The *Variational Principle* for Locally Compact Separable Metrizable Spaces

(joint with Mauro Patrão)

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University of Brasília

3rd July 2016

References

Topological entropy — accepted by ETDS

- `http://andrec.mat.unb.br/publications/`
- `DOI:10.1017/etds.2016.45` (not active, yet)

Topological Pressure

`arXiv:1605.01698`

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One Letter in Binary

A	00
B	01
C	10
D	11

How many bits do we need?

If we don't know the probabilities,

$$\log_2 \#\Gamma$$

With the probabilities, the average bit size is

$$\sum_{\gamma \in \Gamma} \mu(\gamma) s(\gamma).$$

The best s would be

$$s(\gamma) = \log_2 \frac{1}{\mu(\gamma)}$$

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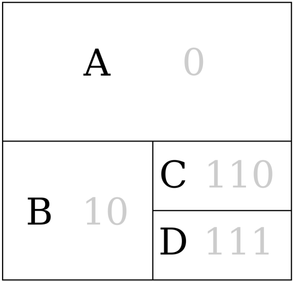

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C	110
D	111

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Measure theoretic "amount of information" per letter

$$H_\mu = \sum_{\gamma \in \Gamma} \mu(\gamma) \log_2 \frac{1}{\mu(\gamma)}$$

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e average bit

Topological "amount of information" per letter

$$H = \log_2 \#\Gamma$$

The best s w

$\mu(\gamma)$

One Letter in Binary

A	0
B	10

Comparing both...

$$H_\mu \leq H$$

How many bits do we need?

If we don't know the probabilities,

... average bit

Measure theoretic "amount of information" per letter

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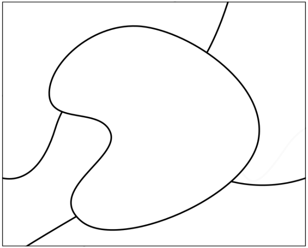
Grab a partition $\mathcal{C}$ and a cover $\mathcal{A}$



- $\mathcal{C}$: Borel measurable partition.
- $\mathcal{A}$: a cover for X.

Definition (Partition and Cover Entropy)

$$H_\mu(\mathcal{C}) = \sum_{C \in \mathcal{C}} \mu(C) \log \frac{1}{\mu(C)}$$

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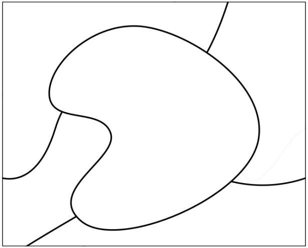

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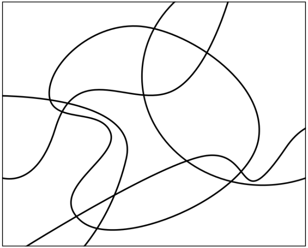

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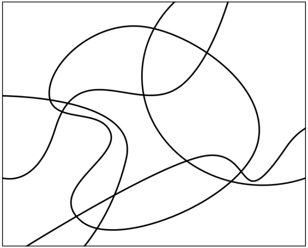

Where $N(\mathcal{A})$ is...

Least cardinality amongst all sobcovers of $\mathcal{A}$.

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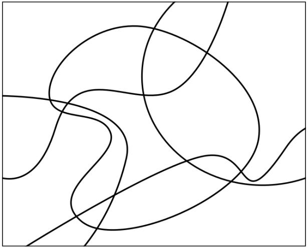

Property

$$H_\mu(\mathcal{C}) \leq H(\mathcal{C})$$

$$\mathcal{A} \prec \mathcal{B} \Rightarrow H(\mathcal{A}) \leq H(\mathcal{B})$$

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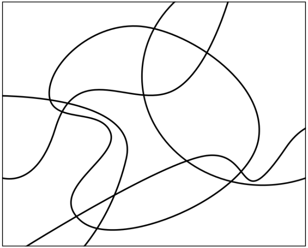

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The First n-Iteractions: $\mathcal{A}^n$

- $T : X \to X$: continuous system.
- μ: T-invariant Borel finite measure.

