Universidade de Brasília (UnB)
Instituto de Ciências Exatas
Departamento de Matemática

A Landesman-Lazer local condition for nonlinear elliptic problems

por

Pedro Manuel Sánchez Aguilar

Orientador: Elves Alves de Barros e Silva

Brasília

2017

Universidade de Brasília (UnB)
Instituto de Ciências Exatas
Departamento de Matemática

A Landesman-Lazer local condition for nonlinear elliptic problems

por

Pedro Manuel Sánchez Aguilar *

Dissertação apresentada ao Departamento de Matemática da Universidade de Brasília como parte dos requisitos necessários para obtenção do grau de

DOUTOR EM MATEMÁTICA

Brasília, 28 de Julho de 2017

Comissão Examinadora:

Prof. Dr. Elves Alves de Barros e Silva (UnB)-Orientador

Prof. Dr. Everaldo Souto de Medeiros (UFPB)

Prof. Dr. Giovany de Jesus Malcher Figueiredo (UnB)

Prof. Dr. Sérgio Henrique Monari Soares (USP)

*O autor foi bolsista do CNPq durante a elaboração deste trabalho.

Ficha catalográfica elaborada automaticamente,
com os dados fornecidos pelo(a) autor(a)

Sánchez Aguilar, Pedro Manuel
SSA194l A Landesman-Lazer local condition for nonlinear elliptic problems / Pedro Manuel Sánchez Aguilar; orientador Elves Alves de Barros e Silva. -- Brasília, 2017.
100 p.

Tese (Doutorado - Doutorado em Matemática) -- Universidade de Brasília, 2017.

1. Matemática. 2. Problemas elípticos não lineares. 3. Métodos variacionais. 4. Condição de Landesman-Lazer. 5. Método de redução de Lyapunov Schmidt. I. Alves de Barros e Silva, Elves , orient. II. Título.

Agradecimentos

À Deus, por permitir mais uma conquista.

Ao Professor Elves Alves de Barros e Silva e a professora Manuela Caetano Martins de Rezende que me orientaram no presente trabalho motivando a estudar e a seguir em frente transmitindo sua experiência de forma pontual e precisa.

Aos membros da banca examinadora, professores Everaldo Souto de medeiros, Giovany de Jesus Malcher Figueiredo e Sérgio Henrique Monari Soares, pela disponibilidade para participar e pelas valiosas sugestões.

Aos meus pais Hermenegildo Sánchez Vega e Gladys Teodonila Aguilar Viteri, a meus avós Emperatrys Viteri Anhuaman, Paula Vega e Pedro Sánchez Azabache, a minha esposa Olinda Cristina Gamboa Murga e a minha filha Gladys Sánchez Gamboa por me apoiarem em todos os momentos.

Aos meus irmãos Alvaro, Mercy, Einer Gladys e Harvey por acreditarem no sucesso desta jornada.

Aos professores e amigos do Departamento de Matemática-UnB, por estarem sempre presentes a pesar de todos os seus compromissos, sempre estiveram à disposição.

Ao Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), pelo apoio financeiro.

Resumo

O objetivo deste trabalho é estudar a existência, multiplicidade e não existência de soluções para problemas elípticos não-lineares dependendo de um parâmetro sob uma hipótese do tipo Landesman-Lazer. Para estabelecer a existência de solução combinamos o Método de Redução de Lyapunov-Schmidt e a técnica de congelamento do termo gradiente com argumentos de truncamento e aproximação através de métodos de bootstrap. Não há restrição de crescimento no infinito sobre o termo não-linear o qual pode mudar de sinal.

Palavras-chave: Problemas elípticos não lineares, métodos variacionais, método de redução de Lyapunov-Schmidt, condição de Landesman-Lazer.

Abstract

The purpose of this work is to study the existence, multiplicity and non existence of solutions for nonlinear elliptic problems depending on a parameter under Landesman-Lazer type hypotheses. In order to establish the existence of solution we combine the Lyapunov-Schmidt Reduction Method and the term gradient freeze technique with truncation and approximation arguments via bootstrap methods. There is no growth restriction at infinity on the nonlinear term and it may change sign.

