# GAP OF THE FIRST TWO EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH NONCONVEX POTENTIAL

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*Dedicated to Professor Manfredo do Carmo on the occasion of his $80^{th}$ birthday*

In this essay, I will extend my previous work [9] on operators whose potential is nonconvex. In particular, the results given here can be applied to the double well potential. I define an invariant associated to the potential in §4. It defines a distance between the point where $\frac{u_2}{u_1}$ achieves its maximum, sup $\frac{u_2}{u_1}$, to the point where $\frac{u_2}{u_1} = \varepsilon \sup \frac{u_2}{u_1}$. Here $u_i$ are the eigenfunctions of the Schrödinger operator. I will show how the gap $\lambda_2 - \lambda_1$ can be estimated from below in terms of this distance. Theorem 6.1 is the main theorem of this essay. It is a very interesting problem to locate the maximum of $\frac{u_2}{u_1}$ and its zeroes. The upper bound of $\lambda_2 - \lambda_1$ depends on the choice of a good trial function, and I shall come back to this question in the future.

This line of research on gradient estimates started from my work on bounded harmonic functions [8] and the method was used by Peter Li [2] and Li-Yau [3] for the Laplacian of a manifold. Li-Yau [3] also applied it to the Schrödinger operator where a distance function similar to the one used here was introduced.

The Li-Yau type distance function was also used by Perelman in his famous work [6].

In the Li-Yau's approach of estimating the first eigenvalue of the Laplacian, it was conjectured by Li-Yau and proved by Zhong-Yang [10] that if $d$ is the diameter of a manifold with nonnegative Ricci curvature, then $\lambda_1 d^2$ has an universal lower bound which is achieved when the manifold is a circle.

Convex domain and convex potential can be considered as an analogue of manifold with nonnegative curvature. In Singer-Wong-Yau-Yau [7], we

improved on the log concavity result of Brascamp-Lieb [1] and proved that $(\lambda_2 - \lambda_1)d^2$ has a universal lower bound. It is natural for us to expect that the interval will give this optimal lower estimate.

I would like to dedicate this work to my friend Manfredo do Carmo whose works on minimal surfaces are very original and influential.

# 1 Generalized log concavity of the first eigen-function

In [7, 9], I used method of continuity to generalize the log concavity result of Brascamp and Lieb [1] when the potential is convex. I generalize it further in this section.

Let $u_1$ be the first eigenfunction of the operator $-\Delta + V$ on a domain $\Omega_1$ with zero boundary valued data. Let $\varphi = -\log u_1$. Then we have the following theorem:

**Theorem 1.1** *When $\Omega$ is strictly convex, the Hessian of $\varphi$ has eigenvalue greater than $g(x)$ where*

$$g(x) = \sup \Big\{ f(x) \mid \quad f \text{ is a smooth function defined on } \Omega, |f| \, d(x, \partial\Omega)$$
$$\text{is bounded by the principle curvature of } \partial\Omega, \text{ and}$$
$$\text{the lowesteigenvalue of the Hessian of } V \text{ plus } \Delta f$$
$$\text{is greater than } f^2 - 2\sum_j \varphi_j f_j \quad \Big\}$$

**Proof:** Differentiating the equation

$$\Delta\varphi = |\nabla\varphi|^2 - V + \lambda_1 \tag{1.1}$$

we obtain

$$\Delta(\varphi_{ii} - f) = \sum \varphi_{ji}^2 - V_{ii} - \Delta f + 2 \sum_j \varphi_j(\varphi_{ii})_j \tag{1.2}$$

where $\varphi_{ji} = \dfrac{\partial^2 u}{\partial x_j \, \partial x_i}$ and $V_{ii} = \dfrac{\partial^2 V}{\partial x_i^2}$.

Note that $\varphi_{ii}$ goes to infinity in a neighborhood of $\partial\Omega$. Therefore $\varphi_{ii} - f$ has a minimum in the interior of $\Omega$ where we have

$$\varphi_{ji} \;=\; 0 \qquad \text{for} \qquad j \neq i \tag{1.3}$$

$$\Delta(\varphi_{ii} - f) \;\geq\; 0 \tag{1.4}$$

$$\nabla(\varphi_{ii} - f) \;=\; 0 \tag{1.5}$$

Hence at such a point,

$$\varphi_{ii}^2 \geq V_{ii} + \Delta f - 2 \sum_j \varphi_j f_j \tag{1.6}$$

Note that the continuity argument was introduced by me and discussed in [7]. (I had lectured on this in 1979 as was noted in [4].) It can be applied in the following way.

Replace the potential $V$ by $V_t = ||x||^2 + tV$. When $t = 0$, the theorem is obviously true. We have to prove the theorem for all $t > 0$.

Suppose the theorem is true for $t < t_0$ and the theorem fails for $t > t_0$. Then at $t = t_0$, there is a function $f$, depending on $t$, so that $\varphi_{ii} \geq f$, $(V_t)_{ii} + \Delta f > f^2 + 2\sum_j \varphi_j f_j$ and at some point, $\varphi_{ii} = f$. By (1.5), we obtain

$$f^2 \geq (V_t)_{ii} + \Delta f - 2 \sum_j \varphi_j f_j \tag{1.7}$$

This contradict the choice of $f$.

Hence $t_0$ is arbitrarily large and we conclude that Theorem 1.1 holds.

**Remark 1.1** *The argument of Theorem 1.1 can be generalized to manifolds with negative curvature.*

In order to apply Theorem 1.1, we shall give a bound on $\sum \varphi_j d_j$ where $d$ is the distance function to $\partial\Omega$.