Definition ($\mathcal{A}^n$ and $\mathcal{C}^n$)

$$\mathcal{A}^n = \left\{ A_1 \cap T^{-1}A_2 \cap \cdots \cap T^{-n+1}A_n \,\middle|\, A_j \in \mathcal{A} \right\}$$

Definition (Partition and Cover Entropy Regarding T)

$$h_\mu\left(T \,\middle|\, \mathcal{C}\right) = \lim \frac{1}{n} H_\mu\left(\mathcal{C}^n\right)$$

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- $T : X$ –
- μ: T-inv

Definition ($\mathcal{A}$

$\mathcal{C}^n$: … $\mathcal{C}\}$

Definition (Entropy)

$$h_\mu(T) = \sup_{\mathcal{C}} h_\mu\left(T \middle| \mathcal{C}\right)$$

$$h(T) = \sup_{\mathcal{A}:open} h\left(T \middle| \mathcal{A}\right)$$

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An open cover $\mathcal{A}$ such that there exists $A \in \mathcal{A}$ with $K = A^c$ compact.

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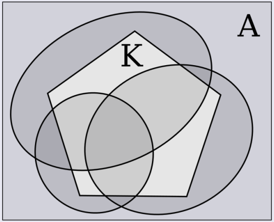

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Entropy of T^n

$$h_\mu(T^n) = nh_\mu(T)$$

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Cover Made of Balls: $\mathcal{B}$

For all balls with radius $\varepsilon > 0$

$$h^d(T) = \sup_{\varepsilon>0} h\left(T \,\middle|\, \mathcal{B}_d(\varepsilon)\right)$$

- It depends on the metric d.
- Can be stated in terms of separated sets.

$$h^d(T) = \sup_{\varepsilon>0} \lim_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon)$$

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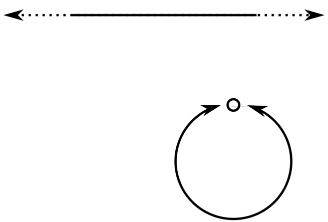

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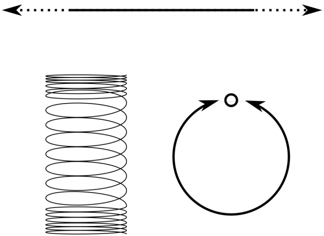

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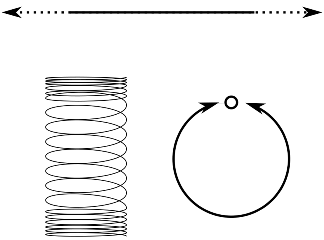

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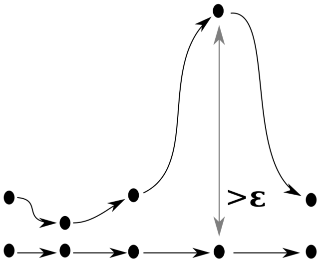

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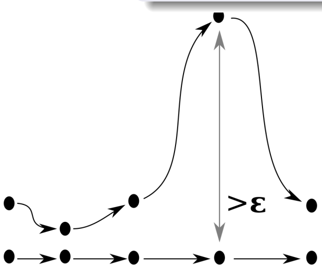
Cover Made of Balls: $\mathcal{B}$

For all balls with radius …

Separated Sets and Ball Covers

$$N\left(\left[\mathcal{B}_d\left(2\varepsilon\right)\right]^n\right) \leq s(n,\varepsilon) \leq N\left(\left[\mathcal{B}_d\left(\frac{\varepsilon}{2}\right)\right]^n\right)$$



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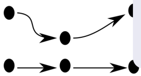
...c d.

...of

Therefore...

Dividing by n, taking the limit for n, and then the supremum for ε,

$$h^d\left(T\right) = \sup_{\varepsilon>0} \lim_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon)$$



$s(n, \varepsilon)$

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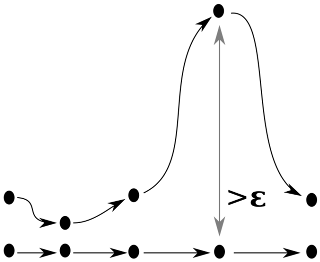

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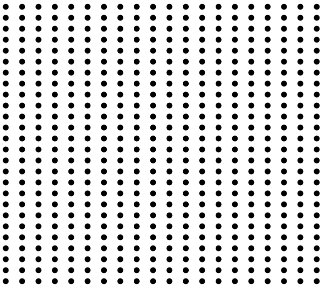
Bowen Entropy Uses Compact Sets



- Depending on the metric d, $s(n, \varepsilon)$ might be ∞.
- Bowen limited the counting to compact sets.
- We denote the Bowen entropy by

$$h_d(T).$$

- If d is totally bounded, $s(n, \varepsilon)$ will never be ∞.

