Diffusive-Ballistic Transition in Random Polymers with Drift and Repulsive Long-Range Interactions

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Abstract

In this note phase transition issues are addressed for random polymers on $\mathbb{Z}^2$ with long-range self-repulsive interactions. It is shown that, in the absence of drift and with power law interactions, the polymer exhibits transition from diffusive to a ballistic behavior. When non-null drifts are added and positive translation invariant interactions are considered, the polymer presents a ballistic behavior. Our results complement some previous studies on the matter and we also derive a Central Limit Theorem for the model.

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1 Introduction

Random polymers can be modelled as connected subsets of $\mathbb{Z}^2$. More precisely, a N-th step polymer S is an element of $\mathbb{W}_N$ given by

$$\mathbb{W}_N := \{S = (S_0, S_1, \dots, S_N) : S_i \in \mathbb{Z}^2, S_0 = 0 \text{ and } \|S_{i+1} - S_i\| = 1\},$$

being $\|\cdot\|$ the ℓ^1 norm. Under Gibbs measure setting at inverse temperature $\beta > 0$ and Hamiltonian $\mathbb{H}_N$ we can write the probability

$$\mathbb{P}_N^{\beta,h}(S) = \frac{\exp\left[-\beta \mathbb{H}_N(S)\right]}{\mathbb{Z}_N^{\beta}(h)}, \qquad \mathbb{H}_N(S) = -\sum_{1 \le i < j \le N} V_{ij} \langle X_i, X_j \rangle + \langle h, S_N \rangle, \tag{1}$$

where $X_i = S_i - S_{i-1}$ stands for the i-th random step, V_{ij} are the prescribed interactions, $\langle \cdot, \cdot \rangle$ denotes the usual inner product, $h \in \mathbb{R}^2$ is the fixed drift vector and $\mathbb{Z}_N^{\beta}(h)$ is the partition function.

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Caracciolo et al. ([3], 1993) introduced a self-repelling random polymer model with Hamiltonian on $\mathbb{W}_N$ given by $\overline{H}_N(S) = g_0 \sum_{0\leq i<j\leq N} V_{ij}\delta_{S_i,S_j}$, where $g_0 > 0$ and the interactions $V_{ij} = |i-j|^{-\alpha}$. Their model interpolates between the lattice Edwards model ($\alpha = 0$) and ordinary SRW ($\alpha = \infty$). Moreover, it was conjectured that for dimension $1 \leq d \leq 4$ there exists a strictly positive exponent $\gamma = \gamma(d, \alpha)$ such that the mean square end-to-end distance satisfies the asymptotics

$$\mathbb{E}_{\overline{P}_N}[\|S_N\|^2] = \sum_{S\in\mathbb{W}_N} \|S_N\|^2 \overline{P}_N(S) \sim cN^{\gamma},$$

where the Gibbs measure $\overline{P}_N$ is given by the Hamiltonian $\overline{H}_N$. In [10], the Hamiltonian $\tilde{H}_N(S) = -\sum_{0\leq i<j\leq N} |i-j|^{-\alpha}\|S_i - S_j\|^2$, where $3 < \alpha \leq 4$, was considered. They proved the existence of positive constants β_1 and β_2 that led to phase transition from diffusive regime ($\beta < \beta_1$) to a ballistic one ($\beta > \beta_2$). However, it was left unknown what undergoes when $\beta \in [\beta_1, \beta_2]$. As usual, the different diffusive regimes are classified according to the asymptotic behavior of the mean square displacement and for our model (1) it reduces in determining $\gamma > 0$ for which the following limit exists, is positive and finite

$$\lim_{N\to\infty} \frac{1}{N^{\gamma}} \mathbb{E}_{\mathbb{P}_N^{\beta,h}}[\|S_N\|^2] = \lim_{N\to\infty} \frac{1}{N^{\gamma}} \sum_{S\in\mathbb{W}_N} \|S_N\|^2 \; \mathbb{P}_N^{\beta,h}(S). \tag{2}$$

We say that the polymer model is *diffusive* if $\gamma = 1$, *superdiffusive* if $1 < \gamma < 2$ and *ballistic* if $\gamma = 2$.