For simplicity, we restrict our attention to the important case where $\Omega$ is a

ball of radius $R$. We shall consider the function

$$
\begin{aligned}
F &= (R^2 - \textstyle\sum_i x_i^2)\sum_i \varphi_i x_i \\
F_j &= -2x_j \sum \varphi_i x_i + (R^2 - \textstyle\sum x_i^2)(\varphi_j + \sum x_i \varphi_{ij}) \\
\Delta F &= -(2n+4)\sum \varphi_i x_i - 2\sum x_i \varphi_{ij} x_j \\
&\quad + (R^2 - \textstyle\sum_i x_i^2)\,(2\Delta\varphi + \sum_i x_i(\Delta\varphi)_i) \\
&= -(2n+4)\sum_i \varphi_i x_i - 2\sum x_i x_j \varphi_{ij} \\
&\quad + (R^2 - \textstyle\sum_i x_i^2)(-2V + 2\lambda + 2|\nabla\varphi|^2 - \sum x_i V_i + 2\sum x_i \varphi_{ij}\varphi_j)
\end{aligned}
\tag{1.8}
$$

When $F$ achieves its maximum, $F_j = 0$ and $\Delta F \le 0$. Hence if we multiply the last equation of (1.8) by $(R^2 - \sum_i x_i^2)^2$, we find

$$
\begin{aligned}
0 \;\ge\; & -(2n+4)(R^2 - \|x\|^2)F - 4\|x\|^2 F + 2(R^2 - \|x\|^2)F \\
& + (R^2 - \|x\|^2)^3(-2\,V + 2\lambda - \textstyle\sum x_i V_i) + 4\,F^2
\end{aligned}
\tag{1.9}
$$

**Proposition 1.1** *Let $\Omega = \{x \,|\, \|x\| < R\}$. Then either,*

$$
\sup\left\{(R^2 - \|x\|^2)\sum_i x_i\varphi_i\right\} \le \sup\left\{(R^2 - \|x\|^2)^{3/2}\big(-V + \lambda - \tfrac{1}{2}\sum x_i V_i\big)^{1/2}\right\}
\tag{1.10}
$$

*or*

$$
\sup\left\{(R^2 - \|x\|^2)\sum_i x_i\varphi_i\right\} \le \left(\frac{n}{2}+1\right)R^2
\tag{1.11}
$$

To construct $f$ that satisfies the inequality

$$
\Delta f > f^2 + 2\sum \varphi_j f_j - \inf_i V_{ii}
\tag{1.12}
$$

we can use the ansatz

$$
f = h(R^2 - \|x\|^2)
\tag{1.13}
$$

Then

$$
\begin{aligned}
\Delta f &= 4\,h''\|x\|^2 - 2\,n\,h' \\
&= -4\,t\,h'' - 2\,n\,h' + 4R^2\,h''
\end{aligned}
\tag{1.14}
$$

where $t = R^2 - \|x\|^2$ and $\sum \varphi_i f_i = -h'\sum x_i\varphi_i$. If we choose $h' \le 0$, then

$$
\begin{aligned}
\sum \varphi_i f_i &\le -\left(\tfrac{h'}{t}\right)(R^2 - \|x\|^2)\sum x_i\varphi_i \\
&\le -\left(\tfrac{h'}{t}\right)\left[(\tfrac{n}{2}+1)R^2 + t^{3/2}\sup_{\|x\|^2 = R^2 - t}\big(-V + \lambda - \tfrac{1}{2}\sum x_i V_i\big)^{1/2}\right]
\end{aligned}
\tag{1.15}
$$

**Theorem 1.2** *Let* $\Omega = \left\{ x \,\middle|\, \|x\| < R \right\}$ *and* $f$ *be a function* $h$ *of* $t = R^2 - \|x\|^2$. *If* $h$ *is chosen to satisfy the inequality*

$$
\begin{aligned}
-4\,t\,h'' \;-\; 2nh' + 4R^2 h'' & \\
> \quad & h^2 - 2 \left( \tfrac{h'}{t} \right) \\
& \left[ \left( \tfrac{n}{2} + 1 \right) R^2 + t^{3/2} \sup_{\|x\|^2 = R^2 - t} \left( -V + \lambda - \tfrac{1}{2} \sum x_i V_i \right)^{1/2} \right] \\
& - \inf_i V_{ii}
\end{aligned}
\tag{1.16}
$$

*then* $f = h(t)$ *satisfies the inequality*

$$
\Delta f > f^2 + 2 \sum_i \varphi_i f_i - \inf_i V_{ii} \tag{1.17}
$$

## 2 Gradient estimate of first eigenfunction

Let $\Omega$ be a convex subdominant. Then we can choose a smooth nonnegative function $\rho$ with compact support.

Let

$$
G = \rho^2 (V + \alpha)^{-1} \, \Delta\varphi \tag{2.1}
$$

where $\alpha$ is a constant to be chosen later. Then according to (1.1), we find

$$
\nabla G = G(2\nabla \log \rho - \nabla \log(V + \alpha)) + \rho^2 (V + \alpha)^{-1} \left( 2\nabla\varphi \cdot \nabla\nabla\varphi - \nabla V \right) \tag{2.2}
$$

$$
\begin{aligned}
\Delta G &= \nabla G(2\nabla \log \rho - \nabla \log(V+\alpha)) + G(2\Delta \log \rho - \Delta \log(V+\alpha)) \\
&\quad + 2\nabla G \cdot \nabla \log \rho - 4|\nabla \log \rho|^2 \, G - \nabla G \cdot \nabla \log(V+\alpha) \\
&\quad + 4G\nabla \log(V+\alpha) \cdot \nabla \log \rho - G|\nabla \log(V+\alpha)|^2 \\
&\quad + \rho^2 \, (V+\alpha)^{-1} \, [2|\nabla\nabla\varphi|^2 + 4\nabla\varphi \cdot \nabla\nabla\varphi \cdot \nabla\varphi - 2\nabla\varphi \cdot \nabla V - \Delta V] \\
&= \nabla G(4\nabla \log \rho - 2\nabla \log(V+\alpha) + 2\nabla\varphi) \\
&\quad + G(2\Delta \log \rho - \Delta \log(V+\alpha) - 4|\nabla \log \rho|^2 - |\nabla \log(V+\alpha)|^2 \\
&\qquad\quad - 4\nabla \log \rho \cdot \nabla\varphi + 2\nabla \log(V+\alpha) \cdot \nabla \log \rho) \\
&\quad + \rho^2(V+\alpha)^{-\frac{1}{2}} \, (|\nabla\nabla\varphi|^2 - \Delta V)
\end{aligned}
\tag{2.3}
$$