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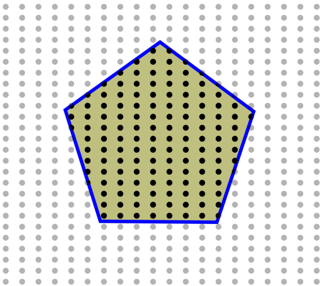

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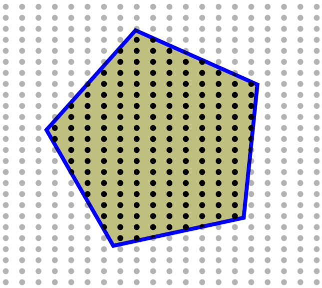

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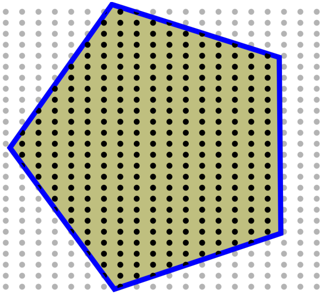

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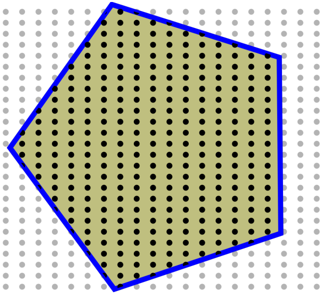

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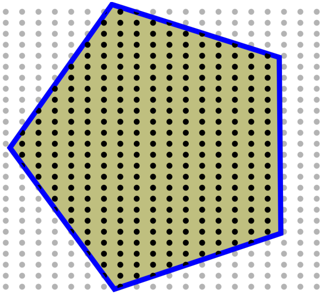

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If we are lucky. . .

$$\sup_{\mu} h_{\mu}(T) = h(T) = \inf_{d} h_d(T) = \inf_{d} h^d(T).$$

Imitate Misiurewicz

Easy Inequality

$$h_\mu(T) \le h(T).$$

- Just imitate Misiurewicz proof!
- $h_\mu\left(T \middle| \mathcal{C}\right) \sim h_\mu\left(T \middle| \mathcal{K}\right) \le h\left(T \middle| \mathcal{A}\right) + M \le h(T) + M.$
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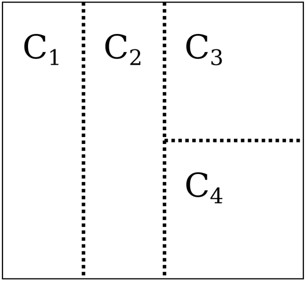

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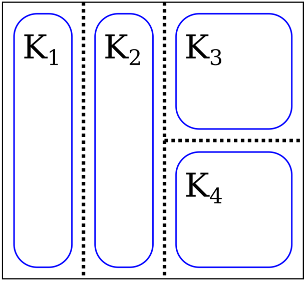

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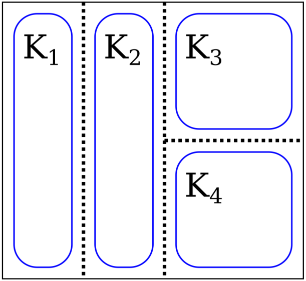

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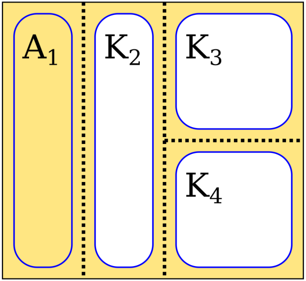

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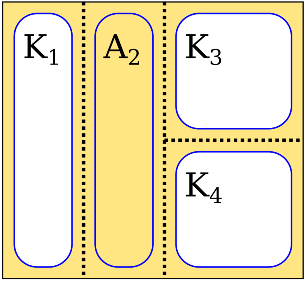

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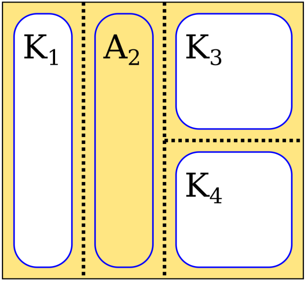

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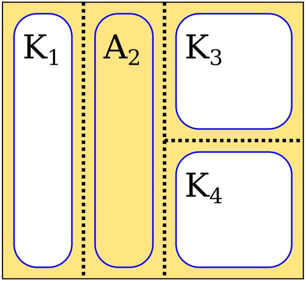

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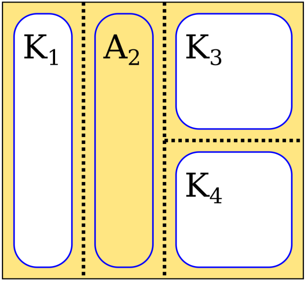

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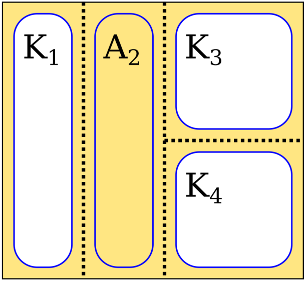

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Because...