Key words: Nonlinear elliptic problems, variational methods, Lyapunov-Schmidt Reduction Method, Landesman-Lazer condition.

Contents

Notations

Throughout this work we use the following notations

- For $k \in \mathbb{N}$ and $0 < \gamma \leq 1$,

$$C^{k,\gamma}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) : H_\gamma(D^\alpha u) < \infty \text{ for } 0 \leq |\alpha| \leq k\},$$

  where $H_\gamma(D^\alpha u) = \sup\{|D^\alpha u(x) - D^\alpha u(y)|/|x-y|^\gamma : x, y \in \Omega, x \neq y\}$, with norm given by

$$\|u\|_{C^{k,\gamma}} = \sum_{|\alpha|\leq k} \max_{x\in\overline{\Omega}} |D^\alpha u(x)| + \sum_{|\alpha|\leq k} H_\gamma(D^\alpha u).$$

- For $1 < q < \infty$,

$$L^q(\Omega) = \left\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^q dx < \infty\right\}$$

  with norm given by

$$\|u\|_q = \left(\int_\Omega |u|^q dx\right)^{1/q}.$$

- $L^\infty(\Omega)$ denotes the space of the measurable functions that are almost every bounded in Ω with norm given by

$$\|u\|_\infty = \inf\{C > 0 : |u(x)| \leq C \text{ a.e. in } \Omega\}.$$

- For $k \in \mathbb{N}$ and $1 \leq q < \infty$,

$$W^{k,q}(\Omega) = \left\{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq k\right\}$$

with norm given by

$$\|u\|_{k,q} = \left(\sum_{0\leq|\alpha|\leq k} \|D^{\alpha}u\|_q^q \right)^{1/q}.$$

- $H_0^1(\Omega)$ is the clousure of $C_0^{\infty}(\Omega)$ in the space $W^{1,2}(\Omega)$ with norm given by

$$\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

- The symbols A, d_q, K and K_i, $i = 1, 2, \cdots$, represent positive constants, reserving d_q for the embedding constant of $H_0^1(\Omega)$ into $L^q(\Omega)$, $q \in [1, 2^*]$.

Introdução

Neste trabalho utilizamos métodos variacionais, o Método de Redução de Lyapunov-Schmidt e a técnica do congelamento do termo gradiente, para o estudo de existência, multiplicidade e não-existência de soluções para problemas elípticos não-lineares dependendo de um parâmetro sob hipóteses do tipo Landesman-Lazer. O nosso principal objetivo é considerar problemas em que não há restrição de crescimento global sobre o termo não-linear e onde este termo pode mudar de sinal.

No Capítulo 1 deste trabalho consideramos a existência e não-existência de soluções fracas para o seguinte problema

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x,u) \text{ em } \Omega, \\ u = & 0 \qquad\qquad \text{sobre } \partial\Omega, \end{cases} \tag{1.1}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 é o segundo autovalor do operador $-\Delta$ em $H_0^1(\Omega)$; $\mu \neq 0$ é um parâmetro real e $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ é uma função Carathéodory satisfazendo

(h_0) para todo $A > 0$ existe $f_A \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, tal que

$$|h(x,s)| \leq f_A(x), \text{ para todo } |s| \leq A, \text{ para quase todo } x \in \overline{\Omega};$$

e

(h_1) dado $A_1 > 0$ existe $\zeta := \zeta(A_1) \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, tal que

$$|h(z, s_1) - h(z, s_2)| \leq \zeta(z)|s_1 - s_2|, \text{ para todos } z \in \overline{\Omega}, \ |s_1|, \ |s_2| \leq A_1.$$

Nosso principal objetivo é estabelecer a existência de múltiplas soluções para o Problema (1.1). Para esse resultado, consideramos que

(h_2) existem $k \in \mathbb{N}$ e $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$ tais que

$$\left[\int_\Omega h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1)\varphi_1 dx\right] < 0,$$

onde φ_1 é uma autofunção positiva associada a λ_1.

