CORRELATION INEQUALITIES AND MONOTONICITY PROPERTIES OF THE RUELLE OPERATOR

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In this paper we provide sufficient conditions for the validity of the FKG Inequality, on Thermodynamic Formalism setting, for a class of eigenmeasures of the dual of the Ruelle operator. We use this correlation inequality to study the maximal eingenvalue problem for the Ruelle operator associated to low regular potentials. As an application we obtain explicit upper bounds for the main eigenvalue (consequently for the pressure) of the Ruelle operator associated to Ising models with a power law decay interaction energy.

Keywords: Thermodynamic Formalism; FKG Inequality; Ruelle Operator.

AMS Subject Classification: 37D35, 28Dxx, 37C30

1. Introduction

The primary aim of this paper is to relate the Fortuin-Kasteleyn-Ginibre (FKG) inequality to the study of the main eigenvalue problem for Ruelle operator associated to an attractive potential A having low regularity (meaning A lives outside of the classical Hölder, Walters and Bowen spaces).

The FKG Inequality [12] is a strong correlation inequality and a fundamental tool in Statistical Mechanics. An earlier version of this inequality for product measures was obtained by Harris in [15]. Holley in [17] generalized the FKG Inequality in the context of finite distributive lattice.

In the context of Symbolic Dynamics the FKG Inequality can be formulated as follows. Let us consider the symbolic space $X = \{-1, 1\}^{\mathbb{N}}$ with an additional structure which is a partial order $\succeq$, where $x \succeq y$, if $x_j \geq y_j$, for all $j \in \mathbb{N}$. A function $f : X \to \mathbb{R}$ is said *increasing* if for all $x, y \in X$, such that $x \succeq y$, we have $f(x) \geq f(y)$. A Borel probability measure μ on X will be said to satisfy the FKG Inequality if for any pair of continuous increasing functions f and g we have

$$\int_X fg\, d\mu - \int_X f\, d\mu \int_X g\, d\mu \geq 0.$$

In Probability Theory such measure are sometimes called *positively associated.*

Establishing FKG Inequality for continuous potentials with low regularity is an important step to study, for example, the Dyson model on the lattice $\mathbb{N}$, within the framework of Thermodynamic Formalism. A Dyson model (see [10]) is a special long-range ferromagnetic Ising model, commonly defined on the lattice $\mathbb{Z}$. This is a very important model in Statistical Mechanics exhibiting the phase transition phenomenon in one-dimension. This model still is a topic of active research and currently it is being studied in both lattices $\mathbb{N}$ and $\mathbb{Z}$, see the recent papers [5, 21, 29] and references therein.

In [21] the authors proved that the Dyson model on the lattice $\mathbb{N}$ presents phase transition. This result is an important contribution to the Theory of Thermodynamic Formalism since very few examples of phase transition on the lattice $\mathbb{N}$ are known (see [2, 8, 13, 16, 20]). They also proved that the critical temperature of the Dyson model on the lattice $\mathbb{N}$ is at most four times the critical temperature of Dyson model on the lattice $\mathbb{Z}$ and actually conjectured that the critical temperature for both models coincides. We remark that the explicit value of the critical temperature for the Dyson model on both lattices still is an open problem. Moreover there are very few examples in both Thermodynamic Formalism and Statistical Mechanics, where the explicit value of the critical temperature is known. A remarkable example where the critical temperatures is explicitly obtained is the famous work by Lars Onsager [25] and the main idea behind this computation is the Transfer Operator.

Although the Ruelle operator $\mathscr{L}_A$ (associated to the potential A) have been intensively studied a little is known about $\mathscr{L}_A$, when A is the Dyson potential. We already know that some of the conclusions of the Ruelle-Perron-Frobenius Theorem can not obtained. Towards to obtain a generalization of this theorem in some sense, here we consider extensions of this operator to larger spaces than $C(X)$, where a weak version of Ruelle-Perron-Frobenius theorem can be obtained. We focus on extensions of the Ruelle operator to the Lebesgue space $L^2(\nu_A) \equiv L^2(X, \mathscr{B}(X), \nu_A)$, where ν_A is an eigenmeasure for $\mathscr{L}_A^*$ (associated to the spectral radius of this operator acting on $C(X)$) and $\mathscr{B}(X)$ is the Borel sigma-algebra of the product space X. We study the problem of existence of the main eigenfunction in such spaces by using the involution kernel and subsequently the Lions-Lax-Milgram theorem.

As an application of our results we show how to use the involution kernel representation of the main eigenfunction and the FKG Inequality to obtain non-trivial upper bound for the topological pressure of potentials of the form

$$A(x) = a_1 x_1 x_2 + a_2 x_1 x_3 + a_2 x_1 x_4 + \ldots + a_n x_1 x_{n+1} + \ldots \tag{1.1}$$

which is associated to a long-range Ising model, when $(a_n)_{n\geq 1}$ is suitable chosen. A particular interesting case occurs when $a_n = n^{-\gamma}$ with $\gamma > 1$. In this case A is the potential of the Dyson model on the lattice $\mathbb{N}$, see [7].

This paper is organized as follows. In Section 2 we state and prove the FKG Inequality in the Thermodynamic Formalism setting and next we discuss some of its consequence for Ising type models. In Section 3 we show how to use the FKG Inequality to obtain maximal spectral eigendata of the Ruelle operator. We present some results about the uniqueness of the conformal measures for potentials having low regularity. We also explore some of the consequences of the FKG inequality to obtain existence and some symmetry properties of the maximal eigenfunctions of the Ruelle operator. In Section 4 we explore the idea of involution kernel to study the problem of the existence of the maximal eigefunctions for the Ruelle operator associated to the Dyson potential. This technique provides a representation of such maximal eigenfunctions on a dense set of the symbolic space. Afterwards, we show how to use this restricted representation and the FKG Inequality to obtain a tight upper bound for the topological pressure of such models.

In Section 5 we proceed with the study of the maximal eingefunction problem not for the Ruelle operator but for an appropriate extension of it. The aim is to extend our dense defined eigenfunction (as obtained by the involution Kernel representation) to an almost surely defined function. For this purpose we reformulate the maximal eigenvalue problem in a weak sense and obtain an existence result in the Hilbert space of square integrable functions with respect to suitable conformal measures.

2. The FKG Inequality in Thermodynamic Formalism Setting

Let $\mathbb{N}$ be the set of positive integers, consider the symbolic space $X = \{-1,1\}^{\mathbb{N}}$ and the left shift mapping $\sigma : X \to X$ which is defined for each $x \equiv (x_1, x_2, \ldots)$ as $\sigma(x) = (x_2, x_3, \ldots)$. As usual we endow X with its standard distance d_X, where $d_X(x,y) = 2^{-N}$, where $N = \inf\{i \in \mathbb{N} : x_i \neq y_i\}$. As mentioned before we consider the partial order $\succeq$ in X, where $x \succeq y$, iff $x_j \geq y_j$, for all $j \in \mathbb{N}$. A function $f : X \to \mathbb{R}$ is called increasing (decreasing) if for all $x, y \in X$ such that $x \succeq y$, we have that $f(x) \geq f(y)$ ($f(x) \leq f(y)$). The set of all continuous increasing and decreasing functions are denoted by $\mathcal{I}$ and $\mathcal{D}$, respectively.

For each $n \geq 1$, $t \in \{-1, 1\}$ and $x, y \in X$ will be convenient in this section to use the following notations

$$[x|y]_n \equiv (x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots) \quad \text{and} \quad [x|t|y]_n \equiv (x_1, \ldots, x_n, t, y_{n+2}, \ldots).$$

A function $A : X \to \mathbb{R}$ will be called a potential. For each potential A, $x \in X$ and $n \geq 1$, we define $S_n(A) \equiv A + \ldots + A \circ \sigma^{n-1}$. In this section a major role will be played by the so-called finite volume Gibbs measures on X, with boundary conditions. They are defined as follows. We fix $y \in X$ and $n \in \mathbb{N}$ and these measures are given by the following expression

$$\mu_n^y = \sum_{x_1,\ldots,x_n = \pm 1} \frac{\exp(S_n(A)([x|y]_n))}{Z_n^y} \delta_{([x|y]_n)}, \tag{2.1}$$

where

$$Z_n^y \equiv \sum_{x_1,\ldots,x_n = \pm 1} \exp(S_n(A)([x|y]_n))$$

and δ_x is the Dirac measure supported on the point $x \in X$. The normalizing factor Z_n^y is called *partition function* (associated to the potential A).

Definition 2.1. Let $\varepsilon > 0$ be given. A function $\widetilde{A} : [-1-\varepsilon, 1+\varepsilon]^{\mathbb{N}} \to \mathbb{R}$ is called a differentiable extension of a potential $A : X \to \mathbb{R}$ if for all $x \in \{-1,1\}^{\mathbb{N}}$ we have $\widetilde{A}(x) = A(x)$ and for all $j, n \in \mathbb{N}$ the following partial derivatives exist and the mappings

$$\big(-1-\varepsilon, 1+\varepsilon\big) \ni t \mapsto \frac{\partial \widetilde{A}}{\partial x_j}(x_1, \ldots, x_n, t, x_{n+2}, \ldots)$$

are continuous for any fixed $x \in [-1,1]^{\mathbb{N}}$.

To avoid a heavy notation, a differentiable extension $\widetilde{A}$ of a potential A will be simply denoted by A. Note that the Ising type potentials are examples of continuous potentials admitting natural differentiable extensions.

Definition 2.2 (Class $\mathcal{E}$ potential). We say that a continuous potential $A : X \to \mathbb{R}$ belongs to class $\mathcal{E}$ if it admits a differentiable extension satisfying:

$$(x_1, x_2, \ldots) \longmapsto \frac{d}{dt} S_n(A)([x|t|y]_n), \tag{2.2}$$

is an increasing function from X to $\mathbb{R}$, for choice of $t \in [-1,1]$, $y \in [-1,1]^{\mathbb{N}}$, $n \geq 1$.

Let $n \geq 1$ be fixed and $f, g : X \to \mathbb{R}$ two real increasing functions, with respect to the partial order $\succeq$, depending only on its first n coordinates. The main result of the next section states that for all potential A in the class $\mathcal{E}$ the probability measure μ_n^y given by (2.1) satisfies the FKG Inequality

$$\int_X fg \, d\mu_n^y - \int_X f \, d\mu_n^y \int_X g \, d\mu_n^y \geq 0, \qquad \forall y \in X. \tag{2.3}$$

Remark 2.1. If for all $n \geq 1$ the probability measure μ_n^y satisfies (2.3) and $\mu_n^y \rightharpoonup \mu$ then μ satisfies (2.3).

An Ising type potential is any real function $A : X \to \mathbb{R}$ of the form $A(x) = hx_1 + x_1 \sum_{i \geq 1} a_i x_{i+1}$, where the parameters $h, a_1, a_2, \ldots$ are fixed real numbers satisfying $\sum_{n \geq 1} |a_n| < \infty$. An interesting family of such potentials is given by

$$A(x) \equiv hx_1 + x_1x_2 + \frac{x_1x_3}{2^\alpha} + \frac{x_1x_4}{3^\alpha} + \ldots, \quad \text{where } \alpha > 1 \text{ and } h \in \mathbb{R}. \tag{2.4}$$

Such potentials are sometimes called Dyson potentials.

It is worth to mention that a Dyson potential is not an increasing, decreasing or Hölder function. On the other hand, a Dyson potential for any fixed $h \in \mathbb{R}$ and $\alpha > 1$ belongs to the class $\mathcal{E}$. Indeed, a straightforward computation shows that

$$S_n(A)([x|t|y]_n) = x_1 \, t \, n^{-\alpha} + x_2 \, t \, (n-1)^{-\alpha} + \ldots + x_{n-1} \, t + D_n,$$

where D_n is a constant that depends only on x and y, but not on t. From this expression one can see that the condition (2.2) is immediately verified. This fact will be used to shown that the probability measure μ_n^y defined from a Dyson potential satisfies the FKG Inequality for any choice of $y \in X$. More generally, any Ising type potential with $a_n \geq 0$, for all $n \geq 1$, satisfies the hypothesis of Theorem 2.1. Such particular potentials are sometimes called ferromagnetic potentials.

