George Boole:
legacy of a mathematics revolutionary

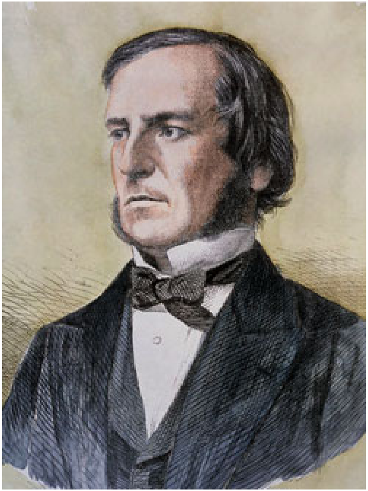
Brasilia, 7 October 2016

Evgeny Khukhro

Sobolev Institute of Mathematics, Novosibirsk, and

UNIVERSITY OF LINCOLN
SCHOOL OF MATHEMATICS AND PHYSICS

George Boole



George Boole (2 November 1815 – 8 December 1864) is a famous mathematician born and bred in Lincoln, England, and later a professor in Cork, Ireland.

His ideas laid the foundations of modern algebra and mathematical logic, as well as paved the way to applications in computer science and technology.

This lecture focuses on the mathematical legacy of Boole's work.

But a short biographical sketch is interesting in its own right.

Self-made man: early years

Father owned a cobbler's shop. Interest in maths, optical instruments, induced in his son George Boole: together built kaleidoscopes, microscopes, telescopes, sundials, etc.

Family could not afford expensive schools or university. George Boole learned by himself Latin, Greek, French, German. Translation of a Latin poem at the age of 14, controversy.

Father's business collapsed. At the age of 16 George Boole became assistant teacher to support family. At 19 he started teaching at Mechanics' Institute in Lincoln. At 23 he became Headmaster of a boarding school in Doncaster. At 25 he opened his own boarding school in Lincoln.

Lincoln: birthplace of George Boole

Great city, known from Roman times, 1st century. Great cathedral since 12th century (but no university until 1996, no maths dept until 2014).

Self-made man: mathematics

In parallel with working as teacher and later running schools, George Boole studied differential and integral calculus, firstly from French books by Lacroix, Lagrange, and Laplace. Also Newton's "Principia..."

Encouraged by Sir Edward Bromhead, first President of the Mechanics' Institute from 1833. Bromhead was a wealthy noble-man, graduated from Cambridge in maths, he had a fine library at his Thurlby Hall. Incidentally, he was also a benefactor of another great self-educated mathematician of that time – George Green (Green function).

Still in Lincoln, George Boole produced his first mathematical results, corresponded with his peers such as Duncan Gregory and Augustus de Morgan. Published several papers, famous book on logic, received Gold Medal of Royal Society.

At the age of 34, George Boole was appointed the first Professor of Mathematics at Queens College Cork, Irelend.

Died at 49 in Cork, 'in the line o duty'.

Famous name

Wikipedia readily supplies us with a list of things Boolean:

- Boolean algebra as a logical calculus
- Boolean algebra as a mathematical structure
- Boolean model (probability theory), a model in stochastic geometry
- Boolean variables and Boolean operations in computer science and computer software
- Boolean gates in computer hardware
- Boolean circuit, a mathematical model for digital logical circuits
- Boolean expression in a programming language that produces a Boolean value when evaluated
- Boolean function, a function that determines Boolean values or operators
- Boolean network, a certain network consisting of a set of Boolean variables whose state is determined by other variables in the network
- Boolean processor, a 1-bit variables computing unit
- Boolean satisfiability problem

Famous name

George Boole has been sometimes credited with many things:

- founding invariant theory
- introduction of mathematical probability
- invention of Boolean algebra
- invention of symbolic logic
- creation of mathematical logic
- discovery of pure mathematics

While some of these claims are exaggerated, Boole deserves to be known for what he really did.

Boole's first mathematical interest

was in mathematical analysis, differential equations, calculus of variations.

Origins in calculus of Newton and Leibniz in 17th century.

If $f(x)$ is a function, a quantity depending on a variable x, then the 'instantaneous' rate of change of $f(x)$ is the derivative $\frac{df}{dx}$.

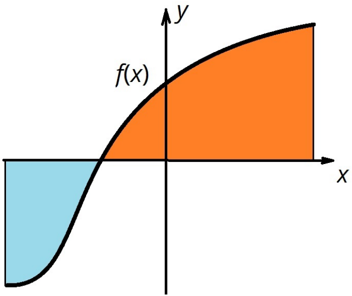
E.g.: if x is time, and $f(x)$ is distance, then $v(x) = \frac{df}{dx}$ is velocity.

The velocity can also change with time, its rate of change is acceleration, the derivative of velocity

$$a(t) = \frac{dv}{dt} = \frac{d\left(\frac{df}{dt}\right)}{dt} = \frac{d^2f}{dt^2}.$$

Note: 'fraction' and 'squaring' in $\frac{d^2f}{dt^2}$ are just notation!

