Tomographic characterization of three-qubit pure states with only two-qubit detectors

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A tomographic process for three-qubit pure states using only pairwise detections is presented.

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The understanding of multipartite entanglement is one of the objectives of the quantum information community. Bipartite entanglement is well understood for pure states, where the Schmidt decomposition[1] plays the central role. Also much has been learnt about mixed states in recent years[2]. The problem one has to face when the numbers of parties grow is the consequent increasing in the complexity of the correlations involved. Recently, a significant progress in three-qubit correlations has been reported by N. Linden, S. Popescu and W. K. Wootters[3], who have shown that "*Almost every pure state of three qubits is completely determined by its two-particle reduced density matrices*". It is important to emphasize the context in which the work is developed which is understanding how information is stored in multipartite systems. Specifically, they show that there is no more information on the three-party state than what is already contained in the three reduced pair states. Their result is, at first sight, surprising. In the case of two qubits it is not difficult to show that several global states give the same reduced states. For bipartite pure states, the Schmidt decomposition theorem[1] asserts that, given a vector $|\Psi\rangle \in \mathsf{V}\otimes\mathsf{W}$, one can choose orthonormal basis $\{|v_i\rangle\}$ for V, and $\{|w_j\rangle\}$ for W such that $|\Psi\rangle = \sum_k \lambda_k |v_k\rangle \otimes |w_k\rangle$. In the case of two qubits, V and W have dimension 2 and the vector state can be written as[9]

$$|\Psi(\theta,\varphi)\rangle = \cos\theta\,|v_1\rangle\otimes|w_1\rangle + e^{i\varphi}\sin\theta\,|v_2\rangle\otimes|w_2\rangle\,, \quad (1)$$

with $\theta \in \left[0,\frac{\pi}{4}\right]$ and $\varphi \in [0,2\pi]$. Writing the density matrix in the basis $\{|v_i\rangle\otimes|w_j\rangle\}$ makes it clear that the relative phase $e^{i\varphi}$ is locally inaccessible, *i.e.*, for fixed θ all reduced density matrices are equal. So, there is more information on the full two-qubit state than on the parts represented by the reduced states. As a clarifying example, the two Bell states[4] $|\Phi_\pm\rangle = \{|00\rangle \pm |11\rangle\}/\sqrt{2}$ originate the same local states[10].

Suppose now that an experimental physicist wants to make tomographic measurements of three qubits with only two detectors. Reference [3] shows that all necessary information is available in the two-qubit reduced matrices, but does not suggest any practical method to do it. In this work we present a tomographic protocol for the complete characterization of generic three-qubit pure states, based only on pairwise detections. The protocol works whenever the result of Ref. [3] holds, *i.e.*: for all pure states except for a restricted class that will be comment latter (see eq. (11)). It must be said that although coincidence measurements are allowed, only local operations should be done, *i.e.*: one can think of three distinct laboratories, two of them equiped with detectors at each time, and a classical electronic coincidence line between them. In this sense, we are only implementing LOCC (Local Operations with Classical Comunnication).

First of all, let us review the authors argument. Consider an arbitrary pure state $|\nu\rangle = \sum_{ijk}\nu_{ijk}|ijk\rangle$ of three qubits A, B, and C. A general state (pure or mixed) that has the same reduced states of $|\nu\rangle$ can be obtained through a pure state $|\Psi\rangle$, describing three qubits plus an environment E (this process is called a *purification*, and its existence can be shown by the Schmidt decomposition). The fact that $|\Psi\rangle$ has the same reduced states than $|\nu\rangle$ when restricted to each two-qubit subspaces puts restrictions in its form. It is shown that for a generic state $|\nu\rangle$, these restrictions determine the form of $|\Psi\rangle$ as $|\Psi\rangle = \sum_{ijk}\nu_{ijk}|ijk\rangle\otimes|E\rangle$, where the environment is factorized. Consequently the environment state is pure, and the three-qubit state is necessarily $|\nu\rangle$. It is a very elegant argument, which also allowed Linden and Wootters to generalize this results for N qubits[5]. It is, however, rather abstract, and gives no clue for the experimentalist to completely characterize his three-qubit pure state.