Our main motivation is to build a self-repelling random polymer model for which we can derive a genuine diffusive-ballistic phase transition, i.e. the existence of a unique positive constant β_c separating the model into two regimes. In this note, assuming zero drift and $V_{ij} = |i-j|^{-\alpha}$ with $1 < \alpha \leq 2$, we prove (Theorem 3) that there exists a unique positive number β_c (the critical temperature of a related one dimensional Ising model) such that the model is diffusive for $\beta < \beta_c$ and ballistic for $\beta > \beta_c$. On the other hand, considering non-null drift and positive, translation invariant and regular interactions, we conclude from Theorem 2 that for all $\beta \in (0, \infty)$ the model is ballistic.

The Lemma 1 is an essential tool in this work. Its proof is similar in spirit to the one introduced for the Potts model by M. Suzuki [13] in 1967. It consists in decoupling the steps of the polymer as two independent Ising random variables. The background idea is the same applied when looking at SRW in lattice $\mathbb{Z}^2$ as two independents SRW's on $\mathbb{Z}$.

In 1983 Newman [9] proved a CLT for block random variables satisfying the FKG inequalities under finite susceptibility hypothesis. In Section 3 we investigate the validity of the CLT for our model. Here assuming non zero drifts and consequently infinite susceptibility, we prove (Theorem 4) a CLT for the projections of suitably normalized displacements. This is obtained by using both the Lee-Yang circle theorem and the C^2-regularity condition from Wu Liming [8].

2 Mean Square Displacement and Phase Transition

For the volume $\Lambda_N = \{1, 2, \ldots, N\}$ consider the one dimensional Ising model with free boundary conditions defined by the Hamiltonian

$$H_{\Lambda_N}(\sigma) = -\sum_{1 \le i < j \le N} V_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} h\sigma_i,$$

where $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N := \Sigma_N$, $V_{ij} \in \mathbb{R}$ are the coupling constants and $h \in \mathbb{R}$ is an external field. To simplify notation for a given a real-valued function $f : \Sigma_N \to \mathbb{R}$ write

$$\langle f \rangle_{\Lambda_N}^{\beta,h} = \mathbb{E}_{P_{\Lambda_N}^{\beta,h}}[f] \quad \text{with} \quad P_{\Lambda_N}^{\beta,h}(\sigma) = \frac{1}{Z_{\Lambda_N}^{\beta}(h)} \exp\left(-\beta H_{\Lambda_N}(\sigma)\right)$$

where $Z_{\Lambda_N}^{\beta}(h)$ is the partition function.

Lemma 1. *For $e_1 = (1, 0)$ and $e_2 = (0, 1)$ define $h_1 = \langle h, e_1 - e_2\rangle$ and $h_2 = \langle h, e_1 + e_2\rangle$. Then*

$$\mathbb{E}_{\mathbb{P}_N^{\beta,h}}[\|S_N\|^2] = \frac{1}{2}\sum_{i,j=1}^{N} \Big[\langle \sigma_i\sigma_j\rangle_{\Lambda_N}^{\frac{\beta}{2},h_1} + \langle \sigma_i\sigma_j\rangle_{\Lambda_N}^{\frac{\beta}{2},h_2} \Big].$$

Proof. The proof follows closely the ideas from [2, 10]. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation $TX_i = (\sigma_i e_1 + \tilde{\sigma}_i e_2)/\sqrt{2}$, with $\sigma_i, \tilde{\sigma}_i \in \{-1, 1\}$. A simple computation shows that

$$\mathbb{P}_N^{\beta,h}(S) = P_{\Lambda_N}^{\frac{\beta}{2},h_1}(\sigma)P_{\Lambda_N}^{\frac{\beta}{2},h_2}(\tilde{\sigma}) \quad \text{and} \quad \|S_N\|^2 = \frac{1}{2}\sum_{i,j=1}^{N}(\sigma_i\sigma_j + \tilde{\sigma}_i\tilde{\sigma}_j).$$