By (1.1)

$$
\begin{aligned}
|\nabla\nabla\varphi|^2 &\geq \frac{1}{n}|\Delta\varphi|^2 \\
&= \frac{1}{n}(V+\alpha)^2 \, \rho^4 \, G^2
\end{aligned}
\tag{2.4}
$$

Hence at the point where $G$ achieves its maximum,

$$
\begin{aligned}
0 &\geq \frac{2}{n}G^2 - \rho^4(V+\alpha)^{-2} \, \Delta V \\
&\quad + \rho^2 \, (V+\alpha)^{-1} \, G[2\Delta \log \rho - \Delta \log(V+\alpha) - 4|\nabla \log \rho|^2 \\
&\qquad\quad - |\nabla \log(V+\alpha)|^2 - 4\log \rho \cdot \nabla\varphi + 2\nabla \log(V+\alpha) \cdot \nabla \log \rho]
\end{aligned}
\tag{2.5}
$$

Note that

$$
\begin{aligned}
\rho^2(V+\alpha)^{-1} \, |\nabla\varphi|^2 &\leq \rho^2 \, (V+\alpha)^{-1} \, (|\nabla\varphi|^2 - V - \lambda_1) \\
&\quad + \rho^2 \, (V+\alpha)^{-1} \, (V - \lambda_1) \\
&= G + \rho^2 \, (V+\alpha)^{-1} \, (V - \lambda_1)
\end{aligned}
\tag{2.6}
$$

Hence

$$
\begin{aligned}
\frac{2}{n}G^2 &\leq \rho^4 \, (V+\alpha)^{-2} \, \Delta V \\
&\quad - \rho^2 \, (V+\alpha)^{-1} \, G[2\Delta \log \rho - \Delta \log(V+\alpha) - 4|\nabla \log \rho|^2 \\
&\qquad\quad - |\nabla \log(V+\alpha)|^2 + 2\nabla \log(V+\alpha) \cdot \nabla \log \rho] \\
&\quad + 4\rho(V+\alpha)^{-\frac{1}{2}}
\end{aligned}
\tag{2.7}
$$

Therefore,

either

$$G \le \frac{4}{n^2} \, \rho^2 (V + \alpha)^{-1} \, |\nabla \log \rho|^2 \tag{2.8}$$

or

$$G \le 3\sqrt{\frac{n}{2}} \, \rho^2 \, (V + \alpha)^{-1} \, \sqrt{(\Delta V)_+} \tag{2.9}$$

or

$$
\begin{aligned}
G \quad \le \quad & \frac{3n}{2} \, \rho^2 \, (V + \alpha)^{-1} \, [-2\Delta \log \rho + 4|\nabla \log \rho|^2 + \Delta \log(V + \alpha) \\
& + |\nabla \log(V + \alpha)|^2 - 2\nabla \log(V + \alpha) \cdot \nabla \log \rho + 4(V - \lambda_1)^{\frac{1}{2}}]
\end{aligned}
\tag{2.10}
$$

**Theorem 2.1** *Let $u_1$ be a positive solution of $(\Delta + V)u_1 = \lambda_1 u$ and $\varphi = -\log u_1$. Then*

$$
\begin{aligned}
& \rho^2 \, (V + \alpha)^{-1}(|\nabla \varphi|^2 - V + \lambda_1) \\
& \quad \le 10n \, \rho^2 \, (V + \alpha)^{-1} \, (|\nabla \log \rho|^2 + |\Delta \log \rho|) \\
& \quad + \tfrac{3n}{2} \, \rho^2 \, (V + \alpha)^{-1} \, \left\{ (\Delta \log(V + \alpha))_+ + 2|\nabla \log(V + \alpha)|^2 + 4(V - \lambda_1)^{\frac{1}{2}} \right\}
\end{aligned}
\tag{2.11}
$$

*In particular,*

$$
\begin{aligned}
\rho^2 \, (V + \alpha)^{-1}|\nabla \varphi|^2 \quad \le \quad & \frac{\sup V - \lambda_1}{\sup V + \alpha} \\
& + 10n^2 \sup(V + \alpha)^{-1} \, \rho^2 \, (|\nabla \log \rho|^2 + |\Delta \log \rho|) \\
& + 3n \sup \left[ (V + \alpha)^{-2} \, ((\Delta V)_+) + (V + \alpha)^{-3} \, |\nabla V|^2 \right] \\
& + 6n \sup(V + \alpha)^{-\frac{1}{2}} \, \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right)^{\frac{1}{2}}
\end{aligned}
$$

# 3   Gradient estimate for $\dfrac{u_2}{u_1}$

Let $u_2$ be the second eigenfunction of $-\Delta + V$ on $\Omega$. Let $u = \dfrac{u_2}{u_1}$.
Then

$$\Delta u = -(\lambda_2 - \lambda_1)u + 2\nabla \varphi \, \nabla u \tag{3.1}$$

Let $c > \sup\limits_{\Omega_2} \dfrac{u_2}{u_1}$ and $\psi = -\ln(c - u)$.