2.1. *The Proof of the FKG Inequality*

The results obtained in this section are inspired in the proof of the FKG Inequality for ferromagnetic Ising models presented in [11]. In that reference this inequality is proved under assumptions on the local behavior of the interactions of the Ising model, while here our hypothesis are about the global behavior of the potential.

Our starting point is the following classical result.

Lemma 2.1. *Let $E \subset \mathbb{R}$ and $(E, \mathscr{F}, \lambda)$ a probability space. If $f, g : E \to \mathbb{R}$ are increasing functions then*

$$\int_E fg \; d\lambda \geq \int_E f \; d\lambda \int_E g \; d\lambda. \tag{2.5}$$

Proof. Since f and g are increasing functions, then for any pair $(s,t) \in E \times E$ we have $0 \leq [f(s) - f(t)][g(s) - g(t)]$. By integrating both sides of this inequality, with respect to the product measure $\lambda \times \lambda$, using the elementary properties of the integral and that λ is a probability measure we finish the proof. □

Now we present an auxiliary combinatorial lemma that will be used in the proof of Theorem 2.1.

Lemma 2.2. *Let $E = \{-1, 1\}$, $y \in X$ fixed and $f : X \to \mathbb{R}$ a continuous function. Then the following identity holds for all $n \geq 1$*

$$\int_E \left[\int_X f \, d\mu_n^{[y|t|y]_n} \right] d\lambda(t) = \int_X f \, d\mu_{n+1}^y,$$

where

$$\lambda \equiv \sum_{t=\pm 1} \exp(A(\sigma^n([y|t|y]_n))) \frac{Z_n^{[y|t|y]_n}}{Z_{n+1}^y} \delta_t.$$

Proof. By using the definitions of λ and μ_n^y, respectively we get

$$\int_E \left[\int_X f\, d\mu_n^{[y|t|y]_n}\right] d\lambda(t)$$

$$= \sum_{t=\pm 1} \exp(A(\sigma^n([y|t|y]_n))) \frac{Z_n^{[y|t|y]_n}}{Z_{n+1}^y} \int_X f\, d\mu_n^{[y|t|y]_n}$$

$$= \sum_{t=\pm 1} \exp(A(\sigma^n([y|t|y]_n))) \frac{Z_n^{[y|t|y]_n}}{Z_{n+1}^y} \sum_{x_1,\ldots,x_n=\pm 1} f([x|t|y]_n) \frac{\exp(S_n(A)([x|t|y]_n))}{Z_n^{[y|t|y]_n}}$$

$$= \sum_{t=\pm 1} \frac{\exp(A(\sigma^n([y|t|y]_n)))}{Z_{n+1}^y} \sum_{x_1,\ldots,x_n=\pm 1} f([x|t|y]_n) \exp(S_n(A)([x|t|y]_n))$$

$$= \frac{1}{Z_{n+1}^y} \sum_{t=\pm 1} \sum_{x_1,\ldots,x_n=\pm 1} f([x|t|y]_n) \exp(S_{n+1}(A)([x|t|y]_n))$$

$$= \frac{1}{Z_{n+1}^y} \sum_{x_1,\ldots,x_{n+1}=\pm 1} f([x|y]_{n+1}) \exp(S_{n+1}(A)([x|y]_{n+1}))$$

$$= \int_X f\, d\mu_{n+1}^y. \qquad \square$$

To shorten the notation in the remaining of this section, we define for each $n \geq 1$, $x, y \in X$ and $t \in [-1, 1]$ the following weights

$$W_n([x|t|y]_n) \equiv \frac{\exp(S_n(A)([x|t|y]_n)}{Z_n^{[y|t|y]_n}} \tag{2.6}$$

Lemma 2.3. *Let $n \geq 1$ and $y \in X$ be fixed, $f : X \to \mathbb{R}$ an increasing function, depending only on its first n coordinates $(x_1, \ldots, x_n)$. If the potential A belongs to the class $\mathcal{E}$ and $\mu_n^{[y|t|y]_n}$ satisfies the FKG Inequality, then*

$$[-1, 1] \ni t \mapsto \int_X f\, d\mu_n^{[y|t|y]_n} \tag{2.7}$$

is an increasing function.

Proof. We first observe that the integral in (2.7) is well-defined because A admits a differentiable extension defined on the product space $[-(1+\varepsilon), 1+\varepsilon]^{\mathbb{N}}$.

By using that f depends only on its first n coordinates we have the following identity for any $y \in X$

$$\int_X f\, d\mu_n^{[y|t|y]_n} = \sum_{x_1,\ldots,x_n=\pm 1} f([x|t|y]_n) W_n([x|t|y]_n) = \sum_{x_1,\ldots,x_n=\pm 1} f([x|y]_n) W_n([x|t|y]_n).$$

Since A belongs to the class $\mathcal{E}$ follows from the expression (2.6) that $W_n([x|t|y]_n)$ has continuous derivative and therefore to prove the lemma is enough to prove that

$$\frac{d}{dt}\int_X f\, d\mu_n^{[y|t|y]_n} = \sum_{x_1,\ldots,x_n=\pm 1} f([x|y]_n)\frac{d}{dt}W_n([x|t|y]_n) \geq 0. \tag{2.8}$$

By using the quotient rule we get that the derivative appearing in the above expression is equal to

$$\begin{aligned}\frac{d}{dt}W_n([x|t|y]_n) &= \frac{d}{dt}\frac{\exp(S_n(A)([x|t|y]_n)}{Z_n^{[y|t|y]_n}}\\ &= \frac{\exp(S_n(A)([x|t|y]_n)}{Z_n^{[y|t|y]_n}}\left[\frac{d}{dt}S_n(A)([x|t|y]_n) - \frac{1}{Z_n^{[y|t|y]_n}}\frac{d}{dt}Z_n^{[y|t|y]_n}\right]\\ &= W_n([x|t|y]_n)\left[\frac{d}{dt}S_n(A)([x|t|y]_n) - \frac{1}{Z_n^{[y|t|y]_n}}\frac{d}{dt}Z_n^{[y|t|y]_n}\right]. \end{aligned} \tag{2.9}$$

Note that the last term in the rhs above is equal to

$$\begin{aligned}\frac{1}{Z_n^{[y|t|y]_n}}\frac{d}{dt}Z_n^{[y|t|y]_n} &= \frac{1}{Z_n^{[y|t|y]_n}}\frac{d}{dt}\sum_{x_1,\ldots x_n=\pm 1}\exp(S_n(A)([x|t|y]_n))\\ &= \frac{1}{Z_n^{[y|t|y]_n}}\sum_{x_1,\ldots x_n=\pm 1}\exp(S_n(A)([x|t|y]_n))\frac{d}{dt}S_n(A)([x|t|y]_n)\\ &= \int_X \frac{d}{dt}S_n(A)([x|t|y]_n)\ d\mu_n^{[y|t|y]}(x). \end{aligned} \tag{2.10}$$

Replacing the expression (2.10) in (2.9) we get that $\frac{d}{dt}W_n([x|t|y]_n)$ is equal to

$$W_n([x|t|y]_n)\left[\frac{d}{dt}S_n(A)([x|t|y]_n) - \int_X \frac{d}{dt}S_n(A)([x|t|y]_n)\ d\mu_n^{[y|t|y]}(x)\right].$$

By replacing the above expression in (2.8) we obtain

$$\begin{aligned}\frac{d}{dt}\int_X f\, d\mu_n^{[y|t|y]_n} &= \int_X f(x)\frac{d}{dt}S_n(A)([x|t|y]_n)\ d\mu_n^{[y|t|y]_n}(x)\\ &\quad - \int_X f\, d\mu_n^{[y|t|y]_n}\int_X \frac{d}{dt}S_n(A)([x|t|y]_n)\, d\mu_n^{[y|t|y]_n}(x)\end{aligned}$$

which is non-negative because f is increasing $A\in\mathcal{E}$ and the probability measure $\mu_n^{[y|t|y]_n}$ satisfies the inequality (2.3) by hypothesis. □

Theorem 2.1. *Let $A: X\to\mathbb{R}$ be a potential in the class $\mathcal{E}$. For any fixed $y\in[-1,1]^{\mathbb{N}}$ and for all $n\geq 1$ the probability measure*

$$\mu_n^y = \sum_{x_1,\ldots,x_n=\pm 1}\frac{\exp(S_n(A)([x|y]_n)}{Z_n^y}\delta_{([x|y]_n)}, \tag{2.11}$$

where Z_n^y is the standard partition function, satisfies the FKG Inequality.

Proof. The proof is by induction in n. The inequality (2.3), for $n = 1$, follows from a straightforward application of Lemma 2.1. Indeed, for any fixed $y \in X$ the mappings $X \ni x \longmapsto f(x_1, y_2, y_3, \ldots)$ and $X \ni x \longmapsto g(x_1, y_2, y_3, \ldots)$ are clearly increasing. By thinking of these maps as functions from $E = \{-1, 1\}$ to $\mathbb{R}$ and μ_1^y as a probability measure over E, we can apply Lemma 2.1 to get the conclusion.

The induction hypothesis is formulated as follows. For some $n \geq 2$ assume that for all $y \in X$ and any pair of real continuous increasing functions f and g, depending only on its first n coordinates, we have

$$\int_X fg\, d\mu_n^y \geq \int_X f\, d\mu_n^y \int_X g\, d\mu_n^y.$$

Now we prove that μ_{n+1}^y satisfies the FKG Inequality. From the definition we have that

$$\int_X fg\, d\mu_{n+1}^y = \sum_{x_1,\ldots,x_{n+1}=\pm 1} f([x|y]_{n+1}) g([x|y]_{n+1}) \frac{\exp(S_{n+1}(A)([x|y]_{n+1})}{Z_{n+1}^y}$$

$$= \exp(A(\sigma^n([y|1|y]_n))) \frac{Z_n^{[y|1|y]_n}}{Z_{n+1}^y} \int_X fg\, d\mu_n^{[y|1|y]_n}$$
$$+ \exp(A(\sigma^n([y|-1|y]_n))) \frac{Z_n^{[y|-1|y]_n}}{Z_{n+1}^y} \int_X fg\, d\mu_n^{[y|-1|y]_n}.$$

By using the induction hypothesis on both terms in the rhs above we get that

$$\begin{aligned}\int_X fg\, d\mu_{n+1}^y &\geq \sum_{t=\pm 1} \exp(A(\sigma^n([y|t|y]_n))) \frac{Z_n^{[y|t|y]_n}}{Z_{n+1}^y} \int_X f\, d\mu_n^{[y|t|y]_n} \int_X g\, d\mu_n^{[y|t|y]_n} \\ &= \int_E \left[\left(\int_X f\, d\mu_n^{[y|t|y]_n} \right) \left(\int_X g\, d\mu_n^{[y|t|y]_n} \right) \right] d\lambda(t), \qquad (2.12)\end{aligned}$$

where $E = \{-1, 1\}$ and λ is defined as in Lemma 2.2. From Lemma 2.3 it follows that both functions

$$t \mapsto \int_X f\, d\mu_n^{[y|t|y]_n} \quad \text{and} \quad t \mapsto \int_X g\, d\mu_n^{[y|t|y]_n}$$

are increasing functions. To finish the proof it is enough to apply Lemma 2.1 to the rhs of (2.12) obtaining

$$\begin{aligned}\int_X fg\, d\mu_{n+1}^y &\geq \int_E \left[\int_X f\, d\mu_n^{[y|t|y]_n} \right] d\lambda(t) \int_E \left[\int_X g\, d\mu_n^{[y|t|y]_n} \right] d\lambda(t) \\ &= \int_X f\, d\mu_{n+1}^y \int_X g\, d\mu_{n+1}^y,\end{aligned}$$

where the last equality is ensured by the Lemma 2.2. □

2.2. *FKG Inequality and the Ising Model*

In this section we recall the classical FKG Inequality for the Ising model as well as some of its applications. For more details see [11, 12] and [23].