- μ is inner regular.
- $h_\mu(T^n) = n h_\mu(T)$.
- $h(T^n) \leq n h(T)$.

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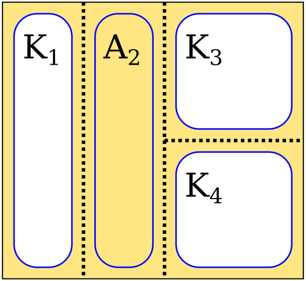

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Comparing with Bowen

Lemma (Lebesgue Number)

If (X, d) is a metric space, then for every admissible cover $\mathcal{A}$, there exists $\varepsilon > 0$ such that

$$\mathcal{A} \prec \mathcal{B}_d(\varepsilon).$$

Proposition

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We have proved that for any metric d of X, and for any Borel measure μ,

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For a metric d restricted from the one point compactification,

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E_n: sequence of (n, ε)-separated sets

We need an invariant measure μ and a partition $\mathcal{C}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \# E_n \leq h_\mu\left(T \middle| \mathcal{C}\right).$$

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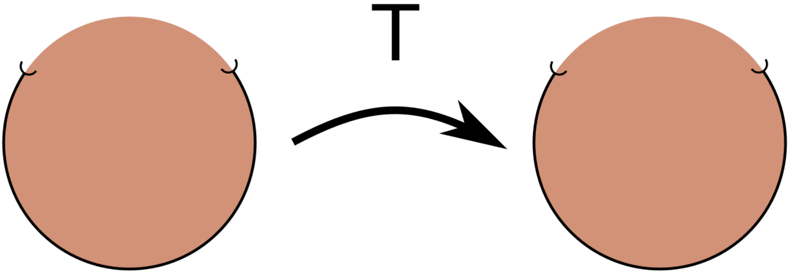
System Compactification



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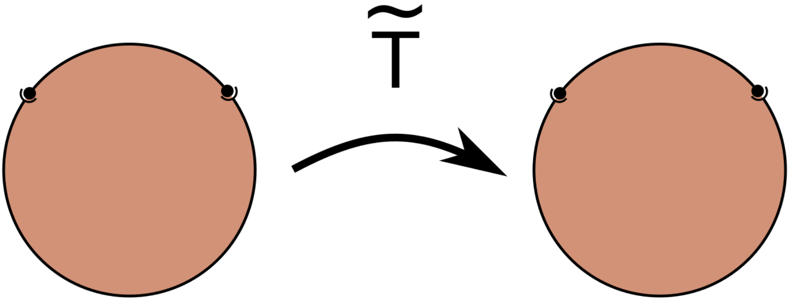

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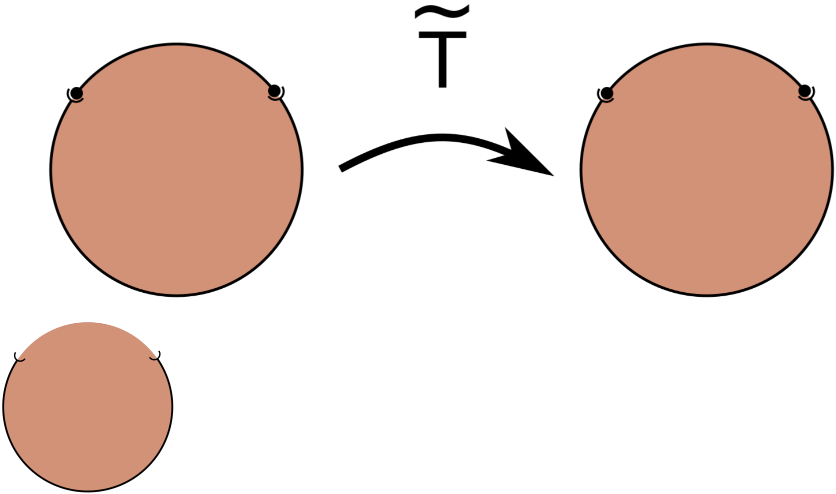