Then

$$\Delta\psi = (\lambda_2 - \lambda_1)(1 - ce^{\psi}) + 2\nabla\varphi\nabla\psi + |\nabla\psi|^2. \tag{3.2}$$

Let

$$F = \rho^2(V + \alpha)^{-1}\left[|\nabla\psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^{\psi})\right] \tag{3.3}$$

when $\alpha > 0$ is a constant to be chosen.

Then

$$\begin{aligned}
\nabla F \;=\;& 2F\nabla\log\rho - F\nabla\log(V + \alpha) \\
&+ \rho^2(V + \alpha)^{-1}\left[2\nabla\psi\,\nabla\nabla\psi - c(\lambda_2 - \lambda_1)e^{\psi}\nabla\psi\right]
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\Delta F \;=\;& \nabla F\left(2\nabla\log\rho - \nabla\log(V + \alpha)\right) + F\left(2\Delta\log\rho - \Delta\log(V + \alpha)\right) \\
&+ 4\rho(V + \alpha)^{-1}\nabla\rho\cdot\nabla\psi\cdot\nabla\nabla\psi - 2\rho^2(V + \alpha)^{-2}\,\nabla V\cdot\nabla\nabla\psi\cdot\nabla\psi \\
&+ 2\rho^2(V + \alpha)^{-1}\,|\nabla\nabla\psi|^2 + 2\rho^2(V + \alpha)^{-1}\,\nabla\psi\cdot\nabla(\Delta\psi) \\
&- 2c\rho(V + \alpha)(\lambda_2 - \lambda_1)e^{\psi}\,\nabla\rho\cdot\nabla\psi + c\rho^2(V + \alpha)^{-2}(\lambda_2 - \lambda_1)e^{\psi}\,\nabla V\cdot\nabla\psi \\
&- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi}\,|\nabla\psi|^2 \\
&- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi}\left[|\nabla\psi|^2 + 2\nabla\varphi\cdot\nabla\psi + (\lambda_2 - \lambda_1)(1 - ce^{\psi})\right] \\
\;=\;& \nabla F(2\nabla\log\rho - \nabla\log(V + \alpha)) + F(2\Delta\log\rho - \Delta\log(V + \alpha)) \\
&+ 2\nabla F\cdot\nabla\log\rho - 4F|\nabla\log\rho|^2 + 2F\nabla\log(V + \alpha)\cdot\nabla\log\rho \\
&+ 2c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi}\,\nabla\psi\cdot\nabla\log\rho - (V + \alpha)^{-1}\,\nabla F\cdot\nabla V \\
&+ 2F\nabla\log\rho\cdot\nabla\log(V + \alpha) - F|\nabla\log(V + \alpha)|^2 \\
&- c\rho^2(V + \alpha)^{-2}(\lambda_2 - \lambda_1)e^{\psi}\,\nabla\psi\cdot\nabla V + 2\rho^2(V + \alpha)^{-1}|\nabla\nabla\psi|^2 \\
&+ 2\rho^2(V + \alpha)^{-1}(2\nabla\psi\cdot\nabla\nabla\psi\cdot\nabla\psi + 2\nabla\psi\cdot\nabla\nabla\psi\cdot\nabla\psi \\
&\qquad + 2\nabla\psi\cdot\nabla\psi\cdot\nabla\nabla\psi - c(\lambda_2 - \lambda_1)e^{\psi}|\nabla\psi|^2) \\
&- 2c\rho(V + \alpha)(\lambda_2 - \lambda_1)e^{\psi}\nabla\psi\cdot\nabla\psi + c\rho^2(V + \alpha)^{-2}(\lambda_2 - \lambda_1)e^{\psi}\,\nabla V\cdot\nabla\psi \\
&- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi}|\nabla\psi|^2 \\
&- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi}\left[|\nabla\psi|^2 + 2\nabla\varphi\cdot\nabla\psi + (\lambda_2 - \lambda_1)(1 - ce^{\psi})\right]
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
= \ & \nabla F(4\nabla \log \rho - 2\nabla \log(V+\alpha)) + F(2\Delta \log \rho - \Delta \log(V+\alpha) - 4|\nabla \log \rho|^2 \\
& \quad + 4\nabla \log(V+\alpha) \cdot \nabla \log \rho - |\nabla \log(V+\alpha)|^2) \\
& + 2\rho^2(V+\alpha)^{-1} \ |\nabla\nabla\psi|^2 + 2\nabla F \cdot \nabla\psi - 2F\nabla \log \rho \cdot \nabla\psi \\
& - 2F\nabla \log(V+\alpha) \cdot \nabla\psi + 4\rho^2(V+\alpha)^{-1} \ \nabla\psi \cdot \nabla\nabla\psi \cdot \nabla\psi + 2\nabla F \cdot \nabla\varphi \\
& - 2F\nabla \log \rho \cdot \nabla\varphi + 2F\nabla \log(V+\alpha)\nabla\varphi \\
& + 2c\rho^2(V+\alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi} \ \nabla\psi \cdot \nabla\varphi \\
& - c\rho^2(V+\alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi}|\nabla\psi|^2 \\
& - c\rho^2(V+\alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi} \left[|\nabla\psi|^2 + 2\nabla\varphi \cdot \nabla\psi + (\lambda_2 - \lambda_1)(1 - ce^{\psi})\right] \\
= \ & \nabla F(4\nabla \log \rho - 2\nabla \log(V+\alpha) + 2\nabla\psi + 2\nabla\psi) \\
& + F(2\Delta \log \rho - \Delta \log(V+\alpha) - 4|\nabla \log \rho|^2 \\
& \quad + 4\nabla \log(V+\alpha) \cdot \nabla \log \rho - |\nabla \log(V+\alpha)|^2 - 2\nabla \log \rho \cdot \nabla\psi \\
& \quad - 2\nabla \log(V+\alpha) \cdot \nabla\psi - 2\nabla \log \rho \cdot \nabla\varphi + 2\nabla \log(V+\alpha) \cdot \nabla\varphi) \\
& + 2\rho^2(V+\alpha)^{-1} \ |\nabla\nabla\psi|^2 + 4\rho^2(V+\alpha)^{-1} \ \nabla\psi \cdot \nabla\nabla\psi \cdot \nabla\psi \\
& - 2c\rho^2(V+\alpha)^{-1}(\lambda_2 - \lambda_1)e^{\psi} \ |\nabla\psi|^2 - c(\lambda_2 - \lambda_1)^2 \ \rho^2 \ (V+\alpha)^{-1} \ e^{\psi}(1 - ce^{\psi})
\end{aligned}
$$