Theorem 1.1. *Suponha que h satisfaz (h_0), (h_1) e (h_2). Então, existem constantes positivas μ^* e ν^* tais que, para todos $0 < |\mu| < \mu^*$ e $|\lambda - \lambda_1| < |\mu|\nu^*$, o Problema* (1.1) *possui k soluções $u_i = \hat{t}_i\varphi_1 + v_i$ de classe $C^{0,\gamma}(\overline{\Omega})$, com $\hat{t}_i \in (t_i, t_{i+1})$ e $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.*

É importante notar que, no Teorema 1.1, a não-linearidade não tem restrição de crescimento global e as projeções das soluções u_i, sobre a direção de φ_1, estão localizadas entre $t_i\varphi_1$ e $t_{i+1}\varphi_1$.

O Teorema 1.1 é uma consequência direta de dois resultados que garantem a existência de um solução para o Problema (1.1) sob um dos seguintes casos particulares de (h_2):

(h_2^+) existem números reais t_1 e t_2, com $t_1 < t_2$, tais que

$$\int_\Omega h(x, t_1\varphi_1)\varphi_1 dx > 0 > \int_\Omega h(x, t_2\varphi_1)\varphi_1 dx,$$

ou

(h_2^-) existem números reais t_1 e t_2, com $t_1 < t_2$, tais que

$$\int_\Omega h(x, t_1\varphi_1)\varphi_1 dx < 0 < \int_\Omega h(x, t_2\varphi_1)\varphi_1 dx.$$

Em nosso primeiro resultado sobre existência, usamos métodos variacionais para estabelecer a existência de uma solução fraca para o Problema (1.1), considerando a hipótese (h_2^+) e menos regularidade na função h. Mais precisamente, temos:

Teorema 1.2. *Suponha que h satisfaz (h_0) e (h_2^+). Então,existem constantes positivas μ^* e ν^* tais que, para todos $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$, o Problema* (1.1) *possui uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{0,\gamma}(\overline{\Omega})$, com $t \in (t_1, t_2)$ e $v \in \langle\varphi_1\rangle^\perp$.*

Para estabelecer a existência de uma solução sob a hipótese (h_2^-), precisamos supor mais regularidade na função h.

Teorema 1.3. *Suponha que h satisfaz (h_0), (h_1) e (h_2^-). Então, existem constantes positivas μ^* e ν^* tais que, para todos $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$, o Problema* (1.1) *possui uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{0,\gamma}(\overline{\Omega})$, com $t \in (t_1, t_2)$ e $v \in \langle \varphi_1 \rangle^\perp$.*

Observação 1.4. *Se assumimos em (h_0) e (h_1) que $\sigma > N$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, a solução u_μ do Problem* (1.1)*, dado no Teorema* 1.2 *ou* 1.3*, é de classe $C^{1,\gamma}(\overline{\Omega})$. Usando este fato podemos provar que u_μ é positiva ou negativa em Ω, desde que $t_1 \geq 0$ ou $t_2 \leq 0$, respectivamente. Além disso, para $|\mu| > 0$ suficientemente pequeno, as soluções do Teorema* 1.1 *são ordenadas (veja os Teoremas* 1.20 *e* 1.21*).*

Observe que a hipótese (h_2) é uma condição do tipo Landesman-Lazer, veja [35]. De fato, no caso onde a não-linearidade h, dada em (1.1), é da forma $h(x, s) = f(x) - g(s)$, onde $f \in L^\sigma(\Omega)$ e $g : \mathbb{R} \to \mathbb{R}$ é uma função continua limitada, com limites finitos $g^- := \lim_{s\to-\infty} g(s)$ e $g^+ := \lim_{s\to\infty} g(s)$, a condição de Landesman-Lazer

$$g^\mp \int_\Omega \varphi_1 dx < \int_\Omega f\varphi_1 dx < g^\pm \int_\Omega \varphi_1 dx$$

nos assegura que

$$\lim_{t\to-\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \int_\Omega (f - g^-)\varphi_1 dx > (<)0$$

e

$$\lim_{t\to+\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \int_\Omega (f - g^+)\varphi_1 dx < (>)0.$$

Consequentemente, existem números reais t_1 e t_2, com $t_1 < 0 < t_2$, tais que a condição (h_2^+) (or (h_2^-)) é válida para t_1 e t_2. Em outras palavras, quando $h(x, s) = f(x) - g(s)$, onde f e g são como acima, obtemos que a condição de Landesman-Lazer implica em (h_2^+) (or (h_2^-)).