Let $\boldsymbol{h} = (h_i)_{i\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ and $\boldsymbol{J} \equiv \{J_{ij} \in \mathbb{R} : i, j \in \mathbb{N} \text{ and } i \neq j\}$ be a collection of real numbers belonging to the set

$$\mathscr{R}(\mathbb{N}) = \Big\{\boldsymbol{J} : \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}\setminus\{i\}} |J_{ij}| < +\infty\Big\}. \tag{2.13}$$

For each $n \in \mathbb{N}$ we define a real function $H_n : X \times X \times \mathscr{R}(\mathbb{N}) \times \ell^\infty(\mathbb{N}) \to \mathbb{R}$ by following expression

$$H_n(x, y, \boldsymbol{J}, \boldsymbol{h}) = \sum_{1\leq i<j\leq n} J_{ij} x_i x_j + \sum_{1\leq i\leq n} h_i x_i + \sum_{\substack{1\leq i\leq n \\ j\geq n}} J_{ij} x_i y_j. \tag{2.14}$$

Note that the summability condition in (2.13) ensures that the series appearing in (2.14) is absolutely convergent and therefore H_n is well defined.

For each $n \geq 1$, $y \in X$ and $(\boldsymbol{J}, \boldsymbol{h}) \in \mathscr{R}(\mathbb{N}) \times \ell^\infty(\mathbb{N})$ we define a probability measure by the following expression

$$\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} = \frac{1}{Z_n^{y,\boldsymbol{J},\boldsymbol{h}}} \sum_{x_1,\ldots,x_n=\pm 1} \exp(H_n(x, y, \boldsymbol{J}, \boldsymbol{h}))\delta_{([x|y]_n)}, \tag{2.15}$$

where $Z_n^{y,\boldsymbol{J},\boldsymbol{h}}$ is the partition function. In the next section we show that for suitable choices of $\boldsymbol{J}$ and $\boldsymbol{h}$ the expression (2.15) can be rewritten in terms of the Ruelle operator.

Theorem 2.2 (FKG Inequality). *Let $n \geq 1$, $\boldsymbol{h} \in \ell^\infty(\mathbb{N})$ and $\boldsymbol{J} \in \mathscr{R}(\mathbb{N})$ so that $J_{ij} \geq 0$ for any pair i, j. If $f, g : X \to \mathbb{R}$ are increasing functions depending only on its first n coordinates, then*

$$\int_X fg\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} - \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} \int_X g\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} \geq 0.$$

Proof. We can prove this theorem using the same ideas employed in the proof of Theorem 2.1. For details, see [11]. □

Note that the Hamiltonian $H_n : X \times X \times \mathscr{R}(\mathbb{N}) \times \ell^\infty(\mathbb{N}) \to \mathbb{R}$ admits a natural differentiable extension to a function defined on $\mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \times \mathscr{R}(\mathbb{N}) \times \ell^\infty(\mathbb{N})$ and so for any f, depending on its first n coordinates, the following partial derivatives exist and are continuous functions

$$\frac{\partial}{\partial h_j} \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}, \quad \frac{\partial}{\partial y_j} \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} \quad \text{and} \quad \frac{\partial}{\partial J_{ij}} \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}$$

Corollary 2.1. *Under the hypothesis of Theorem 2.2 we have*

$$\frac{\partial}{\partial h_i} \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} = \int_X f(x) x_i\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}(x) - \int_X f(x)\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}(x) \int_X x_i\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}(x) \geq 0.$$

In particular, if $\widetilde{\boldsymbol{h}} \succeq \boldsymbol{h}$ *then*

$$\int_X f\, d\mu_n^{y,\boldsymbol{J},\widetilde{\boldsymbol{h}}} \geq \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}.$$

Corollary 2.2. *Under the hypothesis of Theorem 2.2 if* $x \succeq y$ *then*

$$\int_X f\, d\mu_n^{x,\boldsymbol{J},\boldsymbol{h}} \geq \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}.$$

Proof. By considering the natural differentiable extension of H_n to $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathscr{R}(\mathbb{N}) \times \ell^\infty(\mathbb{N})$ we can proceed as in (2.9) obtaining

$$\frac{\partial}{\partial y_i} \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}} = \int_X f(x) \cdot \frac{\partial}{\partial y_i} H_n(x,y,\boldsymbol{J},\boldsymbol{h})\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}(x)$$
$$- \int_X f(x)\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}(x) \int_X \frac{\partial}{\partial y_i} H_n(x,y,\boldsymbol{J},\boldsymbol{h})\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}(x).$$

By using that $J_{ij} \geq 0$ we get from (2.14) that the mapping $x \longmapsto (\partial/\partial y_i) H_n(x,y,\boldsymbol{J},\boldsymbol{h})$ is an increasing function. So we can apply the FKG Inequality to the rhs above to ensure that function

$$y \longmapsto \int_X f\, d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}$$

is coordinate wise increasing and therefore the result follows. □

To lighten the notation $\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}$, when $y = (1,1,1,\ldots) \equiv 1^\infty$ or similarly $y = (-1,-1,-1,\ldots) \equiv -1^\infty$, we will simply write $\mu_n^{+,\boldsymbol{J},\boldsymbol{h}}$ or $\mu_n^{-,\boldsymbol{J},\boldsymbol{h}}$, respectively. If the parameters $\boldsymbol{J}$ and $\boldsymbol{h}$ are clear from the context they will be omitted.

Corollary 2.3. *Under the hypothesis of Theorem 2.2 we have*

$$\int_X f\, d\mu_n^{+,\boldsymbol{J},\boldsymbol{h}} \leq \int_X f\, d\mu_{n-1}^{+,\boldsymbol{J},\boldsymbol{h}} \qquad \textit{and} \qquad \int_X f\, d\mu_{n-1}^{-,\boldsymbol{J},\boldsymbol{h}} \leq \int_X f\, d\mu_n^{-,\boldsymbol{J},\boldsymbol{h}}$$

Proof. The proof of these inequalities are similar, and so we only present the argument for the first one.

Note that from Corollary 2.1 it follows that

$$\int_X f\, d\mu_n^{+,\boldsymbol{J},\boldsymbol{h}} \leq \lim_{h_n \to \infty} \int_X f\, d\mu_n^{+,\boldsymbol{J},\boldsymbol{h}}.$$

By using the definition of $\mu_n^{+,\boldsymbol{J},\boldsymbol{h}}$ we have

$$\int_X f\, d\mu_n^{+,\boldsymbol{J},\boldsymbol{h}}$$
$$= \sum_{t=\pm 1} \frac{Z_{n-1}^{[1^\infty|t|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}}}{Z_n^{1^\infty,\boldsymbol{J},\boldsymbol{h}}} \sum_{x_1,\ldots,x_{n-1}=\pm 1} f([x|t|1^\infty]_n) \frac{\exp(H_n(x,[1^\infty|t|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}))}{Z_{n-1}^{[1^\infty|t|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}}} \tag{2.16}$$

A straightforward computation shows that

$$\lim_{h_n\to\infty} \frac{Z_{n-1}^{1^\infty,\boldsymbol{J},\boldsymbol{h}}}{Z_n^{1^\infty,\boldsymbol{J},\boldsymbol{h}}} = 1 \quad \text{and} \quad \lim_{h_n\to\infty} \frac{Z_{n-1}^{[1^\infty|-1|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}}}{Z_n^{1^\infty,\boldsymbol{J},\boldsymbol{h}}} = 0.$$

To compute the limit when $h_n \to \infty$ in the expression (2.16) one needs to observe that $\exp(H_n(x,[1^\infty|-1|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}))/Z_{n-1}^{[1^\infty|-1|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}}$ is bounded away from zero and infinity, for any choice of $(x_1,\ldots,x_n)\in\{-1,1\}^n$. Finally by using l'Hospital rule one can see that

$$\lim_{h_n\to\infty} \frac{\exp(H_n(x,1^\infty,\boldsymbol{J},\boldsymbol{h}))}{Z_{n-1}^{1^\infty,\boldsymbol{J},\boldsymbol{h}}} = \frac{\exp(H_{n-1}(x,1^\infty,\boldsymbol{J},\boldsymbol{h}))}{Z_{n-1}^{1^\infty,\boldsymbol{J},\boldsymbol{h}}}.$$

Piecing the last four observations together, we have

$$\begin{aligned}\lim_{h_n\to\infty}\int_X f\,d\mu_n^{+,\boldsymbol{J},\boldsymbol{h}} &= \sum_{x_1,\ldots,x_{n-1}=\pm1} f([x|1|1^\infty]_n)\frac{\exp(H_{n-1}(x,[1^\infty|1|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}))}{Z_{n-1}^{[1^\infty|1|1^\infty]_n,\boldsymbol{J},\boldsymbol{h}}}\\ &= \mu_{n-1}^{+,\boldsymbol{J},\boldsymbol{h}}(f).\end{aligned}$$

□

Corollary 2.4. *Under the hypothesis of Theorem 2.2 we have*

$$\int_X f\,d\mu_{n-1}^{-,\boldsymbol{J},\boldsymbol{h}} \le \int_X f\,d\mu_n^{-,\boldsymbol{J},\boldsymbol{h}} \le \int_X f\,d\mu_n^{x,\boldsymbol{J},\boldsymbol{h}} \le \int_X f\,d\mu_n^{+,\boldsymbol{J},\boldsymbol{h}} \le \int_X f\,d\mu_{n-1}^{+,\boldsymbol{J},\boldsymbol{h}}.$$

Proof. These four inequalities follows immediately from the two previous corollaries. □

3. FKG Inequality, Maximal Eigenmeasures and Eigenfunctions

We denote by $C(X)$ the set of all real continuous functions and consider the Banach space $(C(X),\|\cdot\|_\infty)$. Given a continuous potential $A: X\to\mathbb{R}$ we define the Ruelle operator $\mathscr{L}_A : C(X)\to C(X)$ as being the positive linear operator sending $f\longmapsto \mathscr{L}_A(f)$, where for each $x\in X$

$$\mathscr{L}_A(f)(x) \equiv \sum_{a\in\{-1,1\}} e^{A(ax)} f(ax), \quad \text{where } ax \equiv (a,x_1,x_2,\ldots).$$

Let λ_A denote the spectral radius of $\mathscr{L}_A$ acting on $(C(X),\|\cdot\|_\infty)$. If A is a continuous potential, then there always exists a Borel probability ν_A defined over X such $\mathscr{L}_A^*(\nu_A) = \lambda_A\,\nu_A$, where $\mathscr{L}_A^*$ is the dual operator of the Ruelle operator. We refer to any such ν_A as an eigenprobability for the potential A.

Proposition 3.1. *Let $n\ge 1$ and $\boldsymbol{J}\in\mathscr{R}(\mathbb{N})$ such that $J_{ij} = a_{|i-j|}\ge 0$, for some sequence $(a_n)_{n\ge1}$ and $\boldsymbol{h}\in\ell^\infty(\mathbb{N})$ such that $h_i = h$, for all $i\in\mathbb{N}$. Consider the potential $A: X\to\mathbb{R}$ given by $A(x) = hx_1 + x_1\sum_{n\ge2} a_n x_n$. Then for all $x,y\in X$ we have $H_n(x,y,\boldsymbol{J},\boldsymbol{h}) = S_n(A)([x|y]_n)$ and therefore for all continuous $f: X\to\mathbb{R}$ we have*

$$\frac{\mathscr{L}_A^n(f)(\sigma^n y)}{\mathscr{L}_A(1)(\sigma^n y)} = \int_X f\,d\mu_n^y = \int_X f\,d\mu_n^{y,\boldsymbol{J},\boldsymbol{h}}.$$

Proof. We first observe that the hypothesis $\boldsymbol{J} \in \mathscr{R}(\mathbb{N})$ guarantee that the potential A is well-defined since its expression is given by an absolutely convergent series, for any $x \in X$ one can see that A defines a continuous function. By rearranging the terms in the sum $S_n(A)([x|y]_n)$ it is easy to check that it is equal to $H_n(x, y, \boldsymbol{J}, \boldsymbol{h})$. Note that the translation invariance hypothesis placed in J_{ij} is crucial for validity of the previous statement. □

Corollary 3.1. *If* $A(x) = hx_1 + x_1 \sum_{n\geq 2} a_n x_n$*, where* $a_n \geq 0$ *for all* $n \geq 1$ *and* $\sum_n a_n < \infty$*, then*

$$\frac{\mathscr{L}_A^{n-1}(f)(-1^\infty)}{\mathscr{L}_A^{n-1}(1)(-1^\infty)} \leq \frac{\mathscr{L}_A^{n}(f)(-1^\infty)}{\mathscr{L}_A^{n}(1)(-1^\infty)} \leq \frac{\mathscr{L}_A^{n}(f)(\sigma^n(x))}{\mathscr{L}_A^{n}(1)(\sigma^n(x))} \leq \frac{\mathscr{L}_A^{n}(f)(1^\infty)}{\mathscr{L}_A^{n}(1)(1^\infty)} \leq \frac{\mathscr{L}_A^{n-1}(f)(1^\infty)}{\mathscr{L}_A^{n-1}(1)(1^\infty)}.$$

Proof. This follows from Proposition 3.1 and Corollary 2.4. □

3.1. *Uniqueness of the Eingemeasures for $\mathcal{E}$-Potentials*

If A is a potential of the form $A(x) = hx_1 + x_1 \sum_n a_n x_n$, where $a_n \geq 0$ and $\sum_n a_n < \infty$, then the above corollary implies the existence of the following limits

$$\lim_{n\to\infty} \frac{\mathscr{L}_A^{n}(f)(-1^\infty)}{\mathscr{L}_A^{n}(1)(-1^\infty)} \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathscr{L}_A^{n}(f)(1^\infty)}{\mathscr{L}_A^{n}(1)(1^\infty)}, \tag{3.1}$$

for all increasing function f depending only on a finite number of coordinates.

