\epsilon + c - (d^2 + r^2)\eta), \\ u_i &= w_i - 2g(c, w_i)\eta, i = 1, \ \ldots, \ m+1 \end{aligned}$$

*form an $h$-orthonormal basis of $H$ (i. e, form an $h$-orthogonal basis and $q(u_0) = -1,\ q(u_i) = 1,\ i = 1, \ldots, m+1$.)*

# 2 Inversive Product of hyperspheres

We consider two hyperspheres $S(c_i, r_i) \subset I\!R^{n+1}$ of center $c_i$ and radius $r_i$, $i = 1, 2$. The **inversive product** is given by ([2])

$$\langle S_1, S_2 \rangle = \left| \frac{r_1^2 + r_2^2 - (c_1 - c_2)^2}{2r_1 r_2} \right|$$

When $S_1$ and $S_2$ intersect each other we have that $\langle S_1, S_2 \rangle$ is a function of their angle of intersection. Whereas, when they are disjoint, this inversive product is a function of the hyperbolic distance between them. This product is a conformal invariant and generalizes the **inversive distance** given by Coxeter in [1] for circles in the plane.

We remind that if $M$ is a hypersurface in $I\!R^{n+1}$ (locally embedded through $\phi$) and $\Gamma : M \to S^n$ represents its normal Gauss map, the eigenvectors of $D\Gamma(\phi(x))$ are the **principal directions** of curvature of $M$ at the point $\phi(x)$ and the corresponding eigenvalues, $\{K_i(x)\}_{i=1}^n$, are the **principal curvatures**. A curve all of whose tangents point in principal directions at the corresponding points is a **curvature line**. We say that a point $\phi(x) \in M$ is **umbilic** if at least two of the principal curvatures coincide at this point. We shall denote by $U(M)$ the subset of the umbilic points of $M$. For a generic $M$, the subset $M - U(M)$ is an open and dense submanifold of $M$, [7].

Provided $\phi(x) \in M - U(M)$, we can find exactly $n$ **focal hyperspheres** at $\phi(x)$, whose centres are given by $c_i(x) = \phi(x) + r_i(x)N(\phi(x))$, where $N(\phi(x))$ is the normal vector of the hypersurface in the point $\phi(x)$, and whose radii are $r_i(x) = 1/K_i(x)$. We observe that the focal hyperspheres are the tangent hyperspheres whose contact with $M$ is higher than usual. The singular points of $c_i$, known as **ridges**, are the points at which the hypersurface has stronger contact with its focal hyperspheres (see [8]) and we characterized them in [10] by $K_i' = 0$, along the corresponding curvature line. If some of the principal curvatures vanishes, i.e. $\phi(x)$ is a **parabolic point** of $M$, then the corresponding focal hypersphere becomes a tangent hyperplane.

On the other hand, given a curve $\alpha : I\!R \to I\!R^{n+1}$ parameterized by arc-length, let $\{T(t), N_1(t), ..., N_n(t)\}$ denote its Frenet frame and $\{k_i(t)\}_{i=1}^n$ the corresponding curvature functions at the point $\alpha(t)$. The **osculating hyperspheres** of $\alpha$ are those whose centres are given by

$$c_\alpha(t) = \alpha(t) + \sum_{i=1}^{n} \mu_i(t) N_i(t),$$

where $\{\mu_i(t)\}_{i=1}^n$ are rational functions of the curvatures $\{k_i(t)\}_{i=1}^n$ and their derivatives satisfy the following relations (see [9]):

$$\begin{aligned}
&\mu_1(t)k_1(t) = 1,\\
&\mu_2(t)k_2(t) = \mu_1'(t),\\
&\mu_i(t)k_i(t) = \mu_{i-1}'(t) + \mu_{i-2}(t)k_{i-1}(t),\ i = 3, ..., n,
\end{aligned}$$

Again, we have that the osculating hyperspheres are those having higher order of contact with the curve at each point ([9]).