É importante notar que, sob nossas hipóteses, h pode mudar de sinal em Ω, o que caracteriza o Problema (1.1) como indefinido. Esta classe de problemas tem sido objeto de uma intensa pesquisa nas últimas três décadas desde os trabalhos de Alama e Tarantello [2], Berestycki, Capuzzo e Nirenberg [11] e Ouyang [41]. Veja [3, 4, 5, 21, 23, 26, 27, 38, 47] e suas referências para mais detalhes sobre este tipo de problemas.

As demonstrações dos Teoremas 1.2 e 1.3 são inspiradas no Método de Redução de Lyapunov-Schmidt, conforme apresentado nos artigos [19, 20, 22, 36] . No entanto, sob as hipóteses desses teoremas, dito método não pode ser aplicado diretamente, já que não impomos nenhuma restrição de crescimento global no termo não linear h. Para contornar essa dificuldade, combinamos o Método de Redução de Lyapunov-Schmidt com argumentos de truncamento e aproximação via o método bootstrap.

É importante mencionar que em muitas situações a condição (h_2^+) ou (h_2^-) é suficiente para a existência de uma solução $u_\mu = t\varphi_1 + v$, $t \in (t_1, t_2)$, $v \in \langle\varphi_1\rangle^\perp$. A seguir estabelecemos a não-existência de soluções para o Problema (2.1) quando a hipótese (h_2) não é valida. De fato, se assumimos que h satisfaz

(h_3) existe $f \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, tal que

$$|h(x,s)| \leq f(x)(1+|s|), \text{ para todo } s \in \mathbb{R}, \quad \text{para quase todo } x \in \overline{\Omega};$$

e

(h_4) existem números reais t_1 e t_2, com $t_1 < t_2$, tais que

$$\int_\Omega h(x, t\varphi_1)\varphi_1 dx \neq 0, \quad \text{para todo } t \in [t_1, t_2],$$

podemos estabelecer o seguinte resultado.

Teorema 1.5. *Suponha que h satisfaz (h_3) e (h_4). Então, existem constantes positivas μ^* e ν^* tais que, para todos $0 < |\mu| < \mu^*$ e $|\lambda - \lambda_1| < |\mu|\nu^*$, o Problema (1.1) não possui solução fraca $u_\mu = t\varphi_1 + v$, com $t \in [t_1, t_2]$ e $v \in \langle\varphi_1\rangle^\perp$.*

É importante notar que o Teorema 1.5 é consequência de um resultado que estabelece que, sob a hipótese (h_4), o Problema (1.1) não possui solução limitada em $L^\infty(\Omega)$.

Como uma aplicação dos Teoremas 1.2 e 1.3, consideramos a existência de uma solução para o seguinte problema

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x) u^q + b_2(x) u^p \text{ em } \Omega, \\ u = & 0 \qquad\qquad\qquad\qquad \text{sobre } \partial\Omega, \end{cases} \tag{1.2}$$

onde Ω é um domínio suave limitado do $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 é o segundo autovalor do operador $-\Delta$ em $H_0^1(\Omega)$; $\beta > 0$ é um parâmetro real; $p > q$, com $p \neq 1$, e b_1, $b_2 \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ if $N = 1, 2$. Denotemos

$$r_1 := \int_\Omega b_1 \varphi_1^{q+1} dx \quad \text{e} \quad r_2 := \int_\Omega b_2 \varphi_1^{p+1} dx.$$

Considerando a nomenclatura para problemas elípticos usada na literatura, o Problema (1.2) é superlinear ou sublinear no infinito se $p > 1$ ou $0 < p < 1$ e é superlinear, linear ou sublinear na origem se $q > 1$, $q = 1$ ou $0 < q < 1$. Para o caso linear ou superlinear na origem e superlinear no infinito podemos estabelecer o seguinte resultado.