The route we will take uses the generalization of the well known expression for one qubit:

$$\rho = \frac{1}{2}\sum_{\mu=0}^{3} b_\mu\sigma_\mu, \quad (2)$$

where σ_0 is the 2×2 identity matrix, and σ_i are the Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix},$$
$$\sigma_2 = \begin{bmatrix} & -i \\ i & \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad (3)$$

where we leave blank all null entries. The coefficients can be obtained from the expression

$$b_\nu = \mathrm{Tr}\{\rho\sigma_\nu\}\,. \quad (4)$$

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It is important to observe that eq. (4) implies $b_0 = 1$ (normalization of ρ) and that the complete characterization of the state can be achieved with three mean value measurements $b_\nu = \langle\sigma_\nu\rangle$ (*i.e.*: by the three components of the so called Bloch vector). In fact, this is a tomographic scheme for determining a spin-$\frac{1}{2}$ state[6].

To generalize eq. (4) for three qubits, let us define

$$\mathbf{S}_{\gamma\mu\nu} = \sigma_\gamma \otimes \sigma_\mu \otimes \sigma_\nu, \tag{5}$$

and denote the state of three qubits by

$$\rho = \left(\frac{1}{2}\right)^3 a_{\gamma\mu\nu}\mathbf{S}_{\gamma\mu\nu}, \tag{6}$$

where we have adopted the convention of summation over repeated indexes throghout the paper (latin indexes from 1 to 3; greek indexes from 0 to 3). Once again, the coefficients $a_{\gamma\mu\nu}$ can be obtained tomographically by the relation

$$a_{\gamma\mu\nu} = \mathrm{Tr}\left\{\rho\mathbf{S}_{\gamma\mu\nu}\right\}. \tag{7}$$

A first important consequence of eq. (7) is that $a_{000} = 1$. As Pauli matrices are traceless, the reduced density operators are given by

$$\rho_{BC} = \mathrm{Tr}_A(\rho_{ABC}) = \frac{1}{4}a_{0\mu\nu}\mathbf{S}_{\mu\nu}, \tag{8a}$$

$$\rho_{AC} = \mathrm{Tr}_B(\rho_{ABC}) = \frac{1}{4}a_{\gamma0\nu}\mathbf{S}_{\gamma\nu}, \tag{8b}$$

$$\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC}) = \frac{1}{4}a_{\gamma\mu0}\mathbf{S}_{\gamma\mu}, \tag{8c}$$

where $\mathbf{S}_{\mu\nu} = \sigma_\mu \otimes \sigma_\nu$.

To directly determine ρ through eq. (7) one needs to evaluate sixty three mean values. Nine of them (3×3) are the three components of each Bloch vector (a_{i00}, a_{0j0}, a_{00k}), and can be determined by individual detections. Twenty seven (3×9) are the pair correlations (a_{ij0}, a_{i0k}, a_{0jk}) and must be obtained through two-qubit coincidence measurements. The remaining twenty seven are three-qubit correlations, and are directly available only through three-qubit coincidence detections. For a general mixed state the number of mean values to be determined is exactly the same as the number of coefficients in the density operator. However, any previous knowledge on the state of the system is expected to reduce the number of parameters needed. In particular, for pure states we can use the idempotency relation,

$$\rho^2 = \rho, \tag{9}$$

to obtain the coefficients a_{ijk} from those available in the pair states. From expression (9) we get 64 equations, which can be organized in four sets: the first set consists of one solely equation

$$\sum_{ijk}(a_{i00}^2+a_{0j0}^2+a_{00k}^2+a_{ij0}^2+a_{i0k}^2+a_{0jk}^2+a_{ijk}^2) = 7, \tag{10a}$$

the second set is given by

$$3a_{i00} = a_{ij0}a_{0j0} + a_{i0k}a_{00k} + a_{ijk}a_{0jk}, \tag{10b}$$

with similar equations under permutations of indexes.

The third set is consituted of

$$3a_{ij0} = a_{i00}a_{0j0} + a_{00k}a_{ijk} + a_{0jk}a_{i0k} - \frac{1}{2}\epsilon_{ilt}\epsilon_{jmu}a_{lm0}a_{tu0} - \frac{1}{2}\epsilon_{ilt}\epsilon_{jmu}a_{tuk}a_{lmk}, \tag{10c}$$

together with analogous equations under cyclic permutations, and where we use the Levi-Civitta symbol ϵ_{ijk} for the totally antisymmetric tensor. Finally, the fourth group

$$3a_{ijk} = a_{i00}a_{0jk} + a_{0j0}a_{i0k} + a_{00k}a_{ij0} - \epsilon_{ilt}\epsilon_{jmu}a_{tu0}a_{lmk} - \epsilon_{ilt}\epsilon_{knv}a_{t0v}a_{ljn} - \epsilon_{jmu}\epsilon_{knv}a_{0uv}a_{imn}. \tag{10d}$$

The tomographic process is thus constituted by the thirty six mean values measured in individual and two-qubit coincidences, and the sixty four equations (10), that must be solved for a_{ijk}. The Linden, Popescu and Wootters' result guarantee the generic solution of the whole set of equations.