Differential equations are hugely important, including for applications



The simplest differential equation is

$$\frac{df}{dx} = u(x);$$

solution is known as integral:

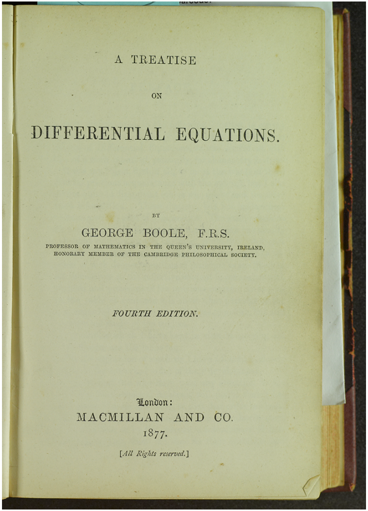
$$f(x) = \int u(x)dx.$$

Connection between indefinite and definite integral is the Newton–Leibniz theorem — 'Fundamental theorem of calculus'.

Can be much more complicated, like $\dfrac{d^2f}{dx^2} + 3\dfrac{df}{dx} + 2f(x) = \sin x$.

Many-many processes in physics, other sciences, engineering are governed by differential equations: wave equation, Maxwell equations, celestial mechanics, heat transfer equations, etc., etc., etc.

Boole's books on differential equations



A TREATISE

ON

DIFFERENTIAL EQUATIONS.

BY

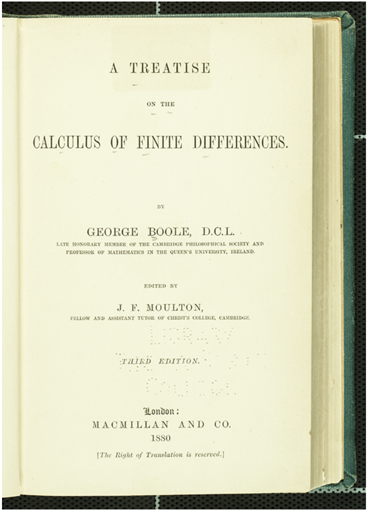
GEORGE BOOLE, F.R.S.

PROFESSOR OF MATHEMATICS IN THE QUEEN'S UNIVERSITY, IRELAND,
HONORARY MEMBER OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

FOURTH EDITION.

London:
MACMILLAN AND CO.
1877.

[*All Rights reserved.*]



A TREATISE

ON THE

CALCULUS OF FINITE DIFFERENCES.

BY

GEORGE BOOLE, D.C.L.

LATE HONORARY MEMBER OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY AND
PROFESSOR OF MATHEMATICS IN THE QUEEN'S UNIVERSITY, IRELAND.

EDITED BY

J. F. MOULTON,

FELLOW AND ASSISTANT TUTOR OF CHRIST'S COLLEGE, CAMBRIDGE.

THIRD EDITION.

London:
MACMILLAN AND CO.
1880

[*The Right of Translation is reserved.*]

Algebraic approach to differential equations

...(already existed by the time of Boole's work). It means that taking derivatives is regarded (denoted) as applying an operator D:

$$\frac{df}{dx} = D(f) \quad \text{and if} \quad D(f) = u, \quad \text{then} \quad f = D^{-1}(u) = \int u.$$

Then differential operators are manipulated by algebraic rules.

Operators are multiplied as performed one after another:

then $\dfrac{d^2 f}{dx^2} = D^2(f)$.

Example: $\dfrac{d^2 f}{dx^2} + 3\dfrac{df}{dx} + 2f = \sin x$ then takes the form

$D^2(f) + 3D(f) + 2f = \sin x$ or $(D^2 + 3D + 2)f = \sin x$.

Algebraic approach:

... treating formally the algebraic expression $D^2 + 3D + 2$ helps to solve the equation:

$D^2 + 3D + 2 = (D + 1)(D + 2).$

So we firstly solve the equation $(D + 1)g(x) = \sin x$,
which is simpler: $g'(x) + g(x) = \sin x$.

Then we solve the equation $(D + 2)f(x) = g(x)$, that is,
$f'(x) + 2f(x) = g(x)$.

Boole's work on differential equations

Boole extended algebraic method in differential equations; in particular, he was the first to apply decompositions into algebraic partial fractions with operators.

One of Boole's papers on this subject, "On a general method in analysis" was published in *Philosophical Transactions of the Royal Society* in 1844

(while he lived in Lincoln, at 29 years of age, running a boarding school).

For this paper he was awarded the Royal Medal of the Royal Society.

Significance of Boole's work on differential equations

Boole applied the algebraic 'D-methods' to both ordinary and partial differential equations.

One of the main new steps was analysing non-commuting operator functions F and G, that is, such that $F(D)G(D) \neq G(D)F(D)$.

