

# A Remark on the Decay of Correlations for Mixed Long Range Spin Vector Models

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## Abstract

In this short note we make it clear under what conditions the Theorem of reference [1] can be extended to spin-spin correlation functions of mixed short-long range ferromagnetic vector spin models in the disordered phase.

In the Concluding Remarks of Ref. [1], the authors claim that their Theorem, which concerns the long distance behavior of the connectivity function  $\tau_{xy}(\beta)$  of a mixed short-long range bond percolation model in the *subcritical regime*, i.e., for values of  $\beta$  such that  $\chi(\beta) \equiv \sum_x \tau_{0x}(\beta)$  is finite, is also valid for short-long range ferromagnetic spin models under the same condition  $\chi(\beta) < \infty$ , i.e., in the *disordered phase*. The aim of this short note is to make it clear under what conditions the claim holds and how the proof presented in Ref. [1] can be adapted for this case.

Here,  $\chi(\beta)$  is the susceptibility function, defined by

$$\chi(\beta) = \sum_x \langle \vec{\sigma}_0 \cdot \vec{\sigma}_x \rangle$$

where  $\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle$  is the infinite volume spin-spin correlation function, taken as the limit of the finite volume one

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle_\Lambda = \frac{1}{Z_\Lambda} \int_{\Omega^{|\Lambda|}} \vec{\sigma}_x \cdot \vec{\sigma}_y e^{-\beta H_\Lambda(\{\vec{\sigma}\})} \prod_{i \in \Lambda} d\mu(\vec{\sigma}_i),$$

with  $x = (x_0, x_1) \in \Lambda \subset \mathbb{Z}^{k+d}$  and  $\vec{\sigma}_x = (\sigma_x^{(1)}, \sigma_x^{(2)}, \dots, \sigma_x^{(N)}) \in \mathbb{R}^N$ .  $Z_\Lambda$  is partition function (normalizing factor) and  $H_\Lambda$  is the Hamiltonian, given by

$$H_\Lambda(\{\vec{\sigma}\}) = - \sum_{u,v \in \Lambda} J_{uv} \vec{\sigma}_u \cdot \vec{\sigma}_v,$$

where  $J_{uv} \geq 0$  is the translation invariant ferromagnetic spin interaction

$$J_{uv} = \begin{cases} K(\|u_1 - v_1\|) & \text{if } u_0 = v_0; \\ 1 & \text{if } u_1 = v_1 \text{ and } \|u_0 - v_0\| = 1; \\ 0 & \text{otherwise,} \end{cases}$$

where the function  $K(r)$  is such that  $\sum_{r \geq 0} r^{d-1} K(r) < \infty$ . Finally, the single spin distribution  $d\mu(\vec{\sigma})$  will be assumed of the form

$$d\mu(\vec{\sigma}) \equiv \frac{e^{-f(\|\vec{\sigma}\|^2)}}{N} d\vec{\sigma}$$

where  $N$  is a normalization factor and the exponent  $f(t)$  is such that multi-point correlation functions make sense.

We wish to find, for which values of the spin dimensionality  $N$ , there are sufficient conditions on the exponent  $f(t)$  and on the long range interaction function  $K(r)$  that guarantee the existence of positive constants  $C = C(\beta)$  and  $m = m(\beta)$  such that

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \leq C K(\|x_1 - y_1\|) e^{-m\|x_0 - y_0\|} \quad (1)$$

holds whenever  $\beta$  is such that  $\chi(\beta) < \infty$ . For a large class of  $N = 1$  spin models, the condition  $\chi(\beta) < \infty$  is equivalent to requiring that  $\beta < \beta_c$ , where  $\beta_c$  is the inverse critical temperature of the model, see [2]. This is particularly true for the  $N = 1$  spin models with spin interactions given, for instance, by

$$K(r) = \frac{2}{1 + r^{d+\varepsilon}}, \quad (2)$$

with  $\varepsilon > 0$ , and

$$K(r) = \frac{2}{(2+r)^d \ln^p(2+r)}, \quad (3)$$

with  $p > 1$ .