□

Theorem 2. *Suppose that V_{ij} is positive and tranlation invariant, i.e. $V_{ij} = V(|i-j|) > 0$ for all $i \neq j$. If $h \in \mathbb{R}^2$ is such that h_1 and h_2 satisfy $h_1h_2 > 0$ and $\beta > 0$, then for some constant $C(\beta, h) > 0$ we have*

$$C(\beta, h) \le \frac{\mathbb{E}_{\mathbb{P}_N^{\beta,h}}[\|S_N\|^2]}{N^2}.$$

Proof. Let $k \in \mathbb{R}$. Since $V_{ij} > 0$ for $i \neq j$ we get from the second Griffiths inequality,

$$\langle \sigma_i\rangle_{\Lambda_N}^{\beta,k,nn}\langle \sigma_j\rangle_{\Lambda_N}^{\beta,k,nn} \le \langle \sigma_i\sigma_j\rangle_{\Lambda_N}^{\beta,k,nn} \le \langle \sigma_i\sigma_j\rangle_{\Lambda_N}^{\beta,k}$$

where the left hand side expected values are taken with respect to the Gibbs measure of the nearest neighbours Ising model on Λ_N with free boundary conditions and Hamiltonian given by $H_{\Lambda_N}(\sigma) = -\sum_{n=1}^{N-1} V(1)\sigma_n\sigma_{n+1} - k\sum_{n=1}^{N}\sigma_n$. Using monotonicity with respect to the volume and classical transfer matrix computation (see [5, p. 107]) we have for any $i \in \Lambda_N$

$$\langle \sigma_i \rangle_{\Lambda_N}^{\beta,k,nn} \underset{N\to\infty}{\longrightarrow} \frac{\sinh \beta k}{\sqrt{\sinh^2(\beta k) + e^{-4\beta V(1)}}}.$$

□

Theorem 3. *Let $1 < \alpha \le 2$, $h = 0$ and $V_{ij} = |i-j|^{-\alpha}$ for $i \neq j$. Then there exist a constant $\beta_c \in (0,\infty)$ and positive numbers $m_*(\beta)$ and $K(\beta)$ such that*

$$\frac{1}{2} m_*^2(\beta) \le \frac{1}{N^2} \mathbb{E}_{\mathbb{P}_N^{2\beta,0}}[\|S_N\|^2] \le 1, \qquad \text{if} \quad \beta > \beta_c \tag{3}$$

and

$$1 \le \frac{1}{N} \mathbb{E}_{\mathbb{P}_N^{2\beta,0}}[\|S_N\|^2] \le K(\beta), \qquad \text{if} \quad 0 < \beta < \beta_c. \tag{4}$$

Proof. For $1 < \alpha \le 2$ the existence of a critical $\beta_c \in (0,\infty)$ for the long range Ising model with coupling V_{ij} is shown in [4, 6]. In this case, we have spontaneous magnetization $m_*(\beta) > 0$ for all $\beta > \beta_c$ and the two-point function with free boundary condition satisfies (cf. [7]),

$$\langle \sigma_i \sigma_j \rangle^{\beta,0} = \frac{1}{2}\Big[\langle \sigma_i \sigma_j \rangle^{\beta,0,+} + \langle \sigma_i \sigma_j \rangle^{\beta,0,-} \Big] \ge m_*^2(\beta) \ge m_*^2(\beta_c).$$