Proposição 1.6. *Suponha que* $p > q \geq 1$ *e* $r_1 r_2 < 0$. *Então, existem constantes positivas* β^* *e* ν^* *tais que o Problema* (1.2) *possui uma solução de classe* $C^{0,\gamma}(\overline{\Omega})$, *para todo* $\beta \in (0, \beta^*)$ *e* $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu^*$.

Para o caso sublinear na origem e superlinear no infinito podemos aplicar o Teorema 2.20.

Proposição 1.7. *Suponha que* $p > q > 0$, $p \neq 1$, $1 > q > 0$ *e* $r_1 > 0 > r_2$. *Então*

(i) *se* $p > 1$, *existem constantes positivas* β_1^* *e* ν_1^* *tais que o Problema* (1.2) *possui uma solução de classe* $C^{0,\gamma}(\overline{\Omega})$, *para todo* $\beta \in (0, \beta_1^*)$ *e* $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_1^*$.

(i) *se* $p < 1$, *existem constantes positivas* β_2^* *e* ν_2^* *tais que o Problema* (1.2) *possui uma solução de classe* $C^{0,\gamma}(\overline{\Omega})$, *para todo* $\beta \in (\beta_2^*, \infty)$ *e* $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_2^*$.

Quando em (1.2) consideramos b_1, $b_2 \in L^\sigma(\Omega)$, com $\sigma > N$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, as soluções dadas pela Proposição 1.6 ou 1.7 é de classe $C^{1,\gamma}(\overline{\Omega})$ e é positiva em Ω (veja Observação 1.27). Além disso, conforme observado no início desta introdução, as funções b_1 e b_2 podem mudar de sinal. Neste caso, o Problema (1.2) é indefinido. Enfatizamos que na Proposição 1.6 e no item (i) da Proposição 1.7, não assumimos a restrição $p < (N + 2)/(N - 2)$ para garantir a existencia de uma solução para (1.2). Para o caso linear ou superlinear na origem e superlinear no infinito referimos o leitor aos artigos de Ouyang [41], Alama e Tarantello [3] e Medeiros, Severo e Silva [38]. Para o caso sublinear na origem citamos os trabalhos de Ambrosetti, Brezis e Cerami [8] e De Figueiredo, Gossez e Ubilla [26].

Inspirados pelo artigo de Brezis e Nirenberg, veja [15], consideramos uma aplicação da Proposição 1.6 para o seguinte problema:

$$\begin{cases} -\Delta u = & \lambda u + b(x) u^p \text{ em } \Omega, \\ u = & 0 \qquad \text{sobre } \partial\Omega, \end{cases} \tag{1.3}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; $\lambda < \lambda_1$, λ_1 é o primeiro autovalor do operador $-\Delta$ em $H_0^1(\Omega)$; $p > 0$ e $b \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, satisfazendo

$$\int_\Omega b(x) \varphi_1^{p+1} dx > 0. \tag{1.4}$$

Proposição 1.8. *Suponha que b satisfaz (1.4), com $p > 0$ e $p \neq 1$, então existe $\underline{\lambda}$ tal que o Problema (1.3) possui uma solução positiva para todo $\underline{\lambda} < \lambda < \lambda_1$.*

Motivados por Landesman e Lazer, veja [35], também apresentamos outra aplicação do Teorema 1.2 ou 1.3. Ou seja, estamos interessados na solubilidade do seguinte problema:

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u)) \text{ em } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\quad \text{sobre } \partial\Omega, \end{cases} \tag{1.5}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; λ e μ são como em (1.1) e $f \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$.