To prove the upper bound (1) in the disordered phase, we will have to assume the existence of positive constants  $\tilde{C}$  and  $\nu$  such that, for  $r > 0$ ,

$$(2/r)^\nu \leq K(r/2) \leq \tilde{C} K(r). \quad (4)$$

We remark that both interactions (2) and (3) satisfy the bounds (4). We can now state our result:

**Theorem** Assume that  $K(r)$  satisfies the bounds (4) for some positive constants  $\tilde{C}$  and  $\nu$  and suppose that there exists  $S > 0$  such that  $\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \leq S^2$  uniformly in  $x$  and  $y$ . If  $\beta$  is such that  $\chi(\beta) < \infty$  and if:

1.  $N = 1$  and the exponent  $f(t)$  is such that  $f''(t) \geq 0$  for  $t \geq 0$ , or;
2.  $N = 2$  and the exponent  $f(t)$  is a polynomial whose coefficients for degree greater than one are positive, or;
3.  $N = 2, 3, 4$  and the spin variables are uniformly distributed on the unitary sphere;

then there are positive constants  $C = C(\beta)$  and  $m = m(\beta)$  such that (1) holds for all  $x, y \in \mathbb{Z}^{k+d}$ .

Then, the above theorem states that spin-spin correlations decay exponentially in the “time direction”  $\mathbb{Z}^k$  and decay like  $K(\|x_1 - y_1\|)$  in the “space direction”  $\mathbb{Z}^d$ , the decay holding whenever  $\chi(\beta) < \infty$ . To understand why we need hypothesis 1. or 2. or 3. in order to prove the Theorem we recall that the proof of Ref. [1] is heavily based on: *positivity* of the connectivity function; on the *Simon-Lieb Inequality* [3, 4]; on the Aizenman-Newman *multi-scale analysis* [5]. Since we are assuming that the exponent  $f(t)$  is even, it is easy to conclude that Griffiths first inequality (G-I) [6, 7]

$$\langle \sigma_A \rangle_\Lambda \geq 0$$

holds for any  $\Lambda \subset \mathbb{Z}^{k+d}$  and any finite set  $A \subset \Lambda$ , with  $\sigma_A = \prod_{k=1}^N \prod_{i \in A} (\sigma_i^{(k)})^{a_i}$ , see [8]. As we will explain below, hypothesis 1. or 2. or 3. will guarantee that some type of Simon-Lieb Inequality is on hold so that the Aizenman-Newman multi-scale analysis will go through as long as we assume the bounds (4). The Simon-Lieb (Rivasseau) inequality [3, 4, 9] states that

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \leq \beta \sum_{\substack{u \in B \\ v \in B^c}} \langle \vec{\sigma}_x \cdot \vec{\sigma}_u \rangle_B J_{uv} \langle \vec{\sigma}_v \cdot \vec{\sigma}_y \rangle, \quad (5)$$

where  $x \in B$ ,  $y \in B^c$ . It follows from [10] that the above inequality holds under the hypothesis 1. or 2. of our theorem. Under the hypothesis 3., i.e., when spin variables are uniformly distributed in the unit sphere  $S^{N-1}$ ,  $N = 2, 3, 4$ , Aizenman and Simon [11] proved that

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \leq (\beta/N) \sum_{\substack{u \in B \\ v \in B^c}} \langle \vec{\sigma}_x \cdot \vec{\sigma}_u \rangle J_{uv} \langle \vec{\sigma}_v \cdot \vec{\sigma}_y \rangle. \quad (6)$$

**Proof of the Theorem:** The proof is similar to the one given in [1] for the  $(k + d)$ -dimensional mixed (exponential-polynomial) bond percolation model. Here, we fill in the steps of the proof which don’t come directly from the above reference. The first step is to prove exponential decay of correlations in the “time direction”  $\mathbb{Z}^k$  whenever  $\chi(\beta)$  is finite

and this is done by replacing  $\tau_{xy}$  by  $\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle$ ,  $p_{uv}$  by  $\beta K_{uv}$  and using G-I and inequality (5) or (6) depending on which hypothesis 1., 2. or 3. is on use, see the beginning of Section 2 of [1]. We get that there are positive constants  $L_0 = L_0(\beta)$  and  $m = m(\beta)$  such that

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \leq S^2 e^{-m\|x_0 - y_0\|}$$

whenever  $\chi(\beta) < \infty$  and  $\|x_0 - y_0\| > L_0$ .

