Cayley-Hamilton Theorem via Cauchy Integral Formula

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November 7, 2009

Abstract

This short note is just a expanded version of [1], where it was obtained a simple proof of Cayley-Hamilton's Theorem via Cauchy's Integral Formula. We remark that non content here is new.

1 Introduction

The aim of this paper is twofold, to introduce the generalization of the Cauchy's Integral Formula for polynomial functions taking values in a square matrix space and show the Cayley-Hamilton's Theorem using this generalization of Cauchy's Integral Formula.

In the first section, we setup the notation and present the integral representation for polynomials of matrices that generalizes the Cauchy's Integral Formula. The proof of this integral representation it was divided in three lemmas. The first and the second lemmas, as stated, are standard in the context of linear algebra. We observe that the results still valid, with slight modifications for complex functions taking values in any Banach algebra. The third lemma is about holomorphic functions in the complex plane. The Cayley-Hamilton Theorem is stated and finally proved in the section 2, using the Lemma 3.

2 Integral Representation

We denote by $\mathbb{M}_{n\times n}(\mathbb{C})$ the set of all $n\times n$ matrices with complex entries. Throughout this paper the identity matrix in $\mathbb{M}_{n\times n}(\mathbb{C})$ is simply denoted by $\mathbb{1}$. If $p(z)$ is a polynomial with complex coefficients given by

$$p(z) = \sum_{j=0}^{n} a_j z^j,$$

it make sense to talk about $p(A)$ for any $A \in \mathbb{M}_{n\times n}(\mathbb{C})$, just by replacing the complex variable z for the matrix A, obtaining

$$p(A) = \sum_{j=0}^{n} a_j A^j.$$

We will consider $\mathbb{M}_{n\times n}(\mathbb{C})$ as a normed vector space, with the norm of $A \in \mathbb{M}_{n\times n}(\mathbb{C})$, notation $\|A\|$, given by

$$\|A\| = \max_{1\leq r,s\leq n} |A_{rs}|.$$

note that $\|A\|$ is the standard maximum norm.

If $\Gamma \subset \mathbb{C}$ is a smooth curve and for $w \in \mathbb{C}$ the function

$$w \mapsto M(w) \in \mathbb{M}_{n\times n}(\mathbb{C})$$

determines n^2 holomorphic functions $M_{rs}(w)$, where $r, s \in \{1, \ldots, n\}$, then we can define the integral of $M(w)$ along Γ, as being an element of $\mathbb{M}_{n\times n}(\mathbb{C})$, where for all pairs of indexes $r, s \in \{1, \ldots, n\}$,

$$\left(\int_\Gamma M(w)dw\right)_{rs} = \int_\Gamma M_{r,s}(w)dw.$$

Lemma 1. *Let be $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ and $w \in \{z \in \mathbb{C}; |z| \geq 2n\|A\|\}$ then the matrix series*

$$\sum_{j=0}^{\infty} \frac{1}{w^{j+1}} A^j$$

converge in the maximum norm for a matrix $L(w) \in \mathbb{M}_{n\times n}(\mathbb{C})$. Moreover each one of its entries converge absolutely and uniformly(with respect to w), i.e. given any $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that, if $k > k_0$ then

$$\left| L_{rs}(w) - \sum_{j=0}^{k} \frac{(A^j)_{rs}}{w^{j+1}} \right| < \varepsilon$$

for all $r, s \in \{1, \ldots, n\}$ and w such that $|w| \geq 2n\|A\|$.

Proof: It follows from the properties of the maximum norm that for any $M, N \in \mathbb{M}_{n\times n}(\mathbb{C})$ we have the following inequality

$$\|M \cdot N\| \leq n\|M\|\|N\|.$$

Applying this inequality iteratively we obtain for all $k \in \mathbb{N}$ the inequality

$$\|A^k\| \leq n^k \|A\|^k. \tag{1}$$

Fix $r, s \in \{1, \ldots, n\}$ and consider $w \in \{z \in \mathbb{C}; |z| \geq 2n\|A\|\}$. By the Triangular Inequality, definition of maximum norm and (1) we have

$$\left| \sum_{j=0}^{k} \frac{(A^j)_{rs}}{w^{j+1}} \right| \leq \sum_{j=0}^{k} \frac{|(A^j)_{rs}|}{|w|^{j+1}} \leq \sum_{j=0}^{k} \frac{n^j \|A\|^j}{|w|^{j+1}} \leq \frac{1}{|w|} \sum_{j=0}^{k} \left(\frac{n\|A\|}{|w|}\right)^j. \tag{2}$$