Para garantir a existência de uma solução para o Problema (1.5), consideramos

(g_1) $g : \mathbb{R} \to \mathbb{R}$ uma função contínua e existe $M > 0$ tal que

$$g(s) \geq -M \text{ se } s \leq 0 \text{ e } g(s) \leq M \text{ se } s \geq 0.$$

Denotando por $g_i^- := \liminf_{s\to-\infty} g(s)$ e $g_s^+ := \limsup_{s\to+\infty} g(s)$, assumimos:

(LL^+)
$$\int_\Omega (f + g_i^-)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+)\varphi_1 dx.$$

Proposição 1.9. *Suponha que g satisfaz (g_1) e (LL^+). Então, existem constantes positivas μ^* e ν^* tais que, para todo $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$, o Problema (1.5) possui uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{0,\gamma}(\overline{\Omega})$, com $t \in \mathbb{R}$ e $v \in \langle\varphi_1\rangle^\perp$.*

Denotando $g_s^- := \limsup_{s\to-\infty} g(s)$ e $g_i^+ := \liminf_{s\to+\infty} g(s)$, fornecemos um resultado análogo quando

$(\hat{g}_1)$ $g : \mathbb{R} \to \mathbb{R}$ é localmente Lipschitz e existe $M > 0$ tal que

$$g(s) \leq M \text{ se } s \leq 0 \text{ e } g(s) \geq -M \text{ se } s \geq 0$$

e

(LL^-)
$$\int_\Omega (f + g_s^-)\varphi_1 dx < 0 < \int_\Omega (f + g_i^+)\varphi_1 dx.$$

Proposição 1.10. *Suponha que g satisfaz $(\hat{g}_1)$ e (LL^-). Então, existem constantes positivas μ^* e ν^* tais que, para todo $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$, o Problema (1.5) possui uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{0,\gamma}(\overline{\Omega})$, com $t \in \mathbb{R}$ e $v \in \langle\varphi_1\rangle^\perp$.*

Observação 1.11. *Os resultados anteriores permitem considerar g tal que $g_i^- = +\infty$ e $g_s^+ = -\infty$ ou $g_s^- = -\infty$ e $g_i^+ = +\infty$, respectivamente. Além disso g pode ter um comportamento oscilante ilimitado, o que não ocorre, por exemplo, em [35].*

Ressaltamos que uma aplicação do Teorema 1.1 é dada quando h é uma função polinomial na segunda variável. Veja Proposição 1.28.

Como mencionado anteriormente, h não tem restrição de crescimento, o que impede a aplicação de métodos variacionais, uma vez que o funcional associado não está bem definido. Para provar o Teorema 1.2, inicialmente assumimos que h é limitada com respeito a $L^\sigma(\Omega)$.

Neste caso, usando um argumento de minimização, existe uma constante positiva ν^* tal que, para $\mu > 0$ suficientemente pequeno e $|\lambda - \lambda_1| < \mu\nu^*$, o Problema (1.1) possui uma solução $u_\mu = t\varphi_1 + v$, com $t \in (t_1, t_2)$ e $v \in \langle\varphi_1\rangle^\perp$. Em seguida, considerando uma função truncamento adequada de h e usando a solubilidade de (1.1), com h limitada com respeito a $L^\sigma(\Omega)$ garantimos a existência de uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{0,\gamma}(\overline{\Omega})$. Posteriormente, via um argumento bootstrap, provamos que $\|v\|_\infty \to 0$, quando $\mu \to 0$, isto nos permite encontrar $\mu^* > 0$ tal que u_μ é uma solução do problema (1.1), para todos $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$.

Chamamos a atenção do leitor para o fato de que, sob a condição (h_2^-), não podemos aplicar o método de minimização utilizado na demonstração do Teorema 1.2. A solução, neste caso, é um ponto crítico minimax.

Para provar o Teorema 1.3, inicialmente assumimos que h é limitada com respeito a $L^\sigma(\Omega)$ e Lipschitz com respeito a $L^\sigma(\Omega)$ na segunda variável.

Sob estas hipóteses, podemos aplicar o Método de Redução de Lyapunov-Schmidt para encontrar um ponto crítico para o funcional associado ao problema truncado. Agora argumentamos como na prova do Teorema 1.2 para mostrar que existem constantes positivas μ^* e ν^* tais que, para todos $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$, $u_\mu = t\varphi_1 + v$, com $t \in (t_1, t_2)$ e $v \in \langle\varphi_1\rangle^\perp$, é uma solução do Problema (1.1).