In modern mathematics this approach to differential operators developed into functional analysis, operator theory, etc., and algebras of operators became a very important new field of research, very much active nowadays.

Thus, Boole contributed to the development of this important part of mathematics, and his work of 1844 was at the very forefront of the early development of this area (although his name was not given to any of the new objects therein).

But more significantly, Boole must have conceived the general idea that algebraic manipulations with symbols (here, representing differential operators) can be a universal calculus that can be applied in other areas....

Invariant theory

In 1841 Boole also published his paper on invariants, one of the very first on this important subject.

Full development of invariant theory is due to Arthur Cayley (1821–1895).

In his paper of 1845 establishing invariant theory, Cayley credits the 1841 paper of George Boole:

> *"...investigations were suggested to me by a very elegant paper on the same subject ... by Mr Boole."*

Logic

Logic can be regarded as part of reasoning, part of definition of human conscience.

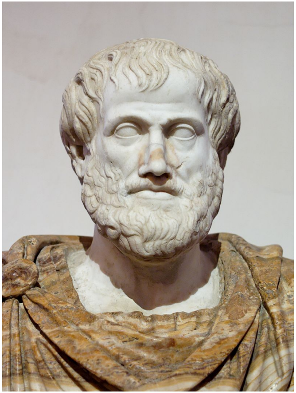


Figure: Aristotle (384–322 BC)

As a theoretical, formal description, one early example is Aristotle's logic.

The central part of it are syllogisms:

Every man is mortal;

Socrates is a man;

hence Socrates is mortal.

Other logic systems

Some formal systems were proposed, among others, by

Figure: G. W. Leibniz (1646–1716)

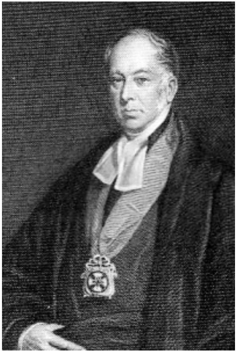
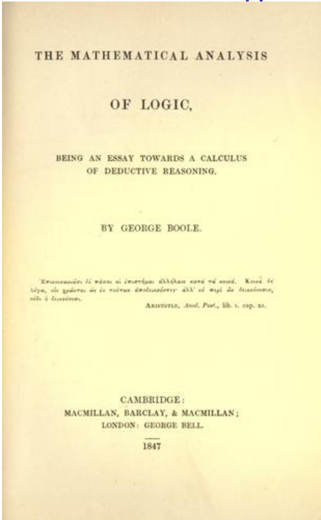


Figure: Richard Whately (1787–1863), book of 1826 *Elements of logic*

Figure: Augustus De Morgan (1787–1863), book of 1847 *Formal Logic*

But none of the forerunners went as far as George Boole in making logic algebraic.

Boole's books on logic


THE MATHEMATICAL ANALYSIS

OF LOGIC,

BEING AN ESSAY TOWARDS A CALCULUS
OF DEDUCTIVE REASONING.

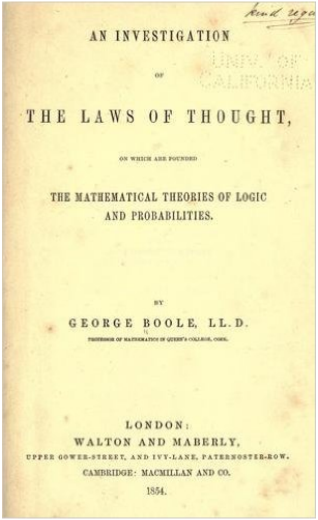
BY GEORGE BOOLE.

ARISTOTLE, *Anal. Post.*, lib. I. cap. XI.

CAMBRIDGE:
MACMILLAN, BARCLAY, & MACMILLAN;
LONDON: GEORGE BELL.

1847

Figure: Mathematical Analysis of Logic, 1847, written in Lincoln


AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.

BY

GEORGE BOOLE, LL.D.

PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, CORK.

LONDON:
WALTON AND MABERLY,
UPPER GOWER-STREET, AND IVY-LANE, PATERNOSTER-ROW.
CAMBRIDGE: MACMILLAN AND CO.
1854.

Figure: Investigation of Laws of Thought, 1854, written in Cork

The second book largely covers and corrects the first.

What laws of thought?

"Laws of thought" in the title does not mean that Boole studied how people actually think! (as in biology, psychology, philosophy).