Taking the limit when k goes to infinity we get from (2) that

$$\begin{aligned}\left|\sum_{j=0}^{\infty}\frac{(A^j)_{rs}}{w^{j+1}}\right| &\leq \frac{1}{|w|}\sum_{j=0}^{\infty}\left(\frac{n\|A\|}{|w|}\right)^j\\ &= \frac{1}{|w|}\cdot\frac{1}{1-\frac{n\|A\|}{|w|}}\\ &= \frac{1}{|w|-n\|A\|}\\ &\leq \frac{1}{n\|A\|}.\end{aligned}$$

Given $\varepsilon > 0$ and a pair of indexes $r, s \in \{1, \dots, n\}$ and $w \in \{z \in \mathbb{C}; |z| \geq 2n\|A\|\}$, it follows from the above inequality that there exist $k_0(r,s) \in \mathbb{N}$(independent of w) and a complex number $L_{r,s}(w)$ such that, if $k > k_0(r,s)$

$$\left|L_{rs}(w) - \sum_{j=0}^{k}\frac{(A^j)_{rs}}{w^{j+1}}\right| < \varepsilon \tag{3}$$

Let be $k_0 \equiv \max\limits_{1\leq r,s\leq n} k_0(r,s)$ and $w \in \{z \in \mathbb{C}; |z| \geq 2n\|A\|\}$, from (3) we have

$$\left\|L(w) - \sum_{j=0}^{k}\frac{A^j}{w^{j+1}}\right\| = \max_{1\leq r,s\leq n}\left|L_{rs}(w) - \sum_{j=0}^{k}\frac{(A^j)_{rs}}{w^{j+1}}\right| < \varepsilon$$

if $k > k_0$. Taking the limit when k goes to infinity in the above inequality we obtain

$$\left\|L(w) - \sum_{j=0}^{\infty}\frac{A^j}{w^{j+1}}\right\| = \lim_{k\to\infty}\left\|L(w) - \sum_{j=0}^{k}\frac{A^j}{w^{j+1}}\right\| \leq \varepsilon$$

since ε is arbitrary positive number, we can take the limit when ε goes to zero and the lemma is proved. □

Lemma 2. *Let be* $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ *if* $w \in \{z \in \mathbb{C}; |z| \geq 2n\|A\|\}$, *then there exist the inverse of* $(w\mathbb{1} - A)$ *and it satisfies*

$$(w\mathbb{1} - A)^{-1} = \sum_{j=0}^{\infty}\frac{A^j}{w^{j+1}} \tag{4}$$

Proof: By the Lemma 2, the series (4) converge in maximum norm to $L(w) \in \mathbb{M}_{n\times n}(\mathbb{C})$. To show that $L(w) = (w\mathbb{1} - A)^{-1}$, since we are dealing with matrices, it is enough to show that

$$(w\mathbb{1} - A)\left(\frac{\mathbb{1}}{w} + \frac{A}{w^2} + \frac{A^2}{w^3} + \dots\right) = \mathbb{1}.$$

This identity is obtained by expanding the product and using the convergence in the maximum norm as follow

$$(w\mathbb{1}-A)\left(\frac{\mathbb{1}}{w}+\frac{A}{w^2}+\frac{A^2}{w^3}+\ldots\right)=\left(\mathbb{1}+\frac{A}{w}+\frac{A^2}{w^2}+\ldots\right)-\left(\frac{A}{w}+\frac{A^2}{w^2}+\ldots\right)=\mathbb{1}.$$

□

Lemma 3. *Let be $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ and $\Gamma = \{z \in \mathbb{C}; |z| = 2n\|A\|\}$, then for all $k \in \mathbb{N}$ we have*

$$A^k = \frac{1}{2\pi i}\int_\Gamma w^k(w\mathbb{1}-A)^{-1}dw.$$

Proof: From the Lemma 2 we have

$$\begin{aligned} w^k(w\mathbb{1}-A)^{-1} &= w^k\left(\frac{\mathbb{1}}{w}+\frac{A}{w^2}+\frac{A^2}{w^3}+\ldots\right) \\ &= \left(w^{k-1}\mathbb{1}+w^{k-2}A+\ldots+\frac{A^k}{w}+\frac{A^{k+1}}{w^2}+\ldots\right). \end{aligned}$$

Integrating both sides in Γ, diving by $2\pi i$ and using the uniform convergence to change the order of the sum and the integral, we obtain

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma w^k(w\mathbb{1}-A)^{-1}dw &= \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_\Gamma \frac{A^{k+1}}{w^{j-k+1}}\,dw \\ &= \sum_{j=0}^{\infty}A^{k+1}\frac{1}{2\pi i}\int_\Gamma \frac{1}{w^{j-k+1}}\,dw. \qquad (5) \end{aligned}$$