The curve $c_\alpha$ is called the **generalized evolute** of $\alpha$. The singular points of $c_\alpha$, known as **vertices**, are precisely the points at which the curve has stronger contact with its osculating hyperspheres and we characterized them in [9] by the relation

$$\mu_n'(t) + \mu_{n-1}(t)k_n(t) = 0.$$

We observe that any inversive map $\varphi : I\!R^{n+1} \cup \{\infty\} \to I\!R^{n+1} \cup \{\infty\}$ transforms hyperspheres into hyperspheres in $I\!R^{n+1} \cup \{\infty\}$, and since it is a diffeomorphism, it must also preserve the contacts of the hyperspheres with submanifolds of $I\!R^{n+1}$. Consequently it takes osculating hyperspheres of a curve, or focal hyperspheres of a hypersurface into the osculating hyperspheres, or the focal hyperspheres of their respective images.

Now, since the conformal maps preserve the inversive product of hyperspheres, we have that the infinitesimal distance between either two nearby osculating hyperspheres, or two nearby focal hyperspheres must also be preserved, i.e.

$$\langle S(t+h), S(t)\rangle = \langle \varphi(S(t+h)), \varphi(S(t))\rangle = \langle \bar{S}(t+h), \bar{S}(t)\rangle,$$

for $h$ tending to 0. So the appropriate manipulation of the inversive product on nearby focal (resp. osculating) hyperspheres may lead to the obtention of conformal invariants on hypersurfaces (resp. curves) in $\mathbb{R}^{n+1}$, as we see next.

## 2.1 Conformal invariants on curves in $\mathbb{R}^{n+1}$

Let $\alpha : \mathbb{R} \to \mathbb{R}^{n+1}$ be a curve parameterized by its arclength. We shall assume in what follows that the vectors $\{\alpha'(t), \alpha''(t), ..., \alpha^{(n)}(t)\}$ are linearly independent at every point, i. e. $\alpha$ is a generic curve (in the sense that most curves in $\mathbb{R}^{n+1}$ satisfy this property).

By applying the inversive product to the osculating hyperspheres on the curve, we obtained in ([9]) the following conformal invariant along it.

**Theorem 1** *Given a curve $\alpha$ in $\mathbb{R}^{n+1}$ parameterized by its arclength, the differential 1-form*

$$\omega_n(t) = \frac{\sqrt{\|c'_n(t)\|^2 - {r'_n}^2(t)}}{r_n(t)}\, dt,$$

*is an inversive invariant, where $c_n$ and $r_n$ are the center and the radius of the osculating hypersphere of $\alpha$.*

In the particular case of circles, the conformal invariant $\sqrt{|k_1'(t)|}dt$ is called *the infinitesimal conformal arc-length.*

**Corollary 2** *The zeroes of this 1-form $\omega_n$ are the vertices of $\alpha$.*

## 2.2 Conformal invariants on hypersurfaces in $\mathbb{R}^{n+1}$

Let $M$ be a hypersurface immersed by $\phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ and we denote by $U(M)$ the subset of the umbilic points of $M$. For a generic $M$, the subset $M - U(M)$ is an open and dense submanifold of $M$ and in $\phi(x) \in M - U(M)$ there exist exactly $n$ linearly independent principal directions associated to the corresponding curvature lines $\varphi_i(t)$, $i = 1, \cdots, n$.

On the other hand, by applying the inversive product to the $i$-th focal hyperspheres on the curvature lines of a hypersurface, we found in ([10]) the following conformal differential 1-forms along them.