Now

$$
|\nabla\nabla\psi|^2 \geq \frac{1}{n}(\Delta\psi)^2 \tag{3.6}
$$

and

$$
\Delta\psi - 2\nabla\psi \cdot \nabla\varphi = \rho^{-2} \ (V+\alpha)F \tag{3.7}
$$

Hence

$$
\begin{aligned}
\rho^2(V+\alpha)^{-1} \ |\nabla\nabla\psi|^2 \geq \ & \rho^2(V+\alpha)^{-1} \\
& \left[(V+\alpha)^2 \ \rho^{-4} \ F^2 + 4\rho^{-2} \ (V+\alpha)F\nabla\psi \cdot \nabla\varphi + 4(\nabla\psi \cdot \nabla\varphi)^2\right]
\end{aligned} \tag{3.8}
$$

When $F$ achieves its maximum, $\nabla F = 0$ and $\Delta F \leq 0$. Therefore,

$$
\begin{aligned}
0 \;\geq\; & F\big(2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 + 4\nabla\log(V+\alpha)\cdot\nabla\log\rho \\
& -|\nabla\log(V+\alpha)|^2 - 2\nabla\log\rho\cdot\nabla\psi - 2\nabla\log(V+\alpha)\cdot\nabla\psi \\
& -2\nabla\log\rho\cdot\nabla\varphi + 2\nabla\log(V+\alpha)\cdot\nabla\varphi\big) \\
& +\frac{2}{n}\rho^{-2}\,(V+\alpha)F^2 + \frac{8}{n}F\,\nabla\psi\cdot\nabla\varphi + \frac{8}{n}\rho^2(V+\alpha)^{-1}\,(\nabla\psi\cdot\nabla\varphi)^2 \quad (3.9) \\
& +4\rho^2(V+\alpha)^{-1}\,\nabla\psi\cdot\nabla\nabla\varphi\cdot\nabla\psi \\
& -2c\rho^2(V+\alpha)^{-1}\,(\lambda_2-\lambda_1)e^{\psi}\left[|\nabla\psi|^2 + (\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\
& +c(\lambda_2-\lambda_1)^2\,\rho^2\,(V+\alpha)^{-1}\,e^{\psi}(1-ce^{\psi})
\end{aligned}
$$

Note that

$$
\rho^2(V+\alpha)^{-1}\,\nabla\psi\cdot\nabla\nabla\varphi\cdot\nabla\psi \geq \inf\big(\rho^2(V+\alpha)^{-1}\,g(x)\big)F \qquad (3.10)
$$

where $g$ is defined in Theorem 1.1.
Hence

$$
\begin{aligned}
0 \;\geq\; & \frac{2}{n}F^2 + 4\inf\big(\rho^2\,(V+\alpha)^{-1}\,g(x)\big)F \\
& -F\big(\rho^2\,(V+\alpha)^{-1}\,|\nabla\psi|^2\big) \\
& \quad \left[\tfrac{8}{n}\big(\rho^2\,(V+\alpha)^{-1}\,|\nabla\varphi|^2\big)^{\frac{1}{2}} + 2\big(\rho^2\,(V+\alpha)^{-1}\,|\nabla\log\rho|^2\big) \right. \\
& \quad\quad \left. +2\big(\rho^2\,(V+\alpha)^{-1}\,|\nabla\log(V+\alpha)|^2\big)^{\frac{1}{2}}\right] \qquad (3.11) \\
& +\rho^2\,(V+\alpha)^{-1}\,F \\
& \quad \left[2\Delta\log\rho - 7|\nabla\log\rho|^2 - 2|\nabla\varphi|^2 - \Delta\log(V+\alpha) - |\nabla\log(V+\alpha)|^2\right] \\
& -2c\rho^2(V+\alpha)^{-1}\,(\lambda_2-\lambda_1)e^{\psi}\,F \\
& +c(\lambda_2-\lambda_1)^2\,\rho^{\psi}\,(V+\alpha)^{-2}\,e^{\psi}\,(1-ce^{\psi})
\end{aligned}
$$