Notice that for $j \in \{0,\ldots,k-1\}$ the function $1/(w^{j-k+1})$ is holomorphic in the open ball $\{z \in \mathbb{C}; |z| < 2n\|A\|\}$ (this function is in fact a polynomial so an entire function), hence by the Morera's Theorem we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{w^{j-k+1}}\,dw = 0.$$

For the other hand, if $j = k$ we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{w^{j-k+1}}\,dw = \frac{1}{2\pi i}\int_\Gamma \frac{1}{w}\,dw = 1$$

and if $j > k$ it follows from the Cauchy Integral Formula, for derivatives that

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{w^{j-k+1}}\,dw = 0.$$

Taking this integrations in account we get

$$\sum_{j=0}^{\infty}A^{k+1}\frac{1}{2\pi i}\int_\Gamma \frac{1}{w^{j-k+1}}\,dw = A^k$$

replacing the above equality in (5) it follows that

$$A^k = \frac{1}{2\pi i}\int_\Gamma w^k (w\mathbb{1} - A)^{-1} dw.$$

$\square$

Theorem 4 (Cauchy's Integral Formula). *Let be* $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ *and* $\Gamma = \{z \in \mathbb{C}; |z| = 2n\|A\|\}$ *then*

$$p(A) = \frac{1}{2\pi i}\int_\Gamma p(w)(w\mathbb{1} - A)^{-1} dw$$

Proof: Apply the Lemma 3 and use the linearity of the integral. $\square$

3 Cayley-Hamilton Theorem

Theorem 5 (Cayley-Hamilton). *Let* $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ *and* $p(z) = \det(z\mathbb{1} - A)$ *the characteristic polynomial of* A *then*

$$p(A) = 0.$$

Proof: If $w \in \{z \in \mathbb{C}; |z| < 2n\|A\|\}$, then we can express the elements $(w\mathbb{1} - A)^{-1}_{rs}$ as follow

$$(w\mathbb{1} - A)^{-1}_{rs} = \frac{1}{\det(w\mathbb{1} - A)} C_{rs}(w),$$

where $C_{rs}(w)$ is the cofactor matrix of $w\mathbb{1} - A$. Recall that each $C_{rs}(w)$ is a polynomial of degree at most $n-1$ in the variable w, this fact will be used at the end of the proof.

Applying the Cauchy integral Formula for the characteristic polynomial of A, $p(z) = \det(z\mathbb{1} - A)$, we get

$$p(A) = \frac{1}{2\pi i}\int_\Gamma \det(w\mathbb{1} - A)(w\mathbb{1} - A)^{-1} dw.$$

We know that $p(A)_{rs} = \langle e_r, p(A)e_s\rangle$, where $\langle\cdot,\cdot\rangle$ is the canonical inner product of vectors in $\mathbb{C}^n$ and $\{e_1, \ldots, e_n\}$ is the standard base of $\mathbb{C}^n$. By the linearity of the integral we have

$$
\begin{aligned}
p(A)_{rs} &= \langle e_r, p(A)e_s\rangle = \left\langle e_r, \frac{1}{2\pi i}\left(\int_\Gamma \det(w\mathbb{1}-A)(w\mathbb{1}-A)^{-1}dw\right)e_s\right\rangle \\
&= -\frac{1}{2\pi i}\left\langle e_r, \left(\int_\Gamma \det(w\mathbb{1}-A)(w\mathbb{1}-A)^{-1}dw\right)e_s\right\rangle \\
&= -\frac{1}{2\pi i}\int_\Gamma \det(w\mathbb{1}-A)\left\langle e_r, (w\mathbb{1}-A)^{-1}e_s\right\rangle dw \\
&= -\frac{1}{2\pi i}\int_\Gamma \det(w\mathbb{1}-A)(w\mathbb{1}-A)^{-1}_{rs}dw \\
&= -\frac{1}{2\pi i}\int_\Gamma \det(w\mathbb{1}-A)\frac{1}{\det(w\mathbb{1}-A)}C_{rs}(w)dw \\
&= -\frac{1}{2\pi i}\int_\Gamma C_{rs}(w)dw \\
&= 0.
\end{aligned}
$$

since this identity is valid for all $r, s \in \{1, \dots, n\}$ the theorem is proved. □

References

[1] Charles A. McCarthy: *The Cayley-Hamilton Theorem* , Amer Math. Monthly, vol. 82, No 4, pp. 390-391, (1975).