**Theorem 2** *The differential 1-form along the curvature line $\varphi_i(t)$ of $M - U(M)$ defined by*

$$\omega_i(t) = \sqrt{\mid K_i{}'(t) \mid}\, dt$$

*is a conformal invariant along the curvature line $\varphi_i(t)$, where $K_i{}'$ represents the derivative of the principal curvature $K_i$ of $M$ restricted to the curve $\varphi_i$, $i = 1 \cdots, n$.*

**Corollary 3** *The zeroes of above 1-forms $\omega_i$, $i = 1 \cdots, n$ are the ridges of $M$.*

**Theorem 3** *Given any curvature line $\varphi_i$, $1 \leq i \leq n-1$ of $M - U(M)$, the 1-forms defined by*

$$\tilde{\omega}_{i,j}(t) = (K_j(t) - K_i(t))dt,\ 1 \leq j \neq i \leq n$$

*are conformal invariants along $\varphi_{i,}$.*

This conformal invariant can be extended on the whole (generic) hypersurface $M$ by assigning the value zero to the (isolate) umbilic points.

# 3 Interpretation of these invariants in $(n+2)$-dimensional de Sitter space

Consider the injective map

$$\begin{array}{llll} j: & I\!R^{n+1} & \longrightarrow & J \\ & x & \longmapsto & \epsilon + x - g(x,x)\eta. \end{array}$$

By using the Proposition 1 and the Corollary 1, we obtain that the image of a hypersphere $S(c,r)$ of radius $r$ and center $c$ in $I\!R^{n+1}$ is a $(n+2)$-dimensional

Minkowski space $W_{c,r} \in H$, i.e. a hyperplane in $H$ generated by the following $h$-orthonormal vectors:

$$\begin{aligned} u_0 &= \frac{1}{r}(\epsilon + c - (d^2 + r^2)\eta), \\ u_{i+1} &= w_i - 2g(c, w_i)\eta, \qquad i = 0, \ldots, n \end{aligned}$$

where $w_j$, $j = 0, 1, \ldots, n$ are a orthonormal basis of $I\!R^{n+1}$ and $d^2 = g(c, c)$.

In the following proposition, we determine a spacelike unit vector $u_{n+2} \in H$ which is pseudo-orthogonal to the above basis.

**Proposition 3** *The vector $u_{n+2} = \frac{1}{r}(\epsilon + c + (r^2 - d^2)\eta) \in H$ is $h$-orthogonal to the $h$-orthonormal basis $\{u_i\}_{i=0}^{n+1}$ and $h(u_{n+2}, u_{n+2}) = 1$. Therefore $u_{n+2}$ belongs to the de Sitter $(n+2)$-space $S_1^{n+2} = \{x \in H : h(x, x) = 1\}$.*

**Proof:** We have that the vector $u_{n+2} = \frac{1}{r}(\epsilon + c + (r^2 - d^2)\eta) \in H$. Moreover:

$$h(u_0, u_{n+2}) = \frac{1}{r^2}\left(-\frac{d^2 + r^2}{2} + \frac{r^2 - d^2}{2} + g(c, c)\right) = \frac{1}{r^2}(-d^2 + d^2) = 0$$

$$h(u_{i+1}, u_{n+2}) = \frac{1}{r}(-g(c, w_i) + g(c, N_i)) = 0, \ \ i = 0, \ldots, \ n,$$

then $u_{n+2}$ is $h$-orthogonal to $\{u_i\}_{i=0}^{n+1}$. On the other hand

$$h(u_{n+2}, u_{n+2}) = q(u_{n+2}) = \frac{1}{r^2}(r^2 - d^2 + g(c, c)) = \frac{1}{r^2}(r^2 - d^2 + d^2) = 1.$$

Therefore $u_{n+2} \in S_1^{n+2}$.