Anyhow, the last set (10d) gives 27 linear equations on the 27 unknowns a_{ijk}. In case they are linearly independent, this specific set can give the complete solution. We numerically checked such independence in all of hundreds of random choices. However, as pointed out by the authors of Ref. [3], there are exceptions, for states like

$$|GHZ(\theta,\varphi)\rangle = \cos\theta\,|000\rangle + e^{i\varphi}\sin\theta\,|111\rangle, \tag{11}$$

with $\theta \in \left(0, \frac{\pi}{2}\right)$, in which the phase $e^{i\varphi}$ is pairwise inaccessible, in the same sense as its analog in the two-qubit states (1). Thus, all the exceptions are the states that, for some choice of local basis, can be written as (11), since for any other state, all the phases can be obtained without involving triorthogonal basis vectors. In such case, eqs. (10d) can not be linearly independent, but we conjecture that a more geometrical argument can show the generic independence, and also point out the exceptions (11).

A parameter counting shows how rare the exceptions are. A vector in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ is given by 8 complex numbers (*i.e.*: 16 real numbers). Normalization and global phase reduce it to 14 real numbers. Local unitary operations are given by the action of $SU(2)$ group in each qubit[11]. $SU(2)$ is parametrized by 3 real numbers (the 3 Euler angles, or the 3 components of a vector $\vec{v}$ for the Lie algebraic parametrization $U(\vec{v}) = \exp\{i\vec{v}\cdot\vec{\sigma}\}$), so the orbit space has real dimension 5 (*i.e.*: $14-3\times 3$)[7]. As the fase φ in (11) can be changed by local unitarities, the GHZ family has just one parameter, θ. So it is like a regular curve in a five dimensional manifold.

The protocol here presented can be described in the following steps: first two-qubit measurements determine the 36 mean values characterizing all the two-qubit reduced states (*i.e.*: the coefficients a_{i00}, a_{0j0}, a_{00k}, a_{ij0}, a_{i0k}, and a_{0jk}); then this experimental data is used as input on the 27 linear equations (10d), and their solution generically determines the 27 remaining coefficients a_{ijk}, *i.e.*: generic three-qubit pure states are completely characterized by these 27 calculated coefficients plus the 36 directly measured ones. In fact, we can check for the purity of the measured state using the obtained coefficients to test the remaining 37 equations (10a, 10b, 10c). If any of these "testing" equations is not satisfied (within experimental precision), one should conclude that the original state is mixed, and can not be determined with only 36 mean values.

Some other questions can be raised on this issue. Is there any other tomographic process, restricted to two-qubit detections, that can determine the state with fewer measurements, without introducing new exceptions? Is the optimal number of measurements, 14, achievable with this kind of restriction? Recently Diósi [8] pointed out that a generic tripartite pure state can be determined by the knowledge of any two constituent pairs. Again this is a *generic* result in which interesting exceptions arise. For example, for three qubits, if an experimentalist decides to directly access ρ_{AB} and ρ_{BC}, and the prepared state is $|\psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |010\rangle + e^{i\varphi}|111\rangle)$, it will be impossible to determine the phase φ. It would be available, however, at ρ_{AC}. In fact, it is an interesting problem to classify the multipartite pure states by the partial information necessary to completely determine them. Such a classification could help understanding the curious geometric structure behind pure states.

In this paper we provide for a feasible experimental prescription to completely characterize a generic three-qubit pure state using only two-qubit detectors.

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[9] It is possible (and usual) to include the phase $e^{i\phi}$ on the basis vectors, however we are exactly interested in its indeterminacy from the local states, which justifies this unusual Schmidt decomposition.

[10] In fact the other two Bell states $|\Psi_{\pm}\rangle = \{|01\rangle \pm |10\rangle\}/\sqrt{2}$ also generate the same local states as $|\Phi_{\pm}\rangle$, but for another reason: degeneracy in Schmidt decomposition corresponding to $\theta = \frac{\pi}{4}$ in eq. (1).

[11] $SU(2)$ is used instead of $U(2)$ because we have already eliminated the global phase.