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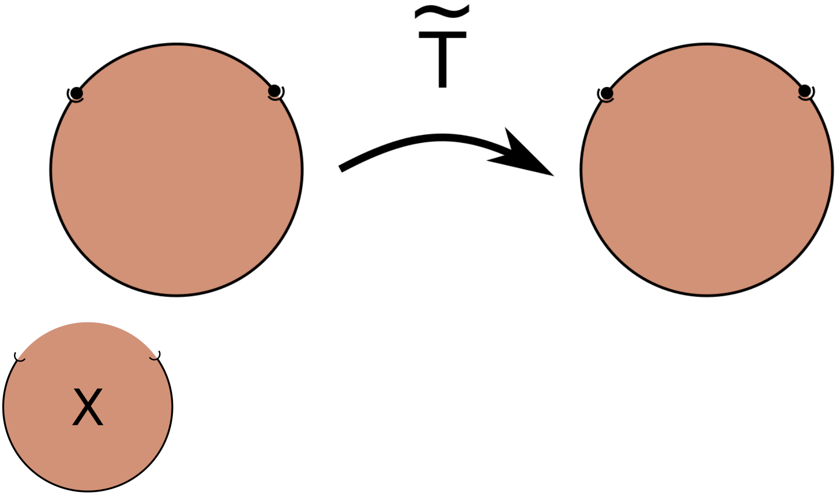

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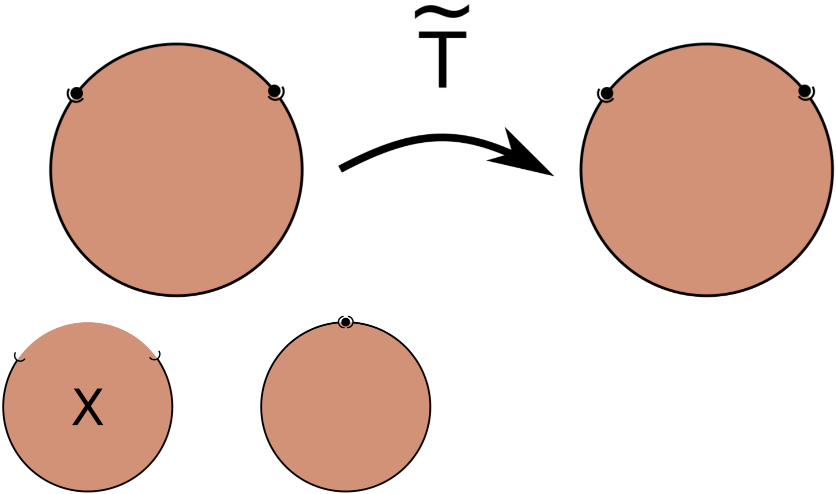

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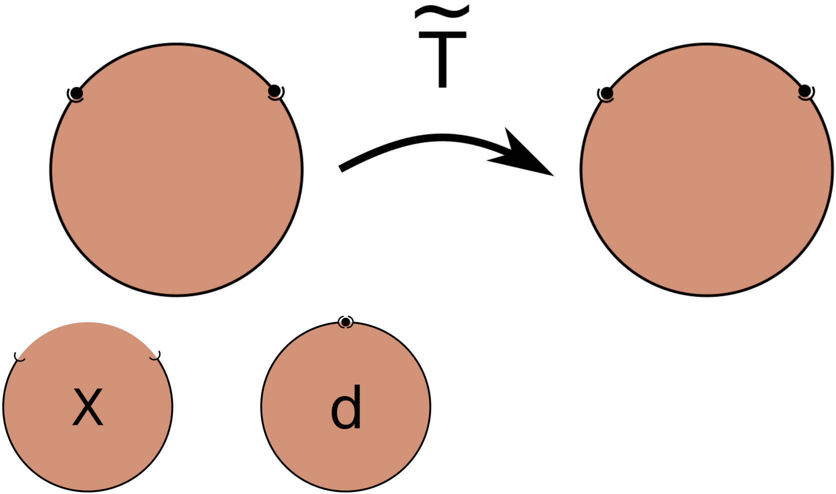