By definition of $F$, either

$$
\rho^2\,(V+\alpha)^{-1}\,|\nabla\psi|^2 \leq 2F \qquad (3.12)
$$

or

$$
F \leq (\lambda_2-\lambda_1)(V+\alpha)^{-1}\,(1-ce^{\psi}) \qquad (3.13)
$$

Let us assume (3.12) first. In that case, either

$$
\begin{aligned}
F^{\frac{1}{2}} \leq{} & \frac{3n}{2}\left[\frac{8}{n}\left(\rho^2\,(V+\alpha)^{-1}\,|\nabla\varphi|^2\right)^{\frac{1}{2}}+2\left(\rho^2\,(V+\alpha)^{-1}\,|\nabla\log\rho|^2\right)^{\frac{1}{2}}\right] \\
& +2\left(\rho^2\,(V+\alpha)^{-1}\,|\nabla\log(V+\alpha)|^2\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.14}
$$

or

$$
\begin{aligned}
& F+6n\inf\left(\rho^2\,(V+\alpha)^{-1}\,g(x)\right) \\
& \leq \frac{3n}{2}\rho^2\,(V+\alpha)^{-1}\Big[2\Delta\log\rho+7|\nabla\log\rho|^2+\Delta\log(V+\alpha) \\
& \qquad +|\nabla\log(V+\alpha)|^2+2|\nabla\varphi|^2+2c(\lambda_2-\lambda_1)e^{\psi}\Big]
\end{aligned}
\tag{3.15}
$$

or

$$
F^2 \leq \frac{3n}{2}c(\lambda_2-\lambda_1)^2\,\rho^2\,(V+\alpha)^{-2}\,e^{\psi}\,|(1-ce^{\psi})|
\tag{3.16}
$$

From this, we conclude:

**Theorem 3.1** *Let $\Omega$ be a domain and $\rho$ be smooth function with compact support in $\Omega$. Let $u_i$ be smooth function satisfying the equation $(-\Delta+V)u_i=\lambda_i\,u_i$ so that $u_1>0$ and $\varphi=-\log u_1$ satisfies the conclusion of Theorem 1.1. Let $c$ be a constant so that $c>\sup u$ where $u=\dfrac{u_2}{u_1}$. Let $\psi=-\log(c-u)$. Then one of the following inequalities hold:*

*(1)*

$$
\begin{aligned}
& \rho^2(V+\alpha)^{-1}\left[|\nabla\psi|^2+(\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\
& \leq (\lambda_2-\lambda_1)\sup(V+\alpha)^{-1}(1-ce^{\psi})
\end{aligned}
\tag{3.17}
$$

*(2)*

$$
\begin{aligned}
& \rho^2\,(V+\alpha)^{-1}\left[|\nabla\psi|^2+(\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\
& \leq 144\left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right)+20n^2\,\sup(V+\alpha)^{-1} \\
& \qquad \rho^2\,\left(|\nabla\log\rho|^2+|\Delta\log\rho|\right) \\
& +4n\,\sup\left((V+\alpha)^{-2}\,(\Delta V)_++(V+\alpha)^{-3}\,|\nabla V|^2\right) \\
& +6n\,\sup(V+\alpha)^{-\frac{1}{2}}\,\left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.18}
$$

*(3)*

$$\rho^2 \left[(V + \alpha)^{-1} |\nabla\psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi)\right]$$
$$+ 6n \inf\left(\rho^2 (V + \alpha)^{-1} g(x)\right)$$
$$\leq \quad 20n^2 \sup(V + \alpha)^{-1} \left(|\nabla\rho|^2 + \rho|\Delta\rho|\right)$$
$$+ 10n \sup \rho^2 \left[(V + \alpha)^{-2} (\Delta V)_+ + (V + \alpha)^{-3} |\nabla V|^2\right] \qquad (3.19)$$
$$+ 10\left(\frac{\sup V - \lambda_1}{\sup V + \alpha}\right) + 6n \sup(V + \alpha)^{-\frac{1}{2}} \left(\frac{\sup V - \lambda_1}{\sup V + \alpha}\right)^{\frac{1}{2}}$$
$$+ 3nc(\lambda_2 - \lambda_1) \sup \rho^2 (V + \alpha)^{-1} e^\psi$$

*(4)*

$$\rho^2 (V + \alpha)^{-1} \left(|\nabla\psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi)\right)$$
$$\leq \quad \sqrt{3nc}\,(\lambda_2 - \lambda_1) \sup \rho^2 (V + \alpha)^{-1} \left[e^\psi \left(1 - ce^\psi\right)\right]^{\frac{1}{2}} \qquad (3.20)$$

# 4   Estimate of the gap $\lambda_2 - \lambda_1$ in terms of the potential

Let $u = \dfrac{u_2}{u_1}$ be defined as in §3. We assume that it is bounded on $\Omega$ and is zero somewhere in $\Omega$.

Assume that for some $\delta > 0$, $u(x_0) = \sup u$ and $u(x_1) = \delta \sup u$. Then for each smooth function $\rho$ with compact support and constant $\alpha \geq 0$, we can define

$$L(\rho, \alpha, \delta) = \inf_x \int_0^1 \left(\rho^{-1}\sqrt{V + \alpha}\right)(x(t)) |\dot{x}| dt \qquad (4.1)$$

where $x$ is any path in $\Omega$ with $x(0) = x_0$ and $x(1) = x_1$.

Now we can define

$$L_\delta \quad = \quad \inf_{\alpha,\rho} L(\rho, \alpha, \delta) \Big\{ 20n^2 \sup(V + \alpha)^{-1} \rho^2 \left(|\Delta\rho| + |\nabla\rho|^2\right)$$
$$+ 10n \sup \rho^2 (V + \alpha)^{-3} \left((\Delta V)_+ + |\nabla V|^2\right) + 10\frac{\sup V - \lambda_1}{\sup V + \alpha}$$
$$+ 6n \sup(V + \alpha)^{-\frac{1}{2}} \left(\frac{\sup V - \lambda_1}{\sup V + \alpha}\right)^{\frac{1}{2}} - 6n \inf\left(\rho^2 (V + \alpha)^{-1} g(x)\right) \Big\}$$

where $\alpha > 0$ is a constant and $\rho$ is any smooth function with compact support in $\Omega$.