'Correct ways of thinking', which is mathematical in nature:

> *"... we ought no longer to associate Logic and Metaphysics, but Logic and Mathematics. ... see Logic resting like Geometry upon axiomatic truths, and its theorems constructed upon that general doctrine of symbols..."* (Boole, MAL, 1847)
>
> *"...logic ... investigates the forms and expressions to which correct reasoning may be reduced and the laws upon which it is founded."* (Boole, The Nature of Logic, 1848)
>
> *"...the ultimate laws of thought are mathematical in their form..."* (Boole, ILT, 1854)

Logic as applied mathematics

Boole saw logic as a tool used in other sciences – just like mathematics is used:

> *"In fact there are two processes in every true science. One is the discovery of true principles, the other the deduction from them of correct conclusions. The former the more difficult and more important task. The latter however equally necessary."* (Boole, Notebooks)

Logic in social life

Even in everyday life, Boole wonders:

> *"That it should be possible for men to believe self-contradictory and mutually contradictory propositions is a remarkable circumstance. That men should in any case be able to reason falsely is remarkable.* **:-)***... The laws of correct inference are not less rigid, not less exact, than are the laws which govern the physical universe...; they are in their ultimate form and essence mathematical."* (Boole, Notebooks)

Boole's 'universe of discourse'

"In every discourse, whether of the mind conversing with its own thoughts, or of the individual in his intercourse with others, there is an assumed or expressed limit within which the subjects of its operation are confined. The most unfettered discourse is that in which the words we use are understood in the widest possible application, and for them the limits of discourse are co-extensive with those of the universe itself. But more usually we confine ourselves to a less spacious field. ... Now, whatever may be the extent of the field within which all the objects of our discourse are found, that field may properly be termed the universe of discourse." (Boole, ILT)

Alexandre Borovik: "This is a concept that you will immediately recognise as obvious, everyone-knows-it kind of things, but which was new, fresh, and perhaps paradoxical in Boole's time."

Boole's algebra of logic

Universe of discourse is denoted by 1.

Every logical statement is an 'elective symbol', like x, y, ... By convention x elects elements X, etc.

Multiplication: product $x \times y$ (or simply xy) is, in modern terms, composition of x and y, applied one after another: elect x, then apply election y to the result. So xy elects elements satisfying both x and y.

In particular, $xx = x$, or shortly $x^2 = x$.

Classes

In ILT Boole moved towards regarding x as a class (in modern terms, a (sub)set). One can say the set of all elements of the 'universe' for which x is true. Then xy is the intersection of the classes, where both x and y are true.

Empty set is 0. So writing $x = 0$ means there are no elements satisfying x.

Negation: $1 - x$ is the class of all non-X. It is postulated that every element is either in x or not; in other words, either x is true or not true.

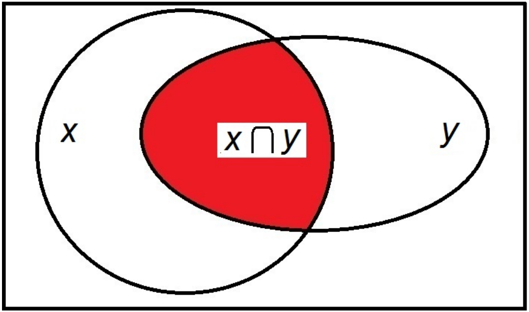
In particular, $x(1 - x) = 0$, which can be regarded as 'principle of contradiction'.

Addition: Boole defined it only for disjoint classes: $x + y$, union of x and y, where either x or y is true.

'Obviously', $x + (1 - x) = 1$, which is the law of excluded middle.

Euler–Venn diagrams

Within the universe represented by rectangular frame,



Intersection, where both x and y are true,

in modern notation $x \cap y$.

Negation (complement)

Within the universe! In this sense Boole's notation is less ambiguous than modern!

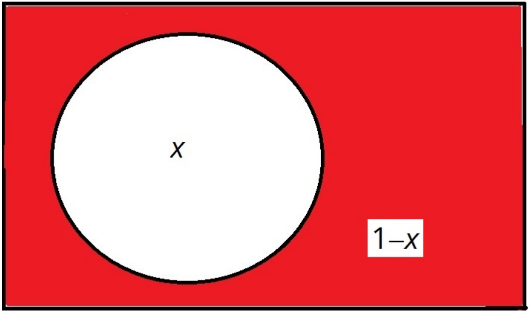


Figure: Negation (complement), where x is not true, $\overline{x}$

Addition as modern disjunction (union)

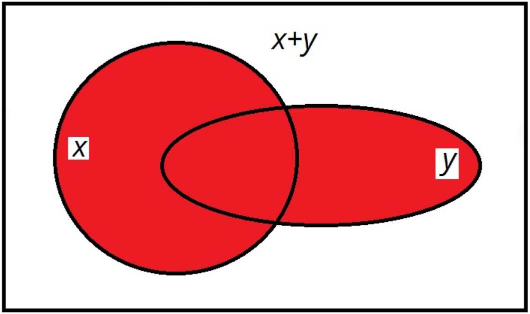


Figure: Union, where either x, or y, or both are true, $x \cup y$

But not used by Boole, who only allowed addition as union of disjoint sets!