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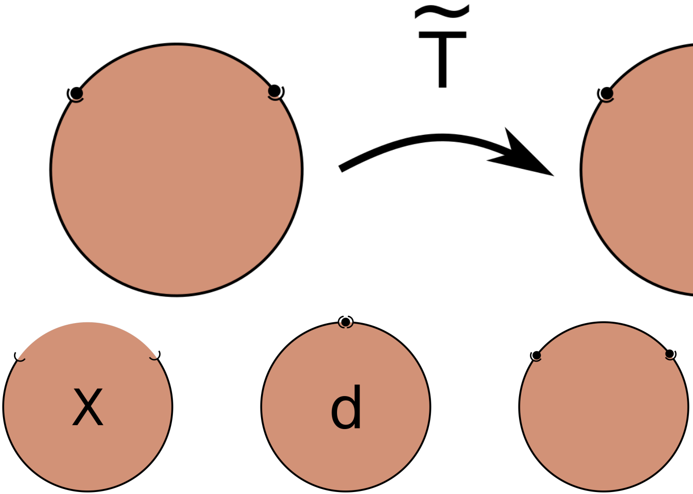

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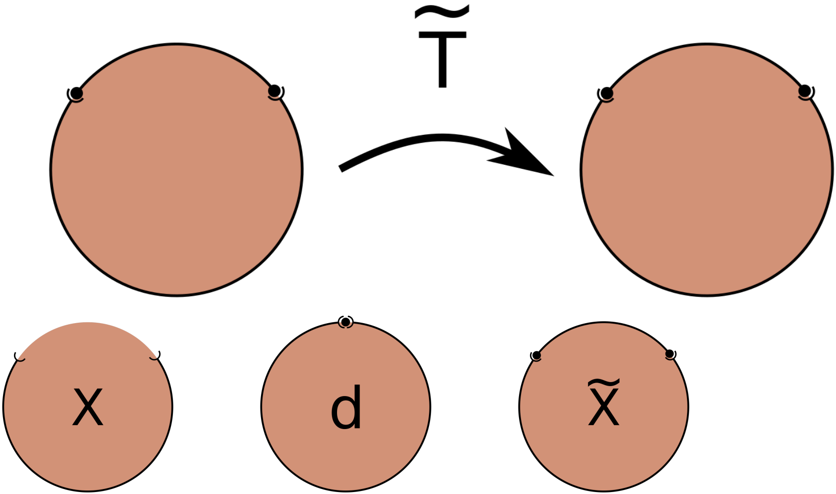

Partition Entropy

Given E_n... we want μ and $\mathcal{C}$...

$$\limsup_{n\to\infty} \frac{1}{n} \log \# E_n \leq h_\mu\left(T \,\middle|\, \mathcal{C}\right).$$

Lemma

- *Constructed μ is $\widetilde{T}$-invariant (and therefore, T-invariant).*
- *$\widetilde{\mathcal{C}}$ is properly choosen...*

$$h_\mu\left(T \,\middle|\, \mathcal{C}\right) = h_\mu\left(\widetilde{T} \,\middle|\, \widetilde{\mathcal{C}}\right).$$

Where $\mathcal{C} = X \cap \widetilde{\mathcal{C}}$.

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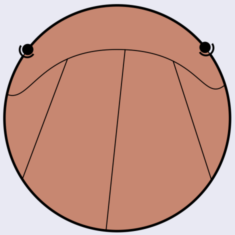

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For $\varepsilon > 0$ and a sequence E_n of (n, ε)-separated sets,

Define...

$$\sigma_n = \frac{1}{\# E_n} \sum_{x \in E_n} \delta_x, \quad \mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \sigma_n \circ T^{-j}$$

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From σ_n to μ

That is...

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \# E_n &\leq \liminf_{n\to\infty} \frac{1}{q} H_{\mu_n}\left(\widetilde{\mathcal{C}}^q\right) \\ &\leq \frac{1}{q} H_{\mu}\left(\widetilde{\mathcal{C}}^q\right) \\ &\to h_{\mu}\left(\widetilde{T} \,\middle|\, \widetilde{\mathcal{C}}\right) \\ &= h_{\mu}\left(T \,\middle|\, \mathcal{C}\right) \leq h_{\mu}\left(T\right)\end{aligned}$$

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Preassure and Topological Pressure

Finaly. . .

A similar variational principle holds to topological pressure for a similar generalization.

Also. . .

We can easily get rid of metrizability!!!

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Thank You

Contact

André Caldas <andre.em.caldas@gmail.com>

References

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- DOI:10.1017/etds.2016.45 (not active, yet)
- Topological pressure: arXiv:1605.01698