Based on Theorem 3.1, we conclude that

**Theorem 4.1**

$$\left|\log \frac{1}{\delta}\right| \le L_\delta + (\lambda_2 - \lambda_1) \left(\frac{1}{\delta} + \inf_{\rho,\alpha} \left\{L(\rho,\alpha,\delta) \sup \frac{\rho^2 (V+\alpha)^{-1}}{\delta}\right\}\right)$$

In particular if for some $\delta$, $\log \dfrac{1-\delta}{\delta} - L_\delta > 0$, there is a lower estimate of $\lambda_2 - \lambda_1$, in terms of $L_\delta$ and $\inf\limits_{\rho,\alpha} \left\{L(\rho,\alpha,\delta) \sup \left(\dfrac{\rho^2 (V+\alpha)^{-1}}{\delta}\right)\right\}$.

# 5  Oscillation of the function $\dfrac{u_2}{u_1}$

Note that in §4, we do not need to assume $u_i$ satisfies any boundary coordinates on $\Omega$.

If we assume $u_i = 0$ on $\partial\Omega$, $u_1 > 0$ and

$$\int_\Omega u_i^2 = 1 \tag{5.1}$$

$$\int_\Omega u_1 u_2 = 0 \tag{5.2}$$

we find

$$\int_\Omega u^2 u_1^2 = 1 \tag{5.3}$$

$$\int_\Omega u u_1^2 = 0 \tag{5.4}$$

The eigenfunction equations give

$$\int_\Omega |\nabla u_i|^2 + \int_\Omega V u_i^2 = \lambda_i \tag{5.5}$$

Note that assuming (5.1), (5.2) can also be written as

$$\int_\Omega (u_1 + u_2)^2 = 2 \tag{5.6}$$

or

$$\int_\Omega (u_1 - u_2)^2 = 2 \tag{5.7}$$

Let

$$\Omega_t = \{x \in \Omega | \, d(x_1 \, \partial\Omega) \geq t\} \tag{5.8}$$

We are interested in the behavior of $\int_{\Omega_t} u_i^2$ and $\int_{\Omega_t} u_1 \, u_2$. Hence we compute

$$
\begin{aligned}
\frac{d^2}{dt^2} \int_{\Omega_t} u_i^2 &= -\frac{d}{dt} \int_{\delta\Omega_t} u_i^2 \\
&= \int_{\partial\Omega_t} u_i^2 \, H_t + 2 \int_{\partial\Omega_t} u_i \frac{\partial u_i}{\partial \nu} \\
&= \int_{\partial\Omega_t} u_i^2 H_t + \int_{\Omega_t} \Delta(u_i^2)
\end{aligned}
\tag{5.9}
$$

where $H_t$ is the mean curvature of $\partial\Omega_t$, measured by the outward normal.

But

$$
\begin{aligned}
\int_{\Omega_t} \Delta(u_i)^2 &= 2 \int_{\Omega_t} |\nabla u_i|^2 + 2 \int_{\Omega_t} u_i \Delta u_i \\
&= 2 \int_{\Omega_t} (|\nabla u_i|^2 + V u_i^2) - 2\lambda_i \int_{\Omega_t} u_i^2 \\
&= 2 \inf(V - \lambda_i)
\end{aligned}
\tag{5.10}
$$

Since $u_i = 0$ on $\partial\Omega$,

$$\frac{d}{dt} \int_{\Omega_t} u_i^2 = 0 \tag{5.11}$$

where $t = 0$.

We conclude that

$$
\begin{aligned}
\int_{\Omega_t} u_i^2 &= \int_0^t \frac{d}{dt} \left( \int_{\Omega_t} u_i^2 \right) + \int_\Omega u_i^2 \\
&= \int_0^t \left( \int_0^s \frac{d^2}{ds^2} \left( \int_{\Omega_s} u_i^2 \right) \right) + \int_\Omega u_i^2 \\
&\geq \inf(V - \lambda_i)t^2 + 1
\end{aligned}
\tag{5.12}
$$

Similarly,

$$
\begin{aligned}
\frac{d^2}{dt^2} &\int_{\Omega_t} (u_1 + u_2)^2 \\
\geq\ & 2\int_{\Omega_t} (\nabla u_1 + \nabla u_2)^2 + 2\int_{\Omega_t} (u_1 + u_2)V(u_1 + u_2) \\
& -2\int_{\Omega_t} (u_1 + u_2)(\lambda_1\, u_1 + \lambda_2\, u_2) \\
\geq\ & 2\int_{\Omega_t} \left(V - \frac{\lambda_1 + \lambda_2}{2}\right)(u_1 + u_2)^2 + (\lambda_2 - \lambda_1)\int_{\Omega_t} u_1^2 + (\lambda_1 - \lambda_2)\int_{\Omega_t} u_2^2 \\
\geq\ & 4\inf\left(V - \frac{(\lambda_1 + \lambda_2)}{2}\right) + (\lambda_1 - \lambda_2) \\
\geq\ & 4\inf(V - \lambda_2)
\end{aligned}
\tag{5.13}
$$

$$
\int_{\Omega_t} (u_1 + u_2)^2 \ \geq\ 2 + 2\left(\inf_\Omega V - \lambda_2\right)t^2
\tag{5.14}
$$

$$
\int_{\Omega_t} (u_1 - u_2)^2 \ \geq\ 2 + 2\left(\inf_\Omega V - \lambda_2\right)t^2
\tag{5.15}
$$

From (5.14), we obtain

$$
\int_{\Omega_t} u_1\, u_2 \geq \left[\inf_\Omega(V - \lambda_2)\right] t^2
\tag{5.16}
$$

From (5.15), we obtain

$$
\int_{\Omega_t} u_1\, u_2 \leq -\left[\inf_\Omega(V - \lambda_2)\right] t^2
\tag{5.17}
$$