Let us consider a very important class of increasing functions. For any finite set $B \subset \mathbb{N}$ we define $\varphi_B : X \to \mathbb{R}$ by

$$\varphi_B(x) = \prod_{i\in B} \frac{1}{2}(1 + x_i). \tag{3.2}$$

For convenience, when $B = \emptyset$ we define $\varphi_B(x) \equiv 1$. The function φ_B is easily seen to be increasing since it is finite product of non-negative increasing functions. For any $i \in \mathbb{N}$ the following holds $(1/2)(1 + x_i)\frac{1}{2}(1 + x_i) = (1/4)(1 + 2x_i + x_i^2) = (1/4)(1 + 2x_i + 1) = (1/2)(1 + x_i)$. Therefore for any finite subsets $B, C \subset \mathbb{N}$ we have $\varphi_B(x)\varphi_C(x) = \varphi_{B\cup C}(x)$. This property implies that the collection $\mathscr{A}$ of all linear combinations of φ_B's is in fact an algebra of functions

$$\mathscr{A} \equiv \left\{ \sum_{j=1}^{n} a_j \varphi_{B_j} : n \in \mathbb{N}, a_j \in \mathbb{R} \text{ and } B_j \subset \mathbb{N} \text{ is finite} \right\}.$$

It is easy to see that $\mathscr{A}$ is an algebra of functions that separate points and contains the constant functions. Of course, $\mathscr{A} \subset C(X)$. Since X is compact it follows from the Stone-Weierstrass theorem that $\mathscr{A}$ is dense in $C(X)$.

Since φ_B depends only on $\#B$ coordinates follows from (3.1) and the linearity of the Rulle operator that we can define a linear functional $F^+ : \mathscr{A} \to \mathbb{R}$ by the following expression

$$F^+(\sum_{j=1}^{n} a_j \varphi_{B_j}) = \sum_{j=1}^{n} a_j \lim_{n\to\infty} \frac{\mathscr{L}_A^{n}(\varphi_{B_j})(1^\infty)}{\mathscr{L}_A^{n}(1)(1^\infty)}.$$

From the positivity of the Ruelle operator it follows that F^+ is continuous. Indeed,

$$F^+(\sum_{j=1}^{n} a_j\varphi_{B_j}) = \sum_{j=1}^{n} a_j \lim_{n\to\infty} \frac{\mathscr{L}_A^n(\varphi_{B_j})(1^\infty)}{\mathscr{L}_A^n(1)(1^\infty)} = \lim_{n\to\infty} \frac{\mathscr{L}_A^n(\sum_{j=1}^{n} a_j\varphi_{B_j})(1^\infty)}{\mathscr{L}_A^n(1)(1^\infty)}$$

$$\leq \lim_{n\to\infty} \frac{\mathscr{L}_A^n(\|\sum_{j=1}^{n} a_j\varphi_{B_j}\|_\infty \cdot 1)(1^\infty)}{\mathscr{L}_A^n(1)(1^\infty)} = \|\sum_{j=1}^{n} a_j\varphi_{B_j}\|_\infty.$$

We prove analogous lower bounds and therefore

$$\Big|F^+(\sum_{j=1}^{n} a_j\varphi_{B_j})\Big| \leq \|\sum_{j=1}^{n} a_j\varphi_{B_j}\|_\infty.$$

Since $\mathscr{A}$ is dense in $C(X)$ the functional F^+ can be extended to a bounded linear functional defined over all $C(X)$. Clearly F^+ is positive bounded functional and $F^+(1) = 1$. Therefore it follows from the Riesz-Markov theorem that there exists a probability measure μ^+ such that

$$F^+(f) = \int_X f\, d\mu^+.$$

Similarly we define F^- and μ^-.

For the functions $\varphi \in \mathscr{A}$ a bit more can be said

$$\lim_{n\to\infty} \frac{\mathscr{L}_A^n(\varphi)(\pm 1^\infty)}{\mathscr{L}_A^n(1)(\pm 1^\infty)} = F^\pm(\varphi) = \int_X \varphi\, d\mu^\pm. \tag{3.3}$$

Of course, both probability measures $\mu^\pm$ depends on A which in turn depends on $(a_n)_{n\in\mathbb{N}}$ and h, but we are omitting such dependence to lighten the notation.

Theorem 3.1. *Let A be a potential as in Corollary 3.1 and $\mu^\pm$ the probability measures defined above. Then*

$$\mu^+ = \mu^- \quad \Longleftrightarrow \quad \int_X x_i\, d\mu^+(x) = \int_X x_i\, d\mu^-(x) \qquad \forall\, i \in \mathbb{N} \tag{3.4}$$

Proof. If $\mu^+ = \mu^-$ then the rhs of (3.4) is obvious. Conversely, assume that lhs of (3.4) holds. Let $\varphi \in \mathscr{A}$ be an increasing function. From the Corollary 3.1 and the identity (3.3) we have

$$0 \leq \lim_{n\to\infty} \frac{\mathscr{L}_A^n(\varphi)(1^\infty)}{\mathscr{L}_A^n(1)(1^\infty)} - \lim_{n\to\infty} \frac{\mathscr{L}_A^n(\varphi)((-1)^\infty)}{\mathscr{L}_A^n(1)((-1)^\infty)} = \int_X \varphi\, d\mu^+ - \int_X \varphi\, d\mu^-. \tag{3.5}$$

Fix a finite subset $B \subset \mathbb{N}$ and define

$$\psi(x) = \sum_{i\in B} x_i - \varphi_B(x).$$

Clearly we have $\psi \in \mathscr{A}$. We claim that ψ is increasing function. To prove the claim take $x, y \in X$ such that $y \succeq x$. If $x_i = y_i$ for all $i \in B$ then $\psi(x) = \psi(y)$ and obviously $\psi(x) \leq \psi(y)$. Suppose that there exist $j \in B$ such that $-1 = x_j < y_j = 1$.

Since φ_B takes only values zero or one, we have $-1 \leq \varphi_B(x) - \varphi_B(y) \leq 1$, by definition of j we have $y_j - x_j = 2$ so

$$\psi(y) - \psi(x) = \sum_{i \in B} y_i - \varphi_B(y) - \sum_{i \in B} x_i + \varphi_B(x) = \sum_{i \in B} (y_i - x_i) + \varphi_B(x) - \varphi_B(y)$$
$$= \sum_{i \in B \setminus \{j\}} (y_i - x_i) + 2 + \varphi(x) - \varphi_B(y) \geq \sum_{i \in B \setminus \{j\}} (y_i - x_i) \geq 0.$$

Since $\psi \in \mathscr{A}$ and increasing follow from (3.5) and the hypothesis that

$$0 \leq \int_X \psi \, d\mu^+ - \int_X \psi \, d\mu^-$$
$$= \int_X [\sum_{i \in B} x_i - \varphi_B(x)] \, d\mu^+(x) - \int_X [\sum_{i \in B} x_i - \varphi_B(x)] \, d\mu^-(x)$$
$$= \int_X \varphi_B(x) \, d\mu^-(x) - \int_X \varphi_B(x) \, d\mu^+(x) \leq 0.$$

Therefore for any finite $B \subset \mathbb{N}$ we have

$$\int_X \varphi_B(x) \, d\mu^-(x) = \int_X \varphi_B(x) \, d\mu^+(x).$$

By linearity of the integral the above indentity extends to any function $\varphi \in \mathscr{A}$. Since $\mathscr{A}$ is a dense subset of $C(X)$ it follows that $\mu^+ = \mu^-$. □

In what follows $\mathcal{G}^*(A)$ denotes the set of eigenprobabilities of $\mathscr{L}_A^*$ associated to its spectral radius. We use the notation $\mathcal{G}^{\mathrm{DLR}}(A)$ for the set of all probability measures satisfying the DLR condition for some specification determined by A, see [9] for more details.

Theorem 3.2 (See [9]). *For all $A \in C(X)$ we have that $\mathcal{G}^*(A) \subset \mathcal{G}^{\mathrm{DLR}}(A)$.*

Theorem 3.3 (Uniqueness). *Let A be a potential as in Corollary 3.1. If $\mu^+ = \mu^-$ then $\mathcal{G}^*(A)$ is a singleton.*

Proof. Since $A(x) = hx_1 + x_1 \sum_{n \geq 2} a_n x_n$ and $\sum_n a_n < \infty$ then A is continuous. For this potential it is very well known that the set $\mathcal{G}^{\mathrm{DLR}}(A)$ is the closure of the convex hull of all the cluster points of the sequences $(\mu_n^y)_{n \in \mathbb{N}}$, for all $y \in X$.

Given a finite subset $B \subset \mathbb{N}$ let $n \geq 1$ be such that $B \subset \{1, \ldots, n\}$. From Corollary 2.4 we get

$$\int_X \varphi_B \, d\mu_n^- \leq \int_X \varphi_B \, d\mu_n^y \leq \int_X \varphi_B \, d\mu_n^+.$$

If μ is any cluster point of $(\mu_n^y)_{n \in \mathbb{N}}$ then follows from the last inequalities that

$$\int_X \varphi_B \, d\mu^- \leq \int_X \varphi_B \, d\mu \leq \int_X \varphi_B \, d\mu^+.$$

The above inequality is in fact an equality by hypothesis. By linearity we can extend the last conclusion to any function $g \in \mathscr{A}$ and therefore follows from the denseness of $\mathscr{A}$ and from the hypothesis that

$$\int_X f\, d\mu^- = \int_X f\, d\mu = \int_X f\, d\mu^+, \qquad \forall\, f \in C(X).$$

Thus proving that the set of the cluster points of $(\mu_n^y)_{n\in\mathbb{N}}$ is a singleton, implying that $\mathcal{G}^{\mathrm{DLR}}(A) \supset \mathcal{G}^*(A)$ is also a singleton. □

3.2. *Properties of the Eigenfunctions of Attractive Potentials*

Definition 3.1. We say that a continuous potential $A : X \to \mathbb{R}$ is mirrored if $A(x) = A(-x)$, for all $x \in X$, where $-x \equiv (-x_1, -x_2, \ldots)$. We denote by $\mathcal{I}$ the set of mirrored potentials.

As an example of a mirrored potential is given by an Ising type potential of the form $A(x_1, x_2, \ldots) = x_1x_2a_1 + x_1x_3a_2 + \ldots + x_1x_{n+1}a_n + \ldots$, where $\sum_n |a_n| < \infty$. Of course, the Dyson potential with $h = 0$ is an element on the above family of potentials.

If in addition we assume that in the above potential that $a_j \geq 0$, for all $j \geq 1$ then we have that $A \in \mathcal{E}$. In this section we established some results for potentials of this form but not living in the space $\mathcal{E}$.