Using the same type of arguments as in Theorem 2 we have for large N

$$\mathbb{E}_{\mathbb{P}_N^{2\beta,0}}[\|S_N\|^2] = \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0} \ge \frac{1}{2} \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle^{\beta,0} \ge \frac{1}{2} m_*^2(\beta) N^2.$$

To prove (4) one needs lower and upper bounds for $\langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0}$. From the monotonicity with respect to the volume and [1], if $\beta < \beta_c$ there are constants $0 < C(\beta) \le C'(\beta) < \infty$ such that $\langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0}$ with free boundary condition satisfies

$$\frac{C(\beta)}{|i-j|^\alpha} \le \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0} \le \langle \sigma_i \sigma_j \rangle_{\mathbb{N}}^{\beta,0} \le \frac{C'(\beta)}{|i-j|^\alpha},$$

where $C(\beta) \equiv (\beta \tanh \beta_c)/\beta_c$. The uniformity of the lower bound is a simple application of FKG inequality. Using Lemma 1 we have

$$N + \sum_{1 \le i < j \le N} \frac{2C(\beta)}{|i-j|^\alpha} \le \mathbb{E}_{\mathbb{P}_N^{2\beta,0}}[\|S_N\|^2] \le N + \sum_{1 \le i < j \le N} \frac{2C'(\beta)}{|i-j|^\alpha}.$$

Inequality (4) follows by observing that $\sum_{1 \le i < j \le N} \frac{1}{|i-j|^\alpha} = \mathcal{O}(N)$. □

3 Central Limit Theorem

To derive a CLT for (1) we make use of Theorem 1.2 and Theorem 3.1 from [8]. It is required that a C^2-regularity condition to be satisfied. We say that a sequence of probability measures $\{\mu_N\}$ satisfies the C^2-regularity condition if for Y_N with probability measure μ_N the following limit exists

$$\Psi(t) = \lim_{N\to\infty} \Psi_N(t) = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}_{\mu_N}[\exp(tNY_N)].$$

Moreover, for some neighborhood $[-\delta,\delta]$ of zero we have $\Psi(\cdot) < \infty$ and

$$\Psi''_N(t) \underset{N\to\infty}{\longrightarrow} \Psi''(t) \text{ uniformly on } [-\delta,\delta].$$

Under these hypotheses Y_N is asymptotically Gaussian.

For our polymer model take $v \in \mathbb{R}^2$ fixed and consider the empirical field projection

$$L_N = \frac{1}{N}\langle S_N, v\rangle = \frac{1}{N}\sum_{j=1}^{N}\langle X_j, v\rangle.$$

Set $\mu_N = \mathbb{P}_N^{\beta,h}$ and define the pressure functional by

$$\Psi^{\beta,h,v}(t) = \lim_{N\to\infty} \Psi_N^{\beta,h,v}(t) = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}_{\mathbb{P}_N^{\beta,h}}[\exp(\beta tNL_N)]$$

Theorem 4. *Assume that the interactions are translation invariant and summable, that is, $V_{ij} = V(|i-j|) > 0$ and $\sum_{i\in\mathbb{N}} V(i) < \infty$. For $h \in \mathbb{R}^2$ with $h_1h_2 \neq 0$ and any fixed $v \in \mathbb{R}^2$ we have*

$$\frac{1}{\sqrt{N}}\Big[\beta\langle S_N, v\rangle - N\mathbb{E}_{\mathbb{P}_N^{\beta,h}}[\beta\langle S_N, v\rangle]\Big] \xrightarrow{\mathscr{D}} N\left(0, \frac{\partial^2}{\partial t^2}\Psi^{\beta,h,v}(0)\right)$$

where "$\xrightarrow{\mathscr{D}}$" stands for convergence in distribution.