$\square$

### 3.1 Conformally invariant 1-forms on curves in $I\!R^{n+1}$

By applying the above methods to the osculating hyperspheres of a curve $\alpha \subset I\!R^{n+1}$ we can define a curve in de Sitter space $S_1^{n+2}$ as follows,

$$\begin{array}{rccl} \gamma : & I\!R & \longrightarrow & S_1^{n+2} \\ & t & \longmapsto & \dfrac{1}{r(t)}\left(\epsilon + c(t) + (r^2(t) - d^2(t))\eta\right) \end{array}$$

where $c(t)$, $r(t)$ are the center and radius respectively of the osculating hypersphere of the curve $\alpha$ at $\alpha(t)$ and $d^2(t) = g(c(t), c(t))$.
Then we obtain the following interpretation of the above conformal invariants:

**Theorem 4** *Given a curve $\alpha$ in $\mathbb{R}^{n+1}$, $n \geq 2$ parameterized by its arclength and with linearly independent first n derivatives, the differential 1-form*

$$\omega_n(t) = \frac{\sqrt{\|c'_n(t)\|^2 - {r'_n}^2(t)}}{r_n(t)}\, dt,$$

*is the infinitesimal arc-length of the curve $\gamma \subset S_1^{n+2}$, defined by the osculating hyperspheres along $\alpha$. The tangent vector $\gamma'(t)$ to the curve $\gamma$ is lightlike if and only if the point $\alpha(t)$ of the curve $\alpha$ is a vertex.*

**Proof:** Let $\alpha$ be a curve in $\mathbb{R}^{n+1}$, we consider the osculating hyperspheres $S(c, r)$ along it and the orthonormal basis $\{N_0, N_1, ..., N_n\}$ of $\mathbb{R}^{n+1}$, defined by the unit tangent and normal vectors of the curve $\alpha$. By using this orthonormal basis and the osculating hyperspheres of the curve, we define a curve $\gamma = \frac{1}{r}(\epsilon + c + (r^2 - d^2)\eta)$, where $d^2 = g(c, c)$, in the de Sitter $(n+2)$-space $S_1^{n+2}$.

By deriving $\gamma \subset S_1^{n+2}$ we obtain that the tangent vector of this curve is given by

$$\gamma' = \frac{-r'}{r} u_0 + \frac{g(c', N_n)}{r} u_{n+1}$$

By [9], we know that $c'$ has the direction of $N_n$, then the norm of $\gamma'$ is

$$\|\gamma'(t)\|^2 = h(\gamma'(t), \gamma'(t)) = \frac{\|c'(t)\|^2 - r'^2(t)}{r^2(t)}$$

Then, the infinitesimal arc-length of $\gamma$ is the conformal invariant $\omega_n(t)$

$$\|\gamma'(t)\|\, dt = \frac{\sqrt{\|c'(t)\|^2 - r'^2(t)}}{r(t)}\, dt$$

It follows from corollary 2 that a point $\alpha(t_0)$ is a vertex of $\alpha$ if and only if $\|\gamma'(t_0)\| = 0$, i.e. $\gamma'(t_0) \in Q$.

$\square$

In the particular case of plane curves it can be easily seen that the curve $\gamma$ is a lightlike curve. In order to treat this case, R. Langevin and J. O'Hara have introduced in [5] the following

**Defininition 1 ([5])** *Let $\gamma$ be a lightlike curve, the non trivial $L^{\frac{1}{2}}$-measure of $\gamma$ is defined by*

$$L^{\frac{1}{2}}(\gamma) = lim_{max|t_{j+1}-t_j|\to 0}\sum_i \sqrt{\|\gamma(t_{i+1}) - \gamma(t_i)\|}.$$

**Proposition 4 ([5])** *Let $\gamma$ be a lightlike curve, then*

$$L^{\frac{1}{2}}(\gamma) = lim_{max|t_{j+1}-t_j|\to 0}\sum_i \sqrt{\|\gamma(t_{i+1}) - \gamma(t_i)\|} = \sqrt[4]{\frac{1}{12}}\int_\gamma \sqrt[4]{|\ h(\gamma'', \gamma'')\ |}\,dt.$$

Then they obtain the following result, whose proof is included here for the sake of completeness.

