Quasilinear Elliptic Problems: a variational approach

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In the last years we have investigated the existence of solutions of the quasilinear elliptic problem

$$(P) \qquad \begin{cases} -\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_t(x,u)|\nabla u|^p \ = \ g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $p > 1$, Ω open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, where the functions $A(x,t)$, $A_t(x,t) = \frac{\partial A}{\partial t}(x,t)$ and $g(x,t)$ are Carathéodory on $\Omega \times \mathbb{R}$.

Taking $G(x,t) = \int_0^t g(x,s)ds$, suitable assumptions on $A(x,t)$ and $g(x,t)$ set off the variational structure of (P) and its related functional is

$$\mathcal{J}(u) \ = \ \frac{1}{p}\int_\Omega A(x,u)|\nabla u|^p dx - \int_\Omega G(x,u)dx,$$

which is C^1 but not verifies the classical Palais–Smale condition on the Banach space $X = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ equipped with the intersection norm $\|\cdot\|_X$.

Anyway, following an approach which exploits the interaction between $\|\cdot\|_X$ and the standard norm on $W_0^{1,p}(\Omega)$, we apply suitable generalizations of classical variational theorems to $\mathcal{J}$ in X so to prove the existence of weak solutions of (P) by comparing the growth of $A(x,t)|\xi|^p$ with that one of $G(x,t)$.

Recently, such results have allowed us to introduce an approximating argument for the quasilinear modified Schrödinger equation

$$-\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_t(x,u)|\nabla u|^p + V(x)|u|^{p-2}u = f(x,u) \quad \text{ in } \mathbb{R}^N.$$

Under "good" hypotheses on potential $V : \mathbb{R}^N \to \mathbb{R}$, the existence of a non-trivial weak bounded solution of such a problem is stated, while if $V(x) \equiv 1$ a dichotomy result occurs.

In order to outline these results, this mini-course will be organized as follows:

- Main tools for a "classical" variational approach
- The Palais-Smale condition and its generalizations
- New setting for the Minimun Theorem and the Mountain Pass Theorem
- A good decomposition for the Sobolev space $W_0^{1,p}(\Omega)$
- Existence and multiplicity results for the quasilinear problem (P) in a bounded domain Ω
- Some results for a quasilinear modified Schrödinger equation in $\mathbb{R}^N$

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