Proof. Under the hypotheses, the existence of the limit $\Psi^{\beta,h,v}(\cdot)$ is proved in [12]. To complete the C^2-regularity verification take complex number $z \in \mathbb{C}$ and express $\Psi_N^{\beta,h,v}(z)$ in terms of partition functions of one-dimensional Ising model. As in Lemma 1 write $\mathbb{Z}_N^{\beta}(h) = Z_{\Lambda_N}^{\beta/2}(h_1)Z_{\Lambda_N}^{\beta/2}(h_2)$. Using the principal-value logarithm identities

$$\ln(zw) = \ln z + \ln w + 2\pi i\mathcal{K}(\ln z + \ln w)$$

$$\ln(z/w) = \ln z - \ln w + 2\pi i\mathcal{K}(\ln z - \ln w)$$

where $\mathcal{K}(x+iy) = -\sum_{n\geq -1} nI((2n-1)\pi < y \leq (2n+1)\pi)$ with $I(\cdot)$ being the indicator function, we have for $v_1 = \langle v, e_1 - e_2\rangle$ and $v_2 = \langle v, e_1 + e_2\rangle$

$$\begin{aligned}\Psi_N^{\beta,h,v}(z) =& \frac{1}{N}\left[\ln \mathbb{Z}_N^{\beta}(h+zv) - \ln \mathbb{Z}_N^{\beta}(h) + 2\pi i \mathcal{K}\left(\ln \mathbb{Z}_N^{\beta}(h+zv) - \ln \mathbb{Z}_N^{\beta}(h)\right)\right] \\ =& \frac{1}{N}\ln Z_{\Lambda_N}^{\beta/2}(h_1 + zv_1) + \frac{1}{N}\ln Z_{\Lambda_N}^{\beta/2}(h_2 + zv_2) \\ &+ \frac{2\pi i}{N}\mathcal{K}\left(\ln Z_{\Lambda_N}^{\beta/2}(h_1 + zv_1) + \ln Z_{\Lambda_N}^{\beta/2}(h_2 + zv_2)\right) \\ &- \frac{1}{N}\ln Z_{\Lambda_N}^{\beta/2}(h_1) - \frac{1}{N}\ln Z_{\Lambda_N}^{\beta/2}(h_2) + \frac{2\pi i}{N}\mathcal{K}\left(\ln \mathbb{Z}_N^{\beta}(h+zv)\right).\end{aligned}$$

By assuming that $\mathrm{Re}(h_i + zv_i) \neq 0$ and $h_1 h_2 \neq 0$ it follows from Lee-Yang Theorem's and standard arguments from [11, p. 111] that

$$\Psi_N^{\beta,h,v}(z) \to \Psi^{\beta,h,v}(z), \text{ locally uniformly in } z.$$

Also, it follows that the derivatives of $\Psi_N^{\beta,h,v}(z)$ converge uniformly on the compact subsets of $\mathbb{C}$. Hence the C^2-regularity condition is satisfied. Since

$$\frac{\partial^2}{\partial t^2}\Psi_N^{\beta,h,v}(t)\bigg|_{t=0} = \frac{1}{N}\ \mathbb{E}_{\mathbb{P}_N^{\beta,h}}\Big[\beta\langle S_N, v\rangle - N\mathbb{E}_{\mathbb{P}_N^{\beta,h}}[\beta\langle S_N, v\rangle]\Big]^2 \to \frac{\partial^2}{\partial t^2}\Psi^{\beta,h,v}(0)$$

we conclude the proof using Theorem 3.1 from [8]. □

Remark 1. *We emphasize that in the above theorem we proved more than C^2-regularity condition. In fact, we proved that the pressure is analytic. Another way to obtain the C^2-regularity condition for our polymer model is to apply both FKG and GHS inequalities, see [8, p. 426].*

4 Concluding Remarks

The random polymer model considered here interpolates between the SRW (infinite temperature) and a deterministic straight line (zero temperature). At very high temperatures this random polymer should be recurrent and transience would occur at very low temperatures, so we expect a recurrence-transience phase transition. It would be interesting to prove the existence of such phase transition and also to determine the critical temperature that separates these two regimes.

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