Proposition 3.2. *If $A \in \mathcal{I}$ and φ is an eigenfunction for $\mathscr{L}_A$ associated to an eigenvalue λ of $\mathscr{L}_A$, then $\tilde{\varphi} : X \to \mathbb{R}$, given by $\tilde{\varphi}(x) \equiv \varphi(-x)$ is also an eigenfunction associated to λ.*

Proof. Indeed, for any $(x_1, x_2, \ldots) \in X$ we have $\lambda\varphi(x) = e^{A(ax)}\,\varphi(ax) + e^{A(-a,x_1,x_2,\ldots)}\,\varphi(-a, x_1, x_2, \ldots)$. Since $A \in \mathcal{I}$ follows from the last equation that $\lambda\tilde{\varphi}(-x) = e^{A(-ax)}\,\tilde{\varphi}(-ax) + e^{A(a,-x_1,-x_2,\ldots)}\,\tilde{\varphi}(a, -x_1, -x_2, \ldots)$. By taking $y_j = -x_j$, for all j, we get $\lambda\tilde{\varphi}(y_0, y_1, \ldots) = e^{A(-a,y_1,y_2,\ldots)}\,\tilde{\varphi}(-a, y_1, y_2, \ldots) + e^{A(ay)}\,\tilde{\varphi}(ay)$, which means that $\tilde{\varphi}$ is an eigenfunction associated to the eigenvalue λ. □

Remark 3.1. If $A \in \mathcal{I}$ and φ is a strictly positive and continuous eigenfunction associated to the spectral radius λ_A then $\varphi(x_1, x_2, \ldots) = \varphi(-x_1, -x_2, \ldots)$. This equality follows from the above and the uniqueness of a strictly positive eigenfunction associated to the maximal eigenvalue for a continuous potential, for details see [26].

Proposition 3.3. *Let $A \in \mathcal{I}$ and ν an eigenprobability for $\mathscr{L}_A^*$, associated to the eigenvalue λ_A. If $\tilde{\nu}$ is the unique Borel probability measure defined by the following functional equation*

$$\int_X f(x)\, d\tilde{\nu}(x) = \int_X f(-x)\, d\nu(x), \qquad \forall f \in C(X)$$

then $\tilde{\nu}$ is also an eigenprobability associated to the eigenvalue λ_A.

Proof. It is enough to prove that for any real continuous function $g : X \to \mathbb{R}$, we have $\lambda_A \int_X g(x)\, d\tilde{\nu}(x) = \int_X e^{A(ax)}\, g(ax) + e^{A(-a,x_1,x_2,\ldots)}\, g(-a,x_1,x_2,\ldots)\, d\tilde{\nu}(x)$. Given any continuous function f it follows from the hypothesis that

$$
\begin{aligned}
\lambda_A \int_X f(x)\, d\nu(x) &= \int_X e^{A(ax)}\, f(ax) + e^{A(-a,x_1,x_2,\ldots)}\, f(-a,x_1,x_2,\ldots)\, d\nu(x) \\
&= \int_X e^{A(-ax)}\, f(ax) + e^{A(a,-x_1,-x_2,\ldots)}\, f(-a,x_1,x_2,\ldots)\, d\nu(x) \\
&= \int_X e^{A(ax)}\, f(-ax) + e^{A(-a,x_1,x_2,\ldots)}\, f(a,-x_1,-x_2,\ldots)\, d\tilde{\nu}(x).
\end{aligned}
$$

By taking $f(x) = g(-x)$ in the above expression, we get

$$
\begin{aligned}
\lambda_A \int_X g(x)\, d\tilde{\nu}(x) = \lambda_A \int_X g(-x)\, d\nu(x) &= \lambda_A \int_X f(x)\, d\nu(x) \\
&= \int_X e^{A(-a,x_1,x_2,\ldots)}\, f(a,-x_1,-x_2,\ldots) + e^{A(ax)}\, f(-ax)\, d\tilde{\nu}(x) \\
&= \int_X e^{A(-a,x_1,x_2,\ldots)}\, g(-a,x_1,x_2,\ldots) + e^{A(ax)}\, g(ax)\, d\tilde{\nu}(x). \qquad \square
\end{aligned}
$$

Remark 3.2. If the eigenprobability ν associated to λ_A, of a mirrored potential is unique, then for any continuous function $f : X \to \mathbb{R}$ we have that $\int_X f(x) d\nu(x) = \int_X f(-x) d\nu(x)$. We shall observe that the results of this section can be applied to the Dyson potential, under appropriate assumptions and restrictions.

3.3. *Monotonic Eigenfunctions and Uniqueness*

In this section we follow closely [18, 22] adapting, to our context, their results for g-measures to non-normalized potentials.

Definition 3.2 (Class $\mathcal{F}$). We say that a potential A belongs to the class $\mathcal{F}$ if for all $y \succeq x$ we have both inequalities $e^{A(-1x)} + e^{A(1x)} \leq e^{A(-1y)} + e^{A(1y)}$, and $e^{A(1y)} - e^{A(1x)} \geq 0$.

Note that the above condition is equivalent to requiring $\mathscr{L}_A(1)(x)$ and $\mathscr{L}_A(1_{[1]})(x)$ be increasing functions. A simple example of a potential belonging to the class $\mathcal{F}$ is given by $A(x) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots + a_nx_n + \ldots$, where $a_n \geq 0$.

Proposition 3.4. *If $A \in \mathcal{F}$ and f is an increasing non-negative function, then $\mathscr{L}_A(f)$ is increasing function.*

Proof. Suppose $A \in \mathcal{F}$, $f \geq 0$, and $f \in \mathcal{I}$. Then follows directly from the definition of the class $\mathcal{F}$ that if $y \succeq x$ then $e^{A(-1x)} - e^{A(-1y)} \leq e^{A(1y)} - e^{A(1x)}$ and $e^{A(1y)} - e^{A(1x)} \geq 0$. By using the above observations and the definition of the Ruelle operator we get for $y \succeq x$ the following inequalities

$$\begin{aligned}\mathscr{L}_A(f)(y) - \mathscr{L}_A(f)(x) &= e^{A(1y)}f(1y) + e^{A(-1y)}f(-1y) \\ &\quad - e^{A(1x)}f(1x) - e^{A(-1x)}f(-1x) \\ &\geq e^{A(1y)}(f(1y) - f(1x)) + e^{A(-1y)}(f(-1y) - f(-1x)) \\ &\quad + f(1x)(e^{A(1y)} - e^{A(1x)}) - f(-1x)(e^{A(1y)} - e^{A(1x)}) \\ &= e^{A(1y)}(f(1y) - f(1x)) + e^{A(-1y)}(f(-1y) - f(-1x)) \\ &\quad + (f(1x) - f(-1x))(e^{A(1y)} - e^{A(1x)}) \\ &\geq 0. \end{aligned}$$

□

Corollary 3.2. *If $A \in \mathcal{F}$ then for any $n \geq 1$ the function $x \longmapsto \mathscr{L}_A^n(1)(x)/\lambda_A^n$ is an increasing function.*

Proof. If $f : X \to \mathbb{R}$ is a non-negative increasing function and $A \in \mathcal{F}$, then follows from the previous corollary that $g(x) \equiv \mathscr{L}_A(f)(x)$ is increasing and from positivity of the Ruelle operator we get $g \geq 0$. Therefore we can ensure that $\mathscr{L}_A^2(f)(x) \geq 0$ and increasing. Finally, by a formal induction we get that $\mathscr{L}_A^n(f)(x) \geq 0$ is monotone for each n. Since $\lambda_A^n > 0$ and $f \equiv 1$ is non-negative increasing function the corollary follows. □

Remark 3.3. If $A \in \mathcal{F}$ and A is a Hölder potential then we know that $\mathscr{L}_A^n(1)(x)/\lambda_A^n \to \varphi(x)$, when $n \to \infty$, uniformly in x, and φ is the main eigenfunction of $\mathscr{L}_A$, associated to λ_A. Since pointwise limit of increasing functions is an increasing function it follows that the eigenfunction φ is an increasing function.

We now consider a more general situation than the one in previous remark. We assume again that $A \in \mathcal{F}$, but now we also assume that the sequence of functions $(\varphi_n)_{n\geq 1}$ given by

$$\varphi_n(x) \equiv \frac{1}{n}\sum_{j=0}^{n-1} \lambda_A^{-j}\, \mathscr{L}_A^j(1)(x)$$

has a pointwise everywhere convergent subsequence $(\varphi_{n_k})_{k\geq 1}$. We also need to assume that

$$0 < \liminf_{n\to\infty} \frac{\mathscr{L}_A^n(1)(1^\infty)}{\lambda_A^n} \leq \limsup_{n\to\infty} \frac{\mathscr{L}_A^n(1)(1^\infty)}{\lambda_A^n} < +\infty.$$

From monotonicity it will follow that $\lambda_A^{-n}\, \mathscr{L}_A^n(1)(x)$ is uniformly bounded away from zero and infinity in n and x.

Under such hypothesis it is simple to conclude that $0 \leq \varphi$ is a $L^1(\nu_A)$ eigenfunction of $\mathscr{L}_A$, associated to its main eigenvalue λ_A. If $\varphi((-1)^\infty) \geq c > 0$, then $0 < c \leq \varphi$.

Since the set of all cylinders of X is countable, up to a Cantor diagonal procedure, we can assume that the following limits exist for any cylinder set C

$$\mu^+(C) \equiv \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{\mathscr{L}_A^j(1_C)(1^\infty)}{\lambda_A^j} \text{ and } \mu^-(C) \equiv \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{\mathscr{L}_A^j(1_C)(-1^\infty)}{\lambda_A^j}. \tag{3.6}$$

By standards arguments one can show that $\mu^\pm$ can be both extended to positive measures on the borelians of X (they are not necessarily probability measures). These measures satisfy for any continuous function f the following identity

$$\int_X f\, d\mu^\pm = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{\mathscr{L}_A^j(f)(\pm 1^\infty)}{\lambda_A^j}.$$

We claim that $\mu^\pm$ are eigenmeasures associated to λ_A. Indeed, they are both non-trivial measures since $0 < \varphi(-1^\infty) = \mu^-(X) \leq \mu^+(X)$ and also bounded measures since $\mu^-(X) \leq \mu^+(X) \leq \varphi(1^\infty) < +\infty$. For any $f \in C(X)$ the condition $\varphi(1^\infty) < +\infty$ implies

$$\limsup_{n\geq 1} \left[\frac{1}{n} \frac{\mathscr{L}_A^n(f)(1^\infty)}{\lambda_A^n} \right] = 0.$$

From the above observations and the definition of the dual of the Ruelle operator, for any continuous function f we have

$$\begin{aligned}
\frac{1}{\lambda_A} \int_X f\, d[\mathscr{L}_A^* \mu^+] &= \frac{1}{\lambda_A} \int_X \mathscr{L}_A(f)\, d\mu^+ = \frac{1}{\lambda_A} \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{\mathscr{L}_A^j(\mathscr{L}_A(f))(1^\infty)}{\lambda_A^j} \\
&= \lim_{k\to\infty} \frac{1}{n_k} \left[\frac{\mathscr{L}_A^{n_k}(f)(1^\infty)}{\lambda_A^{n_k}} - f(1^\infty) + \sum_{j=0}^{n_k-1} \frac{\mathscr{L}_A^j(f)(1^\infty)}{\lambda_A^j} \right] \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{\mathscr{L}_A^j(f)(1^\infty)}{\lambda_A^j} = \int_X f\, d\mu^+.
\end{aligned}$$

The above equation shows that μ^+ is an eigenmeasure. A similar argument applies to μ^- and therefore the claim is proved.

Proposition 3.5. *Let A be a continuous potential, and λ_A the spectral radius of $\mathscr{L}_A$ acting on $C(X)$. Assume that for any continuous function $f : X \to \mathbb{R}$, the following limit exists and is independent of x*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{\mathscr{L}_A^j(f)(x)}{\lambda_A^j} = c(f) \quad \textit{and} \quad \sup_{n\geq 1} \left\| \frac{1}{n} \sum_{j=0}^{n-1} \frac{\mathscr{L}_A^j(f)}{\lambda_A^j} \right\|_\infty < +\infty \tag{3.7}$$

Then $\mathcal{G}^*(A)$ is a singleton.