Subtraction as modern set difference

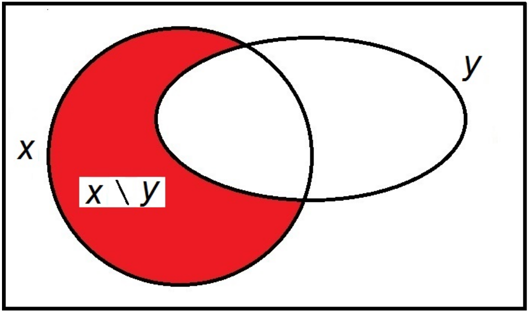


Figure: Difference, where x is true but y is not true, $x \setminus y$

Comparison of notation:

Boole's notation	meaning	modern logic	modern set theory
xy	both x and y are true	$x \wedge y$	$x \cap y$
$x(1-y)+y(1-x)$	either x or y is true but not both	$x \veebar y$	$x \triangle y$ (or $x \ominus y$)
$x+y(1-x)=$ $xy+x(1-y)+y(1-x)$	either x or y is true, or both	$x \vee y$	$x \cup y$
$1-x$	x is not true	$\bar{x}$ (or $\neg x$, (or $\sim x$)	$\bar{x}$

Axioms

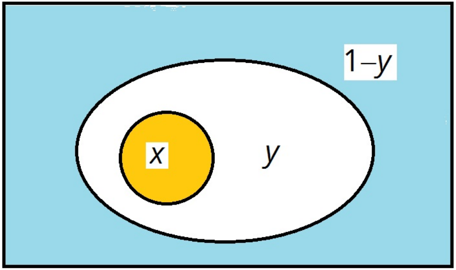
$xy = yx$	$x \cap y = y \cap x$
$x(y+z) = xy + xz$	$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
$x^2 := xx = x$	$x \cap x = x$
$1 \times x = x$	$\mathscr{U} \cap x = x$
$0 \times x = 0$	$\varnothing \cap x = \varnothing$

Inclusion

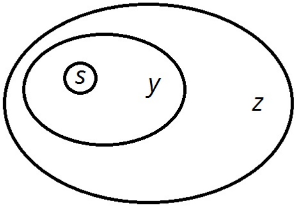
To express that x is contained in y (that is, 'all Xs are Ys'), Boole uses two forms:

either $x = vy$, where v is a special symbol meaning 'some' — in modern notation we write $x \subseteq y$.

But he prefers equations to 'inequalities', and prefers to write this as $x(1 - y) = 0$, which is of course the same as $x \subseteq y$!



Algebraic solution of a syllogism



All Ys (men) are Zs (mortal);
all Ss (Socrates) are Ys (men);
hence all Ss (Socrates) are Zs (is mortal).

Translated in terms of corresponding 'elective symbols', or classes, as

$y \subseteq z$, by Boole: $y(1-z)=0$; and $s \subseteq y$, by Boole: $s(1-y)=0$;

need to show that $s(1-z)=0$.

Substitute $s = s \times 1 = s\big(y + (1-y)\big) = sy + s(1-y)$

into $s(1-z) = \big(sy + s(1-y)\big)(1-z)$

$= sy(1-z) + s(1-y)(1-z) = s \times 0 + 0 \times (1-z) = 0$,

as required: $s(1-z)=0$ means exactly that $s \subseteq z$.

Boolean algebra

Boole states, 'with remarkably little fanfare':

> *"Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values* 0 *and* 1, *and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them."* (Boole, ILT)

This is (almost) the definition of what later became known as Boolean algebra!

Boole's symbolic method

> *"...any system of propositions may be expressed by equations involving symbols x, y, z, which, whenever interpretation is possible, are subject to laws identical in form with the laws of a system of quantitative symbols, susceptible only of the values 0 and 1. But as the formal processes of reasoning depend only upon the laws of the symbols, and not upon the nature of their interpretation, we are permitted to treat the above symbols, x, y, z, as if they were quantitative symbols of the kind above described. We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols, susceptible only of the values 0 and 1; perform upon them as such all the requisite processes of solution; and finally restore to them their logical interpretation."* (Boole, ILT)

We can see here the precursor of symbolic logic, mathematical logic, calculus of Boolean variables taking only values 1 (true) or 0 (false).

One remarkable feature is that Boole allows for intermediate calculations of these logical variables to be even 'uninterpretable' (and compares this to using $\sqrt{-1}$ in ordinary algebra).

Boole's general method

Given any function $f(x, y, z, \dots)$, a logical expression,
in terms of classes $x, y, z, \dots$,
Boole decomposes it in an analogue of Taylor series ('developments'):

First for one variable, $f(x) = f(1)x + f(0)(1 - x)$.