**Theorem 5** *Let $\alpha$ be a plane curve parameterized by its arclength and such that $\alpha'$ and $\alpha''$ are linearly independent along $\alpha$. Then the curve $\gamma$ defined by the osculating circles along $\alpha$ is lightlike and the differential 1-form*

$$\omega_2(t) = \sqrt{|k_1'(t)|}dt,$$

*is $\sqrt[4]{12}$ times the infinitesimal $\frac{1}{2}$-dimensional length element of $\gamma \subset S_1^3$.*

**Proof:** Let $\alpha$ be a curve in $\mathbb{R}^2$. By considering the osculating circles $S(c, r)$ along it, we obtain a curve $\gamma = \frac{1}{r}(\epsilon + c + (r^2 - d^2)\eta)$ in de Sitter 3-space $S_1^3$.

Now, by deriving the curve $\gamma \subset S_1^3$ we get that the tangent vector of this curve is given by:

$$\gamma' = \frac{-r'}{r}u_0 + \frac{g(c', N)}{r}u_2.$$

In this particular case, we know that $c = \alpha + rN$, where $N$ is the normal vector to $\alpha$. By applying the Frenet formulas we obtain that $c' = r'\,N$, then:

$$\|\gamma'(t)\|^2 = h(\gamma'(t), \gamma'(t)) = \frac{r'^2(t) - r'^2(t)}{r^2(t)}$$

and $\|\gamma'(t)\|$ is identically 0, so $\gamma$ is lightlike.

Now, we calculate the second derivative of $\gamma$

$$\gamma'' = \left(\left(\frac{-r'}{r}\right)' + \left(\frac{-r'}{r}\right)^2\right) u_0 - k_1' u_1 + \left(\left(\frac{-r'}{r}\right)' + \left(\frac{-r'}{r}\right)^2\right) u_2.$$

Whose norm is:

$$\|\gamma''(t)\|^2 = h(\gamma''(t), \gamma''(t)) = k_1'(t)^2.$$

Then the differential conformal 1-form $\omega_2(t)$ is $\sqrt[4]{12}$ times the infinitesimal $\frac{1}{2}$-dimensional length of $\gamma$ in $Q$:

$$\sqrt{\|\gamma''(t)\|}\, dt = \sqrt[4]{|\, h(\gamma'', \gamma'')\, |}\, dt = \sqrt{|k_1'(t)|}\, dt.$$

□

## 3.2 Conformally invariant 1-forms on hypersurfaces

Given a hypersurface $M$, we can define the following curves in de Sitter space $S_1^{n+2}$ :

$$\begin{array}{rcl} \gamma_i : \ I\!R & \longrightarrow & S_1^{n+2} \\ t & \longmapsto & \dfrac{1}{r_i(t)}(\epsilon + c_i(t) + (r_i^2(t) - d_i^2(t))\eta) \end{array}$$

where $c_i(t), r_i(t)$ are the center and radius respectively of the focal hyperspheres along the curvature lines $\varphi_i$ on $M$ at $\varphi_i(t)$ and $d_i^2(t) = g(c_i(t), c_i(t))$, $i = 1, \cdots, n$.

Then we interpret the above conformal invariants on $M$ as follows:

**Theorem 6** *Let $M$ be a hypersurface $M$ in $I\!R^{n+1}$. The curves $\gamma_i$ determined in $S_1^{n+2}$ by the focal hyperspheres along the curvature lines $\varphi_i(t)$, $i = 1, \cdots, n$ of $M - U(M)$, are lightlike curves and the differential 1-forms*

$$\omega_i(t) = \sqrt{|\ K_i{}'(t)\ |}\, dt, i = 1, \cdots, n$$

*are $\sqrt[4]{12}$ times the infinitesimal $\frac{1}{2}$-dimensional length elements of $\gamma_i$, $i = 1, \cdots, n$. The tangent vector $\gamma_i''(t)$ to the curve $\gamma_i'$ is lightlike too if and only if the point $\phi(x) = \varphi_i(t)$ is a ridge of the hypersurface $M$.*