Proof. Since A is continuous follows from [9] that $\mathcal{G}^*(A)$ is not empty. If $\nu \in \mathcal{G}^*(A)$, then follows from the basic properties of $\mathscr{L}_A$ that for any $f \in C(X)$ and $j \in \mathbb{N}$ we have

$$\int_X f \, d\nu = \int_X \frac{\mathscr{L}_A^j(f)}{\lambda_A^j} \, d\nu.$$

From this identity and the Lebesgue Dominated Convergence Theorem we have

$$\int_X f \, d\nu = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_X \frac{\mathscr{L}_A^j(f)(x)}{\lambda^j} \, d\nu(x) = c(f).$$

Since the above equality is independent of the choice of ν, we conclude that $\mathcal{G}^*(A)$ has to be a singleton. □

Proposition 3.6. *Let A be a potential $\mathcal{F}$ and λ_A the spectral radius of $\mathscr{L}_A$ acting on the space $C(X)$. If for some $x \in X$ we have $0 < \inf\{\lambda_A^{-n} \mathscr{L}_A^n(1)(x) : n \geq 1\}$ and for all $B \subset \mathbb{N}$ we have*

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{j=0}^{n-1} \lambda_A^{-j} \mathscr{L}_A^j(\varphi_B)(1^\infty) - \frac{1}{n} \sum_{j=0}^{n-1} \lambda_A^{-j} \mathscr{L}_A^j(\varphi_B)(-1^\infty) \right| = 0. \tag{3.8}$$

Then, there exists a continuous positive eigenfunction h for the Ruelle operator $\mathscr{L}_A$, associated to λ_A. Moreover, the measures μ^+ and μ^- defined as in (3.6) are the same and $\mathcal{G}^*(A)$ is a singleton.

Proof. The first step is to show that the sequence $(\varphi_n)_{n\geq 1}$ defined by

$$\varphi_n = \frac{1}{n} \sum_{j=0}^{n-1} \lambda_A^{-j} \, \mathscr{L}_A^j(1)$$

has a cluster point in $C(X)$. The idea is to use the monotonicity of φ_n to prove that this sequence is uniformly bounded and equicontinuous. In fact, for any $j \geq 1$ we have

$$\lambda_A^{-j} \mathscr{L}_A^j(1)(-1^\infty) \leq \int_X \lambda_A^{-j} \mathscr{L}_A^j(1) \, d\nu_A = \int_X 1 \, d\nu_A = 1.$$

Therefore $\varphi_n(-1^\infty)$ is a bounded sequence of real numbers. From the hypothesis (3.8), with $B = \emptyset$, follows that $\varphi_n(1^\infty)$ is also bounded. Since φ_n are increasing function we have the following uniform bound $|\varphi_n(x)| = \varphi_n(x) \leq \sup_{n\geq 1} \varphi_n(1^\infty)$, thus proving that $(\varphi_n)_{n\geq 1}$ is uniformly bounded sequence in $C(X)$. To verify that $(\varphi_n)_{n\geq 1}$ is an equicontinuous family it is enough to use the following upper and lower bounds

$$\varphi_n(-1^\infty) - \varphi_n(1^\infty) \leq \varphi_n(x) - \varphi_n(y) \leq \varphi_n(1^\infty) - \varphi_n(-1^\infty), \qquad \forall \, x, y \in X$$

together with the hypothesis (3.8). Now the existence of a cluster point for the sequence $(\varphi_n)_{n\geq 1}$ is a consequence of Arzela-Ascoli's Theorem, that is, there is some

$\varphi \in C(X)$ such that $\|\varphi_{n_k} - \varphi\|_\infty \to 0$, when $k \to \infty$. Since $0 < \inf\{\lambda_A^{-n} \mathscr{L}_A^n(1)(x) : n \geq 1\}$ follows from the monotonicity of φ that $\varphi(x) \neq 0$. By using the continuity of φ and the argument presented in [26] to prove uniqueness of the eigenfunctions one can see that $\varphi(y) \neq 0$ for every $y \in X$. As we observed next to Remark 4, φ is an eigenfunction of $\mathscr{L}_A$, associated to λ_A.

Now we will prove the statement about $\mathcal{G}^*(A)$. Since φ_B is an increasing function follows from Proposition 3.4 that

$$\frac{1}{n}\sum_{j=0}^{n-1} \lambda_A^{-j} \mathscr{L}_A^j(\varphi_B)(-1^\infty) \leq \frac{1}{n}\sum_{j=0}^{n-1} \lambda_A^{-j} \mathscr{L}_A^j(\varphi_B)(x) \leq \frac{1}{n}\sum_{j=0}^{n-1} \lambda_A^{-j} \mathscr{L}_A^j(\varphi_B)(1^\infty) \quad (3.9)$$

for all $n \geq 1$. From the above inequality and the hypothesis (3.8) is clear that the limit, when $n \to \infty$, of the second sum in (3.9) exist and is independent of x. Therefore the linear mapping

$$\mathscr{A} \ni f \longmapsto \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \lambda_A^{-j} \mathscr{L}_A^j(f)(x) = c(f),$$

defines a positive bounded operator over the algebra $\mathscr{A}$. By using the denseness of $\mathscr{A}$ in $C(X)$ and the Stone-Weierstrass theorem it follows that the above limit is well-defined and independent of x for any continuous function f. Since $\varphi(1^\infty) < +\infty$ all the hypothesis of Proposition 3.5 are satisfied and so we can ensure that $\mathcal{G}^*(A)$ is a singleton, finishing the proof. □

4. Involution Kernel, Eigenfunctions and Pressure

In this section we obtain a semi-explicit expressions for eigenfunctions of the Ruelle operator $\mathscr{L}_A$, associated to the maximal eigenvalue for a large class of potentials A. The main technique used here is the involution Kernel. Before present its definition and some of its basic properties we need set up some notations.

From now on the symbolic space is taken as $X = \{-1,1\}^{\mathbb{N}}$ and we use the notation $\hat{X} \equiv \{-1,1\}^{\mathbb{Z}}$. The set of all sequences written in "backward direction" $\{(\ldots, y_2, y_1) : y_j \in \{-1,1\}\}$ will be denoted by X^*, and given a pair $x \in X$ and $y \in X^*$ we defined $(y|x) \equiv (\ldots, y_2, y_1|x_1, x_2, \ldots)$. Using such pairs we can identify $\hat{X}$ with the cartesian product $X^* \times X$. This bi-sequence space is sometimes called the natural extension of X. The left shift mapping on $\hat{X}$ will be denoted by $\hat{\sigma}$ and defined as usual by

$$\hat{\sigma}(\ldots, y_2, y_1|x_1, x_2, x_3 \ldots) = (\ldots, y_2, y_1, x_1|x_2, x_3, \ldots).$$

Definition 4.1. Let $A : X \to \mathbb{R}$ be a continuous potential (considered as a function on $\hat{X}$). We say that a continuous function $W : \hat{X} \to \mathbb{R}$ is an involution kernel for A, if there exists continuous potential $A^* : X^* \to \mathbb{R}$ (considered as a function on $\hat{X}$) such that for any $a \in \{-1,1\}$, $x \in X$ and $y \in X^*$, we have

$$(A^* + W)(ya|x) = (A + W)(y|ax). \quad (4.1)$$

We say A^* the dual of the potential A (using W) and A is symmetric if for some involution kernel W, we have $A = A^*$.

To simplify the notation we write simply $A(x)$, $A^*(y)$ and $W(y|x)$ during the computations. For general properties of involutions kernels, we refer the reader to the references [4, 14, 24]. In several examples one can get the explicit expression for W and A^* (see Section 5 of [3]). The reader should be warned that the involution kernel W is not unique.

Example 4.1. Let $A : X \to \mathbb{R}$ be a continuous potential (considered as a function on $\hat{X}$) given by $A(x_1, x_2, x_3, ...) = a_1\, x_1 + a_2\, x_2 + ... + x_n\, a_n + \ldots$, where $\sum_k \sum_{n\geq k} |a_n| < \infty$. A large class of such potentials were carefully studied in [6] and spectral properties of the Ruelle operator were obtained there.

We claim that $A^* = A$ (for some choice of W). Indeed, let $k = \sum_{j\geq 2} a_j$. and define for any $(x|y) \in \hat{X}$ the following function

$$\begin{aligned} W(y|x) = [\ldots + (k - (a_2 + a_3 + a_4))\, y_4 + (k - (a_2 + a_3))\, y_3 + (k - a_2)\, y_2 + k\, y_1 \\ + k\, x_1 + (k - a_2)\, x_2 + (k - (a_2 + a_3))\, x_3 + (k - (a_2 + a_3 + a_4))\, x_4 + \ldots]. \end{aligned}$$

Using the hypothesis placed on the coefficients a_n's we can rewrite

$$W(y|x) = \sum_{i\geq 1} (x_i + y_i)\, (a_{i+1} + a_{i+2} + \ldots).$$

A simple computation shows that for any $a \in \{-1, 1\}$, $x \in X$ and $y \in X^*$, we have the following identity

$$A(a\, y) + W(y\, a \,|\, x) = (A + W)(y\, a \,|\, x) = (A + W)(y \,|\, a\, x) = A(a\, x) + W(y \,|a\; x),$$

thus showing that A is simetric, i.e., $A = A^*$.

In [6] is shown that the main eigenfunction of $\mathscr{L}_A$ is given by $\varphi(x) = \exp(\alpha_1 x_1 + \alpha_2 x_2 + \ldots)$, where $\alpha_n = a_{n+1} + a_{n+2} + \ldots$ and the main eigenvalue is $\exp(\sum_{j=1}^{\infty} a_j) + \exp(-\sum_{j=1}^{\infty} a_j)$. If $\beta > 0$ is fixed and the coefficients $(a_n)_{n\geq 1}$ are given by $a_j = \beta\, j^{-\gamma}$ for all $j \in \mathbb{N}$, we get that the main eigenvalue is equals to $2\cosh(\beta\zeta(\gamma))$.

Example 4.2. For an Ising type potential of the form $A(x) = x_1 x_2 a_1 + x_1 x_3 a_2 + \ldots + x_1 x_n a_{n-1} + \ldots$, we can formally write an expression for the involution kernel W, which is

$$W(y|x) = y_1\, (\sum_{j=1}^{\infty} x_j\, a_j)\; + y_2\, (\sum_{j=1}^{\infty} x_j\, a_{j+1})\; + \ldots + y_k\, (\sum_{j=1}^{\infty} x_j\, a_{j+k-1}) + \ldots. \quad (4.2)$$

Of course, to give a meaning for the above expression some restrictions need to be imposed on $(a_n)_{n\geq 1}$. We return to this issue latter.

Theorem 4.1. *Let A be a continuous potential for which there exists an involution kernel W. Let A^* be a continuous potential satisfying the equation* (4.1) *and* ν_{A^*}

an eigenprobability of $\mathscr{L}^*_{A^*}$*, associated to the spectral radius* λ_{A^*}*. Then the function*

$$\varphi(x) \equiv \int_{X^*} e^{W(y|x)} d\nu_{A^*}(y) \tag{4.3}$$

is a continuous positive eigenfunction for the Ruelle operator $\mathscr{L}_A$ *associated to* λ_{A^*}*.*

Proof. Since $\mathscr{L}^*_{A^*}\nu_{A^*} = \lambda_{A^*}\nu_{A^*}$. we have for any continuous function $f : \hat{X} \to \mathbb{R}$ the following identity

$$\int_{X^*} f(y) d\nu_{A^*}(y) = \lambda_{A^*} \int_{X^*} \mathscr{L}_{A^*}(f)(y)\, d\nu_{A^*}(y).$$

On the other hand,

$$\begin{aligned}
\mathscr{L}_A(\varphi)(x) &= \mathscr{L}_A\left(\int_{X^*} e^{W(y|x)} d\nu_{A^*}(y)\right)(x) = \sum_{a=\pm 1} e^{A(a\,x)} \int_{X^*} e^{W(y|ax)} d\nu_{A^*}(y) \\
&= \int_{X^*} \left[\sum_{a=\pm 1} e^{A(a\,x)} e^{W(y|ax)}\right] d\nu_{A^*}(y) \\
&= \int_{X^*} \left[\sum_{a=\pm 1} e^{A^*(y\,a)} e^{W(ya|x)}\right] d\nu_{A^*}(y) \\
&= \int_{X^*} \mathscr{L}_{A^*}\left(e^{W(\cdot|x)}\right)(y)\, d\nu_{A^*}(y) = \lambda_{A^*} \int_{X^*} e^{W(y|x)} d\nu_{A^*}(y) \\
&= \lambda_{A^*}\varphi(x).
\end{aligned}$$

□

4.1. *Involution Kernel and the Dyson model*

Now we consider some continuous Ising type potentials of the form

$$A(x) = \sum_{j=1}^{\infty} \frac{x_1 x_{j+1}}{j^\gamma}, \quad \text{where } \gamma > 1$$

and the formal series

$$W(y|x) = \sum_{k=1}^{\infty} y_k \,(\sum_{j=1}^{\infty} x_j\,(j+k-1)^{-\gamma}) \tag{4.4}$$

Such W is well defined and is continuous, whenever $\gamma > 2$. If the terms in the above formal sum can be rearranged then we can show that $W(x|y) = W(y|x)$. In such cases, a simple algebraic computation give us the following relation

$$A(ay) + W(ya\,|x) = (A+W)(ya|x) = (A+W)(y|\,a\,x) = A(a\,x) + W(y\,|\,a\,x) \tag{4.5}$$

for any $a \in \{-1, 1\}$, showing that A is symmetric. By multiplying both sides of the above equation by $\beta > 0$ we get that βW is an involution kernel for βA.