To obtain the decay in the “space direction”  $\mathbb{Z}^d$ , we first observe that a modified version of Simon’s Inequality holds for  $T_{m'}(x, y) \equiv e^{m'\|x_0 - y_0\|} \langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle$ ,  $m' > 0$ :

$$T_{m'}(x, y) \leq \sum_{\substack{u \in \mathcal{C}_L(x) \\ v \in \mathcal{C}_L^c(x)}} T_{m'}(x, u) J_{uv} T_{m'}(v, y), \quad (7)$$

where  $\mathcal{C}_L(x) \equiv \{z \in \mathbb{Z}^{k+d}; \|x_1 - y_1\| \leq L\}$  is the “vertical cylinder” of radius  $L$  centered at  $x$  and  $y \in \mathcal{C}_L^c(x)$ . As in [1], we take  $m' = m - \delta$ , with  $\delta > 0$ . Iteration of inequality (7), for  $\beta$  such that  $\chi(\beta) < \infty$ , allows for the wished decay but the choice of  $\alpha$ , see Ref. [1], has to be slightly modified: since we are assuming the existence of  $\tilde{C}$  and  $\nu$  such that the bounds (4) are valid,  $\alpha$  should belong to the interval  $(0, 1/(2^\nu \tilde{C}))$ . Take  $x$  and  $y$  such that  $\|x_1 - y_1\| > L_0$  and define  $L \equiv \|x_1 - y_1\|/4$ . Proceeding with the argument exactly as in [1], if  $\chi(\beta)$  is finite then we get the following inequality for  $\mathbb{T}_{m'}(L) \equiv \sup_{u \in \mathcal{C}_L^c(0)} T_{m'}(0, u)$ :

$$\mathbb{T}_{m'}(L) \leq \chi_{m'}^2 K_{x_1 y_1} \sum_{j=0}^{n-1} (\tilde{C}^2 \alpha)^j + \alpha^n \mathbb{T}_{m'}(L/2^n). \quad (8)$$

Taking  $n$  such that  $L2^{-n} \leq L_0$ , it follows that

$$\alpha^n < \frac{1}{2^{\nu n}} = \frac{1}{K_{x_1 y_1}} \frac{K_{x_1 y_1}}{2^{\nu n}} \leq K_{x_1 y_1} \frac{(4L)^\nu}{2^{\nu n}} \leq K_{x_1 y_1} 4^\nu L_0^\nu,$$

where we have used the lower bound in (4). Using this upper bound in (8), together with  $T_{m'}(x, y) \leq \mathbb{T}_{m'}(L)$  if  $y \in \mathcal{C}_L^c(x)$  and  $\mathbb{T}_{m'}(L') \leq \chi_{m'}$  for any  $L' > 0$ , we conclude that

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \leq CK_{x_1 y_1} e^{-m_0\|x_0 - y_0\|}$$

if  $\chi(\beta) < \infty$ , proving the upper bound (1). □

**Remark** Under the hypothesis 1. or 2. of the Theorem, Griffiths Second Inequality also holds and one can then conclude that there exist positive constants  $C_1 = C_1(\beta)$  and  $m_1 = m_1(\beta)$  such that

$$\langle \vec{\sigma}_x \cdot \vec{\sigma}_y \rangle \geq C_1 K_{x_1 y_1} e^{-m_1\|x_0 - y_0\|}$$

for any  $x, y \in \mathbb{Z}^{k+d}$  and any  $\beta > 0$ .

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