For two, or more variables, do the same consecutively:

$$f(x, y) = f(1, y)x + f(0, y)(1 - x)$$

$$= \big(f(1,1)y + f(1,0)(1 - y)\big)x + \big(f(0,1)y + f(0,0)(1 - y)\big)(1 - x)$$

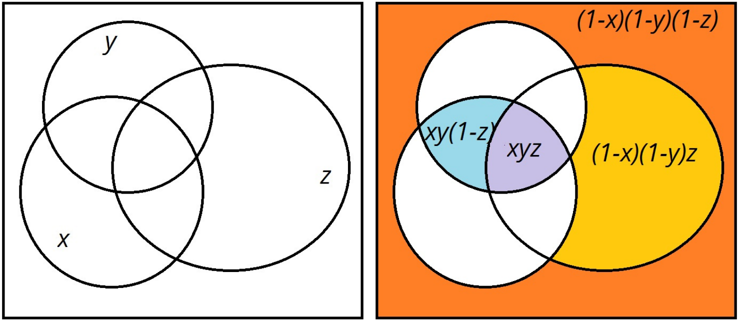
$$= f(1,1)xy + f(1,0)x(1 - y) + f(0,1)(1 - x)y + f(0,0)(1 - x)(1 - y).$$

Products like $x(1 - y)$, or $(1 - x)y$, are 'constituents',

in fact, treated as independent basis vectors, and algebraic calculations are performed with coefficients $f(0,0)$, $f(0,1)$, etc. (values of $f(x, y)$ for fixed values of x, y, in all cases either 1 or 0).

‘Developments’

These basis constituents have clear interpretation by regions on Venn’s diagrams (here, for three variables):



Then systems of equations appear, which Boole was solving, expressing one variable in terms of the others, similarly to solving linear systems of equation. Just that $\frac{0}{0}$ had to be used at times...

Probability

Large part of Boole's ILT book was devoted to computing probabilities of combinations of independent events, on the basis of his calculus of logical propositions.

Nowadays this is known as 'geometric probability'; it must be said that Boole not always clearly stated the assumptions that he made on independence of events.

Boole's name appears in probability theory in Boole's inequality: *for any finite or countable set of events, the probability that at least one of the events happens is no greater than the sum of the probabilities of the individual events*:

$$\mathbb{P}\Big(\bigcup_i A_i\Big) \leqslant \sum_i \mathbb{P}(A_i).$$

Imperfections

Boole's algebra of logic largely achieved its purpose of resolving logical equations by means of algebra similar to elimination and substitution methods for simultaneous linear equations.

Imperfections that attracted criticism were:

- dubious rules for division of coefficients like $\dfrac{0}{0}$,
- awkward rule for addition (later replaced by inclusive disjunction),
- being limited to propositional calculus.

Modern logic relies on quantifiers, rules of inference. Moreover, in second-order predicate calculus quantifiers can be applied not only to 'subject' symbols (as implicitly in Boole's propositional logic) but also to predicates: e.g.

$$\forall x \left(\exists y \in D \left(B(x,y) \right) \vee C(c,y) \right); \qquad \forall P \, \forall x \Big(x \in P \vee x \notin P \Big).$$

Start of development of mathematical logic

It was Boole's algebra of logic that truly initiated the continuous development of mathematical logic.

In 19th century:

- Work of De Morgan, Venn, et al. elaborated on Boole's ideas.
- Later work of Peirce, Frege, Schröder, Cantor, et al. established set theory and mathematical logic with quantifications as we know it today.

In 20th century:

- Logic, including set theory, was used to build foundations of mathematics (Peano, Hilbert, Russell and Whitegead, Gödel, et al.).
- Logic was used to solve problems in other parts of mathematics (Tarski, Mal'cev, et al.).
- Logic (primarily Boolean logic) became a theoretical basis for computer science and even technical realizations of electronic circuits and computers (Shannon, von Neumann, Turing, et al.)

Great things are better seen from a distance

Boole's books attracted a lot of interest when they appeared.

But later not all successors appreciated Boole's contribution as much as we do now. Some criticized it for the aforementioned imperfections, some even dismissed it.

Although undoubtedly influenced by Boole's work, some authors probably did not quite realize this fact, and formally could get away without giving references to Boole in expositions of their theories, which indeed superseded Boole's logic.

> *"If I understand him aright, Boole wanted to construct a technique for resolving logical problems systematically, similar to the technique of elimination and working out the unknown that algebra teaches. To this end, he represents judgements in the form of equations that he constructs out of letters and arithmetical signs such as +, 0 and 1. ... In the main these means fulfil their purpose, at least as far as the range of problems that Boole has in mind are concerned. But one may think of logical problems lying outside this range."* G. Frege

True revolutionary

It was not only the development of mathematical logic that was initiated by Boole.

We now realize that Boole made a giant step towards mathematics as a truly abstract discipline, causing a paradigm shift, giving modern mathematics enormous scope and potency.

Once seen, this path could not be 'unseen', and for successors this paradigm shift, once it was realized and adopted, became self-evident, clear as day, natural as air they breathed....

Much later, Bertrand Russell wrote that "pure mathematics was discovered" in Boole's book ILT.