Let $u = \dfrac{u_2}{u_1}$. Then from (5.12),

$$
\begin{aligned}
\int_{\Omega_t} u^2\, u_1^2 \ &=\ \int_{\Omega_t} u_2^2 \\
&\geq\ 1 + \inf_\Omega(V - \lambda_2)t^2
\end{aligned}
\tag{5.18}
$$

Hence

$$
\sup_{\Omega_t} u^2 \geq 1 + \inf_\Omega(V - \lambda_2)t^2
\tag{5.19}
$$

On the other hand,

$$
\begin{aligned}
\inf_{\Omega_t} |u| &\leq \left| \int_{\Omega_t} u\, u_1^2 \right| \\
&\leq t^2 \left[ -\inf_{\Omega}(V - \lambda_2) \right]
\end{aligned}
\tag{5.20}
$$

Combining (5.19) and (5.20), we conclude

$$
\frac{\inf_{\Omega_t} |u|}{\sup_{\Omega_t} |u|} \leq \frac{t^2 \left[ -\inf_{\Omega}(V - \lambda_2) \right]}{1 + \inf_{\Omega}(V - \lambda_1) t^2}
\tag{5.21}
$$

In the next section, we shall apply the estimates in §4.

# 6   Distance function and the estimate of the gap

For simplicity, we shall assume that $\Omega$ to be convex in this section. We also assume that

$$
|\nabla V| \leq c_\alpha (V + \alpha)^{\frac{3}{2}}
\tag{6.1}
$$

$$
|(\Delta V)_+| \leq c_\alpha (V + \alpha)^3
\tag{6.2}
$$

We introduce

$$
d_\alpha(x_0, x_1) = \inf \int_0^1 \sqrt{V + \alpha}(x(t))\, |\dot{x}| dt
\tag{6.3}
$$

where the infinitum is taken over all paths $x : [0,1] \to \Omega$ joining $x_0$ to $x_1$.
If

$$
u(x_0) = \sup u
\tag{6.4}
$$

and

$$
u(x_1) = \delta \sup u
\tag{6.5}
$$

We define $L_\Omega(\delta)$ to be $d(x_0, x_1)$. It is of course dominated by

$$
L(\Omega) = \sup_{\tilde{x_0}, \tilde{x_1} \in \Omega} d(\tilde{x_0}, \tilde{x_1})
\tag{6.6}
$$

Using the terminology of §4 and §5, we set $\rho$ to be function of $t$ so that $\rho = 0$ on $\partial\Omega$ and $\rho = 1$ when $t \geq \left( \inf_{\partial\Omega} V \right)^{-\frac{1}{2}}$.

We can assume

$$\rho^2 \left( |\nabla \log \rho|^2 + |\Delta \log \rho| \right) \leq 3 \left( \inf_{\partial\Omega} V \right) \tag{6.7}$$

Note that

$$
\begin{aligned}
\frac{1}{V + \alpha} - \frac{1}{\inf_{\partial\Omega} V + \alpha} &\leq \frac{|\nabla V|}{(V + \alpha)^2} t \\
&\leq \frac{c_\alpha t}{(V + \alpha)^{\frac{1}{2}}} \\
&\leq \frac{c_\alpha t}{(\inf_{\partial\Omega} V + \alpha)^{\frac{1}{2}}} + \frac{1}{2} c^2 t^2
\end{aligned}
\tag{6.8}
$$

Hence,

$$\sup(V + \alpha)^{-1} \rho^2 (\Delta\rho + |\nabla\rho|^2) \leq \frac{3(\inf V)}{(\inf_{\partial\Omega} V + \alpha)} + \frac{c_\alpha(\inf_{\partial\Omega} V)^{\frac{1}{2}}}{(\inf_{\partial\Omega} V + \alpha)^{\frac{1}{2}}} + \frac{3c_\alpha^2}{2} \tag{6.9}$$

From (5.21), we obtain

$$\frac{\inf_{\Omega_t} |u|}{\sup_{\Omega_t} |u|} \leq \frac{-\inf_\Omega(V - \lambda_2)}{\inf_{\partial\Omega} V + \inf_\Omega(V - \lambda_1)} \tag{6.10}$$

Now assume

$$|\inf_\Omega(V - \lambda_2)| \leq \varepsilon \inf_{\partial\Omega} V \tag{6.11}$$

Then

$$\frac{\inf_{\Omega_t} |u|}{\sup_{\Omega_t} |u|} \leq \frac{\varepsilon}{1 - \varepsilon} \tag{6.12}$$

According to Theorem 4.1, we have proved

**Theorem 6.1** *Assume (6.1), (6.2) and (6.11) and $t = (\inf_{\partial\Omega} V)^{-\frac{1}{2}}$*

$$\left| \log\left( \frac{\varepsilon}{1 - \varepsilon} \right) \right| \leq \tilde{c}_\alpha L(\Omega_t) + \frac{2(\lambda_2 - \lambda_1)}{\varepsilon} \left[ 1 + L(\Omega_t) \sup_\Omega (V + \alpha)^{-1} \right] \tag{6.13}$$

*Here $\tilde{c}_\alpha$ depends on $c_\alpha$ and $-\inf((V + \alpha)^{-1} g(x))$.*

Note that when $V$ grows fast

$$\left| \log \left( \frac{\varepsilon}{1 - \varepsilon} \right) \right| > \tilde{c}_\alpha \, L(\Omega_t)$$

and we have a lower bound for $\lambda_2 - \lambda_1$ from (6.13).

If we know the location of the points to achieve $\inf_{\Omega_t}(u) = u(x_0)$ and $\sup_{\Omega_t} |u| = u(x_1)$, then we can replace $L(\Omega_t)$ by $d_\alpha(x_0, x_1)$. This can be applied when we have the double well potential.

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