A natural question: is the involution kernel $\nu_A \times \nu_A$ almost everywhere well-defined, where ν_A is some eigenprobability ? If the answer is affirmative, then above formula for W provides an measurable eigenfunction.

Let $\widetilde{X}$ be the subset of all $x = (x_1, x_2, \ldots)$ in $X = \{-1, 1\}^{\mathbb{N}}$ such that there exist and N such that $x_j = -x_{j+1}$ for all $N \le j$. Note that the set $\widetilde{X}$ is dense subset of X and if $x \in \widetilde{X}$, then their preimages are also in $\widetilde{X}$.

Suppose $1 < \gamma < 2$, then, for each k we have that $\sum_{j=1}^{\infty} x_j\,(j+k-1)^{-\gamma}$ converges and it is of (at most) order $k^{-\gamma}$, when $k \to \infty$. In this way for such $x \in \widetilde{X}$ we get that $W(y|x)$ is well defined for all y.

Theorem 4.2. *Consider the potential $A(x) = \sum_{j=1}^{\infty} j^{-\gamma} x_1 x_{j+1}$, where $1 < \gamma < 2$. There exist a non-negative function $\widetilde{\varphi} : \widetilde{X} \to \mathbb{R}$ such that for any $x \in \widetilde{X}$ we have $\mathscr{L}_A(\widetilde{\varphi})(x) = \lambda_A\, \widetilde{\varphi}(x)$, where λ_A is the spectral radius of $\mathscr{L}_A$, acting on $C(X)$.*

Proof. Let ν_A the eingenprobability of $\mathscr{L}_A^*$, associated to the spectral radius λ_A, that is, $\mathscr{L}_A^* \nu_A = \lambda_A \nu_A$. We denote by ν_{A^*} the eingenprobability of $\mathscr{L}_{A^*}$. Since $A = A^*$, we get that $\nu_{A^*} = \nu_A$.

Note that for all $x \in \widetilde{X}$ the expression (4.4) for $W(y|ax)$ is well-defined for any a and y. Finally, for such x we can proceed as in the proof of Theorem 4.2 to verify that $0 \le \widetilde{\varphi}(x) \equiv \int_{X^*} e^{W(y|x)} d\nu_A(y)$ is a solution to the eigenvalue problem $\mathscr{L}_A(\widetilde{\varphi})(x) = \lambda_A \widetilde{\varphi}(x)$. □

4.2. *Topological Pressure of some Long-Range Ising Models*

Now we show how to use the involution kernel representation of the main eigenfunction, to obtain bounds on the main eigenvalue $\lambda_{\beta A}$, where A is an Ising type potential, of the form $A(x) = \sum_{j=1}^{\infty} j^{-\gamma} x_1 x_{j+1}$, where $\gamma > 2$ and $\beta > 0$. For such potentials the main eigenfunction φ_β, associated to the main eigenvalue λ_β, is given by $\varphi_\beta(x) = \int_{X^*} e^{\beta\, W(y|x)} d\nu_{\beta A}(y)$ and positive everywhere. From the definitions we have

$$\lambda_\beta \varphi_\beta(1,1,1,..) = e^{\beta A(1,1,1,..)} \varphi_\beta(1,1,...) + e^{\beta A(-1,1,1,..)} \varphi_\beta(-1,1,...) \tag{4.6}$$

and therefore

$$\lambda_\beta = e^{\beta A(1,1,1,..)} + e^{\beta A(-1,1,1,..)} \frac{\varphi_\beta(-1,1,...)}{\varphi_\beta(1,1,1,..)} = e^{\beta\zeta(\gamma)} + e^{-\beta\zeta(\gamma)} \frac{\varphi_\beta(-1,1,...)}{\varphi_\beta(1,1,1,..)}$$

$$= e^{\beta\zeta(\gamma)} + e^{-\beta\zeta(\gamma)} \frac{\int_{X^*} \exp(\beta W(y|-11^\infty)) d\nu_{\beta A}(y)}{\int_{X^*} \exp(\beta W(y|1^\infty)) d\nu_{\beta A}(y)}.$$

Let us foccus on the integrals appearing above. By using the expression (4.4) we have

$$\frac{\int_{X^*} \exp\Big(-y_1\beta(\zeta(\gamma)-1) + \beta \sum_{n\ge 2} y_n \left(-n^{-\gamma} + \sum_{j\ge 2} (j+n-1)^{-\gamma}\right)\Big) d\nu_{\beta A}(y)}{\int_{X^*} \exp\Big(y_1 \beta\zeta(\gamma) + \beta \sum_{n\ge 2} y_n \left(n^{-\gamma} + \sum_{j\ge 2}(j+n-1)^{-\gamma}\right)\Big) d\nu_{\beta A}(y)}$$

Note that the numerator is a product of an decreasing function by an increasing and the denominator is a product of two increasing functions therefore we can use the FKG Inequality to get an upper bound to the above fraction which is given by

$$\frac{\int_{X^*} e^{-y_1\beta(\zeta(\gamma)-1)} d\nu_{\beta A}(y)}{\int_{X^*} e^{y_1\beta\zeta(\gamma)} d\nu_{\beta A}(y)} \times \frac{\int_{X^*} \exp\left(\beta \sum_{n\geq 2} y_n \left(-n^{-\gamma} + \sum_{j\geq 2}(j+n-1)^{-\gamma}\right)\right) d\nu_{\beta A}(y)}{\int_{X^*} \exp\left(\beta \sum_{n\geq 2} y_n \left(n^{-\gamma} + \sum_{j\geq 2}(j+n-1)^{-\gamma}\right)\right) d\nu_{\beta A}(y)}$$

The first quotient above can be explicit computed as follows

$$\frac{\int_{X^*} e^{-y_1\beta(\zeta(\gamma)-1)} d\nu_{\beta A}(y)}{\int_{X^*} e^{y_1\beta\zeta(\gamma)} d\nu_{\beta A}(y)} = \frac{e^{\beta(\zeta(\gamma)-1)}\nu_{\beta A}(y_1=-1) + e^{-\beta(\zeta(\gamma)-1)}\nu_{\beta A}(y_1=1)}{e^{\beta\zeta(\gamma)}\nu_{\beta A}(y_1=1) + e^{-\beta\zeta(\gamma)}\nu_{\beta A}(y_1=-1)}$$

$$= \frac{\cosh(\beta(\zeta(\gamma)-1))}{\cosh(\beta\zeta(\gamma))} < 1,$$

where we have used that $\mathcal{G}^*(A)$ is a singleton and Remark 3.2 to conclude that the probabilities $\nu_{\beta A}(y_1 = \pm 1) = 1/2$. The second quotient can be bounded by one using again the FKG Inequality and Remark 3.2. The above estimates implies that the pressure of this long range Ising model is bounded by

$$P(\beta A) = \log \lambda_{\beta A} < \log(2\cosh(\beta\zeta(\gamma))).$$

Tights lower bounds are much harder to obtain. Anyway this computation let it clear the relevance of the involution kernel to obtain upper bounds for the pressure functional.

5. Extensions of the Ruelle Operator and Its Eigenfunctions

It is well-known that the space $C(X)$ is not always a suitable space to solve the main eigenvalue problem for the Ruelle Operator $\mathscr{L}_A$ for a general continuous potential A. In [6] and [9] the authors exhibit a family of potentials for which the Ruelle operator has no continuous main eigenfunction. For example, if the potential A is of the form

$$A(x) = \sum_{n\geq 1} \frac{x_n}{n^\alpha}, \quad \text{where } \ 1 < \alpha < 2$$

the authors prove the existence of a measurable non-negative "eigenfunction" taking values from zero to infinity in any fixed cylinder set of X. Of course, such function can not be extended or modified to become a continuous function defined in whole space X. Therefore some extension of the operator, for example, to some Lebesgue space, ought to be considered in order to view these functions as legitimate eigenfunctions. In this section we study the main eigenvalue problem for the natural extension of the Ruelle operator to the Hilbert space $L^2(\nu_A)$.

We begin by proving that the Ruelle operator can be extended to a positive operator defined on $L^2(\nu_A)$ for any continuous potential A.

Lemma 5.1. *Let $A : X \to \mathbb{R}$ be a continuous potential and ν_A any element in $\mathcal{G}^*(A)$. Then the Ruelle operator $\mathscr{L}_A : C(X) \to C(X)$ can be extended to a bounded linear operator defined on $L^2(\nu_A)$.*

Proof. Since $\sigma : X \to X$ is non-sigular, that is, $\nu_A \circ \sigma^{-1}(E) = 0 \Leftrightarrow \nu_A(E) = 0$ we can ensure that equivalence classes are mapped into equivalence classes so it is enough to prove that the Ruelle operator is bounded on a dense subspace $L^2(\nu_A)$, with respect to the $L^2(\nu_A)$-norm. Since X is a compact metric space, we have that $C(X)$ is a dense subset of $L^2(\nu_A)$. Let ψ be a continuous function. By using the positivity and duality relation of the Ruelle operator we get

$$\begin{aligned}\|\mathscr{L}_A(\psi)\|^2_{L^2(\nu_A)} &= \int_X \mathscr{L}_A(\psi)\mathscr{L}_A(\psi)\, d\nu_A \leq \int_X \mathscr{L}_A(|\psi|)\mathscr{L}_A(|\psi|)\, d\nu_A \\ &= \int_X \mathscr{L}_A(\mathscr{L}_A(|\psi|)\circ\sigma\cdot|\psi|)\, d\nu_A = \lambda_A \int_X \mathscr{L}_A(|\psi|)\circ\sigma\cdot|\psi|\, d\nu_A.\end{aligned}$$

Developing the integrand by using the definition of the Ruelle operator and the continuity of the potential A we get

$$\begin{aligned}\mathscr{L}_A(|\psi|)\circ\sigma\cdot|\psi| &= |\psi(1x_2x_3\ldots)|\cdot\exp(A(1x_2x_3\ldots))\cdot|\psi(x_1x_2x_3\ldots)| \\ &\qquad + |\psi(-1x_2x_3\ldots)|\cdot\exp(A(-1x_2x_3\ldots))\cdot|\psi(x_1x_2x_3\ldots)| \\ &\leq \exp(\|A\|_\infty)\Big[|\psi(1x_2x_3\ldots)|\cdot|\psi(x_1x_2x_3\ldots)| \\ &\qquad + \exp(\|A\|_\infty)|\psi(-1x_2x_3\ldots)|\cdot|\psi(x_1x_2x_3\ldots)|\Big].\end{aligned}$$

By using this upper bound and the Cauchy-Schwarz Inequality we can conclude from the above inequalities that

$$\begin{aligned}\|\mathscr{L}_A(\psi)\|^2_{L^2(\nu_A)} &\leq \lambda_A \exp(\|A\|_\infty)\left(\int_X \psi^2(1x_2\ldots)\, d\nu_A\right)^{\frac{1}{2}}\|\psi\|_{L^2(\nu_A)} \\ &\qquad + \lambda_A \exp(\|A\|_\infty)\left(\int_X \psi^2(-1x_2\ldots)\, d\nu_A\right)^{\frac{1}{2}}\|\psi\|_{L^2(\nu_A)} \\ &\leq \lambda_A \exp(\|A\|_\infty)\left(\int_X 1_{\{x_1=1\}}\psi^2\, d\nu_A\right)^{\frac{1}{2}}\|\psi\|_{L^2(\nu_A)} \\ &\qquad + \lambda_A \exp(\|A\|_\infty)\left(\int_X 1_{\{x_1=-1\}}\psi^2\, d\nu_A\right)^{\frac{1}{2}}\|\psi\|_{L^2(\nu_A)} \\ &\leq 2\lambda_A \exp(\|A\|_\infty)\|\psi\|^2_{L^2(\nu_A)}.\end{aligned}$$

Thus proving that the Ruelle operator can be extended in $L^2(\nu_A)$ to a bounded linear operator. □

Given a continuous potential A, a point $z_0 \in X$ and $n \in \mathbb{N}$, is natural to approximate A by a potential A_n defined by the mapping $x = (x_1, \ldots, x_n, \ldots) \longmapsto A(x_1, x_2, \ldots, x_n, z_0)$. Note that A_n depends on a finite number of coordinates and therefore belongs to the Hölder class. Typical choices of z_0 could be either 1^∞ or $(-1)^\infty$.