No Capítulo 2, consideramos a existência e não-existência de soluções fracas para o seguinte problema:

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x, u, \nabla u) \text{ em } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad \text{sobre } \partial\Omega, \end{cases} \tag{2.1}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 é o segundo autovalor do operador $-\Delta$ em $H_0^1(\Omega)$; $\mu \neq 0$ é um parâmetro real e $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ é uma função Carathéodory satisfazendo:

$(h_\nabla)_0$ para todo $A > 0$, existe $f_A \in L^\sigma(\Omega)$, com $\sigma > N$ se $N \geq 3$ e $\sigma > 2$ se $N = 1, 2$, tal que

$$|h(x, s, \xi)| \leq f_A(x), \text{ para todos } |s| \leq A, \ |\xi| \leq A, \text{ para quase todo } x \in \overline{\Omega};$$

e

$(h_\nabla)_1$ para todos A_1, $A_2 > 0$, existem $\zeta_1 = \zeta_1(A_1)$, $\zeta_2 = \zeta_2(A_2) \in L^\sigma(\Omega)$, com $\sigma > N$ se $N \geq 3$ e $\sigma > 2$ se $N = 1, 2$, tais que

$$|h(z, s_1, \xi) - h(z, s_2, \xi)| \leq \zeta_1(z)|s_1 - s_2|, \text{ para todos } z \in \overline{\Omega}, \ |s_1|, |s_1| \leq A_1, \ |\xi| \leq A_2$$

e

$$|h(z, s, \xi_1) - h(z, s, \xi_2)| \leq \zeta_2(z)|\xi_1 - \xi_2|, \text{ para todos } z \in \overline{\Omega}, \ |s| \leq A_1, \ |\xi_1|, |\xi_2| \leq A_2.$$

Nosso principal resultado estabelece a existência de uma solução para o Problema (2.1). Para este resultado, consideramos que:

$(h_\nabla)_2$ existem t_1 e $t_2 \in \mathbb{R}$, $t_1 < t_2$, tais que

$$\left[\int_\Omega h(x, t_1\varphi_1, t_1\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_2\varphi_1, t_2\nabla\varphi_1)\varphi_1 dx\right] < 0,$$

onde φ_1 é uma autofunção positiva associada a λ_1.

Teorema 2.1. *Suponha que h satisfaz $(h_\nabla)_0$, $(h_\nabla)_1$ e $(h_\nabla)_2$. Então, existem constantes positivas μ^* e ν^* tais que, para todos $\mu \in (0, \mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$, o Problema* (2.1) *possui uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{1,\gamma}(\overline{\Omega})$, com $t \in (t_1, t_2)$ e $v \in \langle\varphi_1\rangle^\perp$.*

É importante notar que no Teorema 2.1 não impomos restrição de crescimento global sobre o termo não-linear em relação à segunda e terceira variável. Também observe que a projeção da solução u_μ na direção de φ_1 está localizada entre $t_1\varphi_1$ e $t_2\varphi_2$.

Como consequência direta do Teorema 2.1 estabelecemos a existência de múltiplas soluções para Problema (2.1). Na verdade, assumindo que

$(\hat{h}_\nabla)_2$ existem $k \in \mathbb{N}$ e $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$, tais que

$$\left[\int_\Omega h(x, t_i\varphi_1, t_i\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1, t_{i+1}\nabla\varphi_1)\varphi_1 dx\right] < 0,$$

podemos estabelecer o seguinte resultado:

Proposição 2.2. *Suponha que* h *satisfaz* $(h_\nabla)_0$, $(h_\nabla)_1$ *e* $(\hat{h}_\nabla)_2$. *Então, existem constantes positivas* μ^* *e* ν^* *tais que, para todos* $0 < |\mu| < \mu^*$ *e* $|\lambda - \lambda_1| < \mu\nu^*$, *o Problema* (2.1) *possui* k *soluções* $u_i = \hat{t}_i\varphi_1 + v_i$ *de classe* $C^{1,\gamma}(\overline{\Omega})$, *com* $\hat{t}_i \in (t_i, t_{i+1})$ *e* $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.

Observação 2.3. *A solução* u_μ, *dada pelo Teorema* 2.1, *é positiva ou negativa em* Ω *desde que* $t_