Copernicus of mathematics

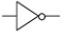
In the words of Augustus De Morgan, who himself devoted efforts to logic:

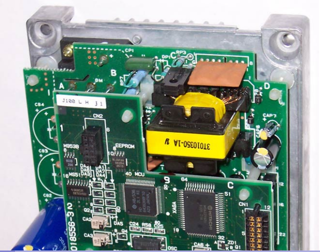
> *"Booles system of logic is among ... paradoxes, or things counter to general opinion ... a paradox which, like that of Copernicus, excited admiration from its first appearance. ... Boole's name will be remembered in connection with one of the most important steps towards the attainment of this knowledge."*

Thus Boole was indeed, Copernicus of algebra of logic, as well as of digital age to come hundred years later!

Boole's pure mathematics found applications in computers

Boole's algebra of variables taking only values 1 (true) or 0 (false) proved to be a perfect match for computer science and technical realization of computers. Claude Shannon (1916–2001) showed back in 1937 that electrical currents can realize any Boolean function. Electric telephone switchboards, or vacuum bulbs, or transistors perform Boolean operations, with 1 (true) corresponding to current, and 0 (false) to its absence.

Gate	Symbol	Operator
and		A · B
or		A + B
not		$\bar{A}$
nand		$\overline{A \cdot B}$



Role of logic in mathematics

Nowadays, there are, roughly, three aspects of logic within mathematics.

1. Mathematical logic as one of mathematical disciplines, with its axioms, theory, results, journals, conferences, etc. (Other branches of mathematics often can largely ignore it, as often happens between specialized branches of mathematics.)
2. Logical foundations of mathematics, where logic is used as a basis for the whole mathematical edifice. (To an extent, never ending process, but most mathematicians are satisfied with the foundations that exist since about 100 years ago.)
3. Logic essentially used in other branches of mathematics. (Like continuum hypothesis, decidablity of theories, local theorems, model theory tools, etc.)

Lack of rigor in early mathematics

Why does mathematics need mathematical logic? Is not mathematics already 'logical', the most, if not the only, rigorous science, which commands absolute truth in its proofs?

Nowadays, it may sound like a paradox, or even a heresy: but not so long ago, 150–200 years ago, mathematics was not as rigorous as today.

Many notions of calculus remained without precise definitions,

like a limit $\lim_{x \to a} f(x) = b$, or an infinite sum $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Mathematicians often relied on their intuition to arrive at correct results.

Ambiguities in definitions and methods of proof opened opportunities to errors; there was obvious quest for more solid foundations, more precise definitions.

One of the tools for introducing higher levels of rigor was mathematical logic, which included set theory.

Applying logic to mathematics

As we saw, Boole's successfully applied algebra, mathematics, to logic.

Mathematical logic in turn gave a very fruitful payback to mathematics.

Set-theoretical basis for mathematics

The language of sets and logic became a solid foundation for mathematics later in 19th century, and this largely remains to be the case today ("paradise" in the words of Hilbert).

Everything in mathematics is expressed in terms of sets.

For example, what is a function $f(x)$? A function can of course be described (and used to be, historically) as a rule, a procedure, a formula, that with every x (in a domain D) associates the value $f(x)$, special notation stresses this by arrows $x \mapsto f(x)$.

But the ultimate rigorous notion of a function $f(x)$ is the set of all pairs $\{(x, f(x)) \mid x \in D\}$.

The rigorous definition of a limit appeared only in the second half of 19th century: $\lim_{x \to a} f(x) = b$ by definition means that

$\forall \varepsilon > 0 \; \exists \delta = \delta(\varepsilon)$ such that $|b - f(x)| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Logical foundations of mathematics

Mathematical logic with set theory ultimately provided axiomatic foundations of mathematics (in the works of Frege, Peano, Hilbert et al.)

Moreover, Russell and Whitehead tried to reduce all mathematics to mathematical logic,

but Gödel showed that any more or less interesting mathematical system (which would include positive integers) cannot be both consistent and complete, in a certain precise sense.

Decidable and undecidable theories

Logic can give an overview of whole theories.

Decidable or undecidable theories: By definition, a theory is decidable if there is an algorithm, fixed set of rules, to decide if an arbitrary given formula is true in this theory.

Example of a decidable theory: The first-order theory of algebraically closed fields of a given characteristic, established by Alfred Tarski in 1949.

Example of an undecidable theory: The first-order theory of groups, established by Tarski in 1953. Remarkably, not only the general theory of groups is undecidable, but also several more specific theories, for example the theory of finite groups (as established by Mal'cev 1961).

'Silver lining' of this 'negative' result is that at any moment in time, there will always be open problems in group theory! **:-)**

Metamathematical arguments

Example

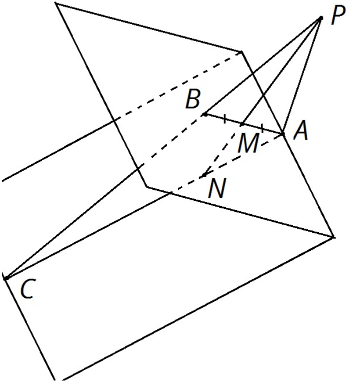
Geometric construction on the plane with compass and straight edge.