Lemma 5.2. *Let A be a continuous potential and for each $n \in \mathbb{N}$ we define $A_n(x) = A(x_1, \ldots, x_n, 1, 1, \ldots)$. Let φ_n denotes the main eigenfunction of $\mathscr{L}_{A_n}$ normalized so that $\|\varphi_n\|_{L^1(\nu_n)} = 1$, where ν_n is the unique eigenprobability of $\mathscr{L}^*_{A_n}$. Then there exist a σ-invariant Borel probability measure μ_A (called asymptotic equilibrium state) such that, up to subsequence,*

$$\lim_{n\to\infty} \int_X \mathscr{L}_{A_n}(\varphi_n)\psi \, d\nu_n = \lambda_A \int_X \psi \, d\mu_A, \quad \forall \psi \in C(X)$$

Proof. Let λ_{A_n} be the main eigenvalue of the Ruelle operator associated to the potential A_n and φ_n its normalized eigenfunction, i.e., $\|\varphi_n\|_{L^1(\nu_n)} = 1$. By the Corollary 2 of [9] we get that $\lambda_{A_n} \to \lambda_A$, when $n \to \infty$. Since $\varphi_n \geq 0$ and $\|\varphi_n\|_{L^1(\nu_n)} = 1$ the measure $\mu_n = \varphi_n \nu_n$ is a probability measure for each $n \in \mathbb{N}$. Since X is compact, there is a probability measure μ_A such that, up to subsequence, $\mu_n \rightharpoonup \mu_A$. Therefore for all $\psi \in C(X)$ we have

$$\int_X \mathscr{L}_{A_n}(\varphi_n)\psi \, d\nu_n = \lambda_n \int_X \varphi_n \psi \, d\nu_n = \lambda_n \int_X \psi \, d\mu_n \xrightarrow{n\to+\infty} \lambda_A \int_X \psi \, d\mu_A. \qquad \square$$

Theorem 5.1 (See [9]). *Let $A : X \to \mathbb{R}$ be a continuous potential. Suppose that $\{A_n : n \in \mathbb{N}\}$ is a sequence of Hölder potentials such that $\|A_n - A\|_\infty \to 0$, when $n \to \infty$. Then any the asymptotic equilibrium state μ_A given by Lemma 5.2 is an equilibrium state for A.*

Definition 5.1 (Weak-Solution). Let ν_A be an eigenprobability for $\mathscr{L}^*_A$ and μ_A be given by the Lemma 5.2. We say that a non-negative function $\varphi_A \in L^2(\nu_A)$ is a weak solution to the eigenvalue problem for the Ruelle operator, if $\|\varphi_A\|_{L^1(\nu_A)} = 1$ and

$$\int_X \mathscr{L}_A(\varphi_A)\psi \, d\nu_A = \lambda_A \int_X \psi \, d\mu_A, \qquad \forall \psi \in C(X).$$

Proposition 5.1. *If A is a Hölder potential and φ_A is the main eigenfunction of $\mathscr{L}_A$, then φ_A is a weak solution to the eigenvalue problem in the sense of the Definition 5.1.*

Proof. We first consider a sequence of potentials $A_n : X \to \mathbb{R}$ defined, for each $n \in \mathbb{N}$ and $x \in X$, by $A_n(x) = A(x_1, \ldots, x_n, 1, 1, \ldots)$. Note that A is Hölder and one can immediately check that $\|A_n - A\|_\infty \to 0$, when $n \to \infty$. As in the Lemma 5.2, let φ_n denotes the main eigenfunction of $\mathscr{L}_{A_n}$ normalized so that $\|\varphi_n\|_{L^1(\nu_n)} = 1$, where ν_n is the unique eigenprobability of $\mathscr{L}^*_{A_n}$. Then there exist a σ-invariant Borel probability measure μ_A such that, up to subsequence,

$$\lim_{n\to\infty} \int_X \mathscr{L}_{A_n}(\varphi_n)\psi \, d\nu_n = \lambda_A \int_X \psi \, d\mu_A, \quad \forall \psi \in C(X)$$

From the Theorem 5.1 we have that μ_A is an equilibrium state for A. Since the potential A is Hölder its unique equilibrium state is known to be given by the probability measure $\gamma_A = \varphi_A \nu_A$ and therefore $\mu_A = \gamma_A$. This last equality together with the hypothesis give us for all $\psi \in C(X)$ that

$$\begin{aligned}\int_X \mathscr{L}_A(\varphi_A)\psi \, d\nu_A = \int_X \lambda_A \varphi_A \psi \, d\nu_A = \lambda_A \int_X \psi \, d[\varphi_A \nu_A] = \lambda_A \int_X \psi \, d\gamma_A \\ = \lambda_A \int_X \psi \, d\mu_A.\end{aligned}$$

□

The main tool in this section is the Lions-Lax-Milgram Theorem and it is used to provide weak solutions to the eigenvalue problem for the Ruelle operator, see [28] for a detailed proof of this result.

Theorem 5.2 (Lions-Lax-Milgram). *Let H be a Hilbert space and V a normed space, $B : H \times V \to \mathbb{R}$ so that for each $v \in B$ the mapping $h \mapsto B(h,v)$ is continuous. The following are equivalent: for some constant $c > 0$,*

$$\inf_{\|v\|_V = 1} \sup_{\|h\|_H \leq 1} |B(h,v)| \geq c;$$

for each continuous linear functional $F \in V^*$, there exists $h \in H$ such that

$$B(h,v) = F(v) \quad \forall v \in V.$$

Theorem 5.3. *Let $A : X \to \mathbb{R}$ be a continuous potential, ν_A be an element of $\mathcal{G}^*(A)$ and μ_A as constructed in Lemma 5.2. Assume that there is $K > 0$ such that for all $v \in C(X)$ we have $\|v\|_{L^2(\nu_A)} \leq K\|v\|_{L^2(\mu_A)}$. Then there exist a weak solution $\varphi_A \in L^2(\nu_A)$ to the eigenvalue problem for the Ruelle operator.*

Proof. We will prove the theorem assuming that: there is $K > 0$ such that for all $v \in C(X)$ we have $\|v\|_{L^2(\nu_A)} \leq K\|v\|_{L^2(\mu_A)}$.

The main idea of the proof is to use the Lions-Lax-Milgram Theorem with the space $H = L^2(\nu_A)$, $V = (C(X), \|\cdot\|_{L^2(\nu_A)})$, $B : L^2(\nu_A) \times C(X) \to \mathbb{R}$ and $F : C(X) \to \mathbb{R}$ given respectively, by

$$B(h,v) = \int_X \mathscr{L}_A(h) v \, d\nu_A \qquad \text{and} \qquad F(v) = \int_X v \, d\mu_A.$$

In the sequel we prove the coercivity condition of the Lions-Lax-Milgram theorem and then the continuity of the bilinear form B. For any $v \in V$ such that

$\|v\|_{L^2(\nu_A)} = 1$ we have

$$\int_X v^2\, d\nu_A = \frac{1}{\lambda_A}\int_X \mathscr{L}_A(v^2)\, d\nu_A = \frac{1}{\lambda_A}\int_X (v^2\circ\sigma)\mathscr{L}_A(1)\, d\nu_A$$

$$\geq \frac{\exp(-\|A\|_\infty)}{\lambda_A}\int_X (v^2\circ\sigma)\, d\nu_A$$

and therefore

$$\frac{1}{\|v\circ\sigma\|_{L^2(\nu_A)}} \geq \frac{\exp(-\|A\|_\infty)}{\lambda_A}.$$

Similarly we prove that for all $h \in L^2(\nu_A)$ we have $h\circ\sigma \in L^2(\nu_A)$. So it follows from the elementary properties of the Ruelle operator that

$$\sup_{\|h\|\leq 1}\int_X \mathscr{L}_A(h)v\, d\nu_A \geq \int_X \mathscr{L}_A\left(\frac{v\circ\sigma}{\|v\circ\sigma\|_{L^2(\nu_A)}}\right) v\, d\nu_A$$

$$= \frac{1}{\|v\circ\sigma\|_{L^2(\nu_A)}}\int_X \mathscr{L}_A(1)v^2\, d\nu_A \geq \frac{\exp(-2\|A\|_\infty)}{\lambda_A}\int_X v^2\, d\nu_A.$$

From the last inequality we get

$$\inf_{\|v\|_V=1}\sup_{\|h\|_H\leq 1}|B(h,v)| = \inf_{\|v\|_{L^2(\nu_A)}=1}\sup_{\|h\|\leq 1}\int_X \mathscr{L}_A(h)v\, d\nu_A \geq \frac{\exp(-2\|A\|_\infty)}{\lambda_A}$$

which proves the coercivity hypothesis.

Now we prove the continuity of the mapping $h \longmapsto B(h,v)$, where $v \in L^2(\mu_A)$ is fixed. From Lemma 5.1 we have that $\mathscr{L}_A(h) \in L^2(\nu_A)$ for every $h \in L^2(\nu_A)$. So we can use Cauchy-Schwarz inequality to bound $B(h,v)$ as follows

$$|B(h,v)| = \left|\int_X \mathscr{L}_A(h)v\, d\nu_A\right| \leq \left(\int_X [\mathscr{L}_A(h)]^2\, d\nu_A\right)^{\frac{1}{2}}\left(\int_X v^2\, d\nu_A\right)^{\frac{1}{2}}$$

$$\leq (2\lambda_A\exp(\|A\|_\infty))^{\frac{1}{2}}\|h\|_{L^2(\nu_A)}\cdot\|v\|_{L^2(\nu_A)},$$

where the last inequality comes the Lemma 5.1 proof's. The above inequality proves that $h \longmapsto B(h,v)$ is continuous.

The hypothesis $\|v\|_{L^2(\nu_A)} \leq K\|v\|_{L^2(\mu_A)}$ guarantees the continuity of the functional F so we can apply the Lions-Lax-Milgram theorem to ensure the existence of a function $\overline{\varphi_A} \in L^2(\nu_A)$ so that

$$\int_X \mathscr{L}_A(\overline{\varphi_A})v\, d\nu_A = \int_X v\, d\mu_A \qquad \forall\ v \in L^2(\mu_A). \tag{5.1}$$

By using the identity (5.1) with $v \equiv 1$ we get

$$1 = \int_X \mathscr{L}_A(\overline{\varphi_A})d\nu_A = \int_X \overline{\varphi_A}\ d\big[\mathscr{L}_A^*\nu_A\big] \leq \lambda_A\|\overline{\varphi_A}\|_{L^1(\nu_A)}. \tag{5.2}$$

Therefore the following function

$$\varphi = \frac{\overline{\varphi_A}}{\lambda_A\|\overline{\varphi_A}\|_{L^1(\nu_A)}}$$

is a non-trivial weak solution for the eigenvalue problem. □

Remark 5.1. We can weaken the hypothesis $\|v\|_{L^2(\nu_A)} \leq K\|v\|_{L^2(\mu_A)}$ for all $v \in C(X)$ of Theorem 5.3 by only requiring that

$$\limsup_{n\to\infty} \int_X \varphi_n^2 \, d\nu_n \quad < +\infty,$$

where φ_n and ν_n are chosen as in Lemma 5.2. Indeed, in order to get the inequality $\|v\|_{L^2(\nu_A)} \leq K\|v\|_{L^2(\mu_A)}$ from the above condition it is enough to note that $\nu_n \rightharpoonup \nu_A$ and then to apply the Cauchy-Schwarz inequality (we leave the details of the proof to the reader).

Acknowledgments

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. L. Cioletti would like to acknowledge financial support by CNPq through projects PQ 310818/2015-0 and and FAP-DF for financial support. A. O. Lopes thanks CNPq for partial support.

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