**Proof:** Let $M$ be a hypersurface in $I\!R^{n+1}$, we consider the focal hyperspheres $S_i(c_i, r_i)$ along the curvature lines $\varphi_i(t)$, $i = 1, \cdots, n$ of $M - U(M)$. The orthonormal basis of the affine spaces $W_i$ of $I\!R^{n+1}$ is defined by the unit principal directions of $M$ and the normal vector of the hypersurface in $\phi(x) = \varphi_i(t)$ : $\{X_1, X_2, ..., X_n, N\}$. By using this basis and applying the corollary 1, we define the curves $\gamma_i = \frac{1}{r_i}(\epsilon + c_i + (r_i^2 - d_i^2)\eta)$, where ${d_i}^2 = g(c_i, c_i)$, $i = 1, \cdots, n$, in the de Sitter (n+2)-space $S_1^{n+2}$.

By deriving the curves $\gamma_i \subset S_1^{n+2}$ along the corresponding curvature lines $\varphi_i(t)$, $1 \leq i \leq n$, we obtain that the tangent vectors of these curves are given by:

$$\gamma_i' = \frac{-r_i'}{r_i} u_0 + \frac{g(c_i', N)}{r_i} u_{n+1}$$

And its norm is:

$$\|\gamma_i'(t)\|^2 = h(\gamma_i'(t), \gamma_i'(t)) = -\left(\frac{-r_i'}{r_i}\right)^2 + \left(\frac{g(c_i', N)}{r_i}\right)^2.$$

In this case, we know that $c_i = \phi + r_i N$, where $N$ is the normal vector of the hypersurface $M$. By applying the Olinde Rodrigues formula we obtain $c_i' = r_i' N$, then $\|\gamma_i'(t)\|^2 = 0$ is identically 0, so $\gamma_i$ is lightlike.

We calculate the second derivative of $\gamma_i$:

$$\gamma_i'' = \left(\left(\frac{-r_i'}{r_i}\right)' + \left(\frac{-r_i'}{r_i}\right)^2\right) u_0 - K_i' u_{i+1} + \left(\left(\frac{-r_i'}{r_i}\right)' + \left(\frac{-r_i'}{r_i}\right)^2\right) u_{n+1}.$$

And their norms are:

$$\|\gamma_i''(t)\|^2 = h(\gamma_i''(t), \gamma_i''(t)) = K_i'(t)^2.$$

Then the differential conformal 1-forms are $\sqrt[4]{12}$ times the infinitesimal $\frac{1}{2}$-dimensional arc-lengths of $\gamma_i$ in $Q$ :

$$\sqrt{\|\gamma_i''(t)\|} dt = \sqrt[4]{|\ h(\gamma_i'', \gamma_i'')\ |}\, dt = \sqrt{|K_i'(t)|} dt,\ i = 1, \cdots, n.$$

It follows from corollary 3 that a point $\phi(x_0) = \varphi_i(t_0)$ of the hypersurface $M$ is a ridge if and only if $K_i'(t_0) = 0$, i.e. $\gamma_i''(t_0) \in Q$.

□

**Theorem 7** *Suppose that $M$ is a generic surface in the sense that the subset of umbilics $U(M)$ has zero measure. Then the differential 1-forms*

$$\tilde{\omega}_{i,j}(t) = (K_j(t) - K_i(t))dt,\ 1 \leq j \neq i \leq n$$

*correspond to the infinitesimal length element of the curves $\gamma_{i,j} \subset S_1^{n+2}$ associated to the focal hyperspheres $S_j$ along the curvature lines $\varphi_i(t)$, $1 \leq j \neq i \leq n$ of $M - U(M)$. The tangent vector to the curve $\gamma_{i,j}$ is lightlike at a point if and only if this point is an umbilic ($K_i = K_j$) of the hypersurface $M$.*

**Proof:** Let $M$ be a hypersurface in $\mathbb{R}^{n+1}$, if we consider the focal hyperspheres $S_j(c_j, r_j)$, $j \neq i$ along the curvature lines $\varphi_i(t)$, $i = 1, \cdots, n$ of $M - U(M)$, we define the curves $\gamma_{i,j} = \frac{1}{r_j}(\epsilon + c_j + (r_j^2 - d_j^2)\eta)$ in the De Sitter $(n+2)$-space $S_1^{n+2}$.