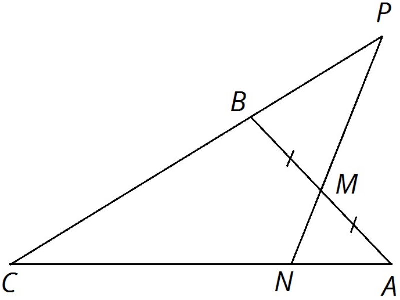
Mohr in 1672 and Mascheroni in 1797: possible by a compass alone (meaning that for a straight line it is enough to produce two points on it).

Question: can all these constructions be performed by straightedge alone? (meaning that for a circle it is enough to produce the center and a point on it).

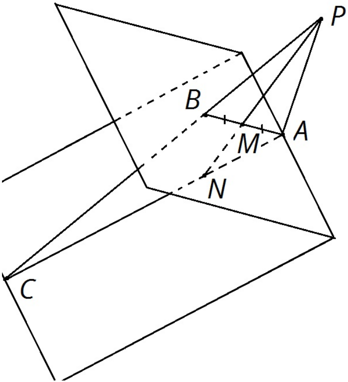
The answer is no, by a ‘metamathematical’ argument.

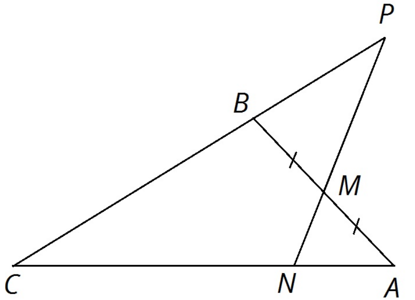
Proof: Suppose that there is a method to find the midpoint M of a given line segment AB. Central projection from a point P above the plane onto another plane at an angle. Choose P and the other plane in such a way M is projected **not** to the midpoint of the image AC of AB.





Second picture is the section of the first 3D figure along the lines AB and BC through point P.





But every step of the construction will be also projected as the same construction step on the second plane! Therefore, if the method worked on the first plane, then it must work also on the second plane, producing the projection N of the point M as a result. But N is not the midpoint of AC! Therefore the 'method' cannot be right.

Continuum hypothesis

The set of all rational numbers $\mathbb{Q}$ is countable, which means all of them can be enumerated by positive integers. One of the discoveries of set theory was that there are 'bigger' infinite sets: in particular the set of all real numbers $\mathbb{R}$ is uncountable.

The Continuum hypothesis stated in 1879 was that any set between $\mathbb{Q}$ and $\mathbb{R}$ is either countable, of the same cardinality as $\mathbb{Q}$, or of the same cardinality as $\mathbb{R}$.

$$\aleph_0 < ? < 2^{\aleph_0}.$$

This problem was solved only in 1963 by Paul Cohen by means of mathematical logic. The amazing result: this can be neither proved, nor refuted! In other words, either this hypothesis can be adopted as an additional axiom, or its negation.

(Similarly to non-Euclidean geometry, where the 5th postulate on parallel lines can be replaced by its negation.)

Local theorems in group theory

Suppose that a certain property σ holds on every finitely generated subgroup of a group G. Does this necessarily imply that the whole group enjoys this property?

If yes, then we say that the local theorem holds for the property σ. The local theorem holds for some properties but not for others. Separate papers were written proving of disproving the local theorem for particular properties,

... until Anatolii Mal'cev in 1941 and 1959 applied mathematical logic. He proved a general result: if a property can be written, as a logical statement, in a certain way, as a so-called quasi-universal formula, then the local theorem does hold for this property.

In this way he obtained all interesting local theorems of group theory in one blow, using, of course certain tricks to encode the properties in that form.

This was an impressive example of applying logic to algebra.

Local theorem used in a recent paper

Just an example:

G. Cutolo, E.I. Khukhro, J.C. Lennox, S. Rinauro, H. Smith and J. Wiegold, Locally finite groups all of whose subgroups are boundedly finite over their cores, *Bull. London Math. Soc.* **29** (1997), no. 5, 563–570.

....

Theorem 1. *If G is a locally finite group such that every subgroup contains a normal subgroup of index at most n, then the group G has an abelian subgroup of index bounded in terms of n.*

Proof (reduction to finite groups): The property to have an abelian subgroup of finite index bounded by a given number can be written by a quasi-universal formula. Therefore, by the Mal'cev Local Theorem, it is sufficient to prove the theorem assuming that the group G is finite.........

....

Among giants on whose shoulders we stand

Boole's work has put him among the giants on whose shoulders we stand to see further.

Celebrating the mathematical legacy of George Boole, we celebrate the triumphs of human intellectual endeavour, triumphs of mathematics as one of the most creative sciences which not only enables us to see logical structures in the present but also helps to create the future.

Thank you!