By deriving the curves $\gamma_{i,j} \subset S_1^{n+2}$ along the curvature lines $\varphi_i(t)$, $1 \leq j \neq i \leq n$ and by applying the Olinde Rodrigues formula we obtain that the tangent vectors of these curves are given by:

$$\gamma'_{i,j} = \frac{-r'_j}{r_j}u_0 + \frac{1}{r_j}\left(1 - \frac{K_i}{K_j}\right)u_{j+1} + \frac{1}{r_j}\left(\frac{{K_j}'}{K_j^2}\right)u_{n+1}$$

And their norms are:

$$\|\gamma'_{i,j}(t)\|^2 = -\left(\frac{-r'_j}{r_j}\right)^2 + \left(\frac{1}{r_j}\left(1 - \frac{K_i}{K_j}\right)\right)^2 + \left(\frac{1}{r_j}\left(\frac{{K_j}'}{K_j^2}\right)\right)^2 = (K_j - K_i)^2$$

Then the squared root to the infinitesimal conformal arc-length of $\gamma_{i,j}$ in $S_1^{n+2}$ is the differential conformal 1-form:

$$\sqrt{\|\gamma'_{i,j}(t)\|^2}dt = \sqrt{(K_j - K_i)^2}dt = (K_j - K_i)\, dt,\ 1 \leq j \neq i \leq n.$$

A point $\phi(x_0) = \varphi_i(t_0)$ is a umbilic point ($K_i = K_j$) on the generic hypersurface $M$ if and only if $\|{\gamma_{i,j}}'(x_0)\| = 0$ i.e. ${\gamma_{i,j}}'(t_0) \in Q$.

$\square$

# References


[1] Coxeter, H. S. M., *Inversive distance*, Ann. Mat. Pura Apli. 4, 71 (1966), 73-83.

[2] Beardon, A. F., *The geometry of discrete groups,* Springer-Verlag, 1983.

[3] Bryant, R. L., *A Duality Theorem for Willmore surfaces*, Differencial Geometry 20 (1984), 23-53.

[4] Deheuvels, R., *Groupes conformes et algèbres de Clifford*, Rend. Sem. Mat. Univers. Politecn. Torino. 43, 2 (1985), 205-226.

[5] Langevin, R. J.; O'Hara, J., *Conformal arc-lenght via osculating circles.* http://math.u-bourgogne.fr/topolog/langevin/preprints.html, Preprint 2008.

[6] Montesinos Amilibia, A.; Romero Fuster, M. C.; Sanabria Codesal, E., *Conformal curvatures of curves in* $I\!R^{n+1}$, Indag. Mathem. N.S. 12, 3 (2001), 369-382.

[7] Moraes, S. M.; Romero-Fuster, M. C.; Sánchez-Bringas, F., *Principal configurations and umbilicity of submanifolds in* $\mathbb{R}^N$. Bull. Belg. Math. Soc. Simon Stevin 11 (2004), no. 2, 227–245.

[8] Porteous, I. R., *The normal singularities of a submanifold.* J. Differential Geometry 5 (1971), 543–564.

[9] Romero-Fuster, M. C.; Sanabria-Codesal, E., *Generalized evolutes, vertices and conformal invariants of curves in* $I\!R^{n+1}$, Indag. Mathem. N.S. 10, 2 (1999), 297–305.

[10] Romero-Fuster, M. C.; Sanabria-Codesal, E., *Lines of curvature, ridges and conformal invariants of hypersurfaces*, Beitrage zur algebra und geometrie 2 (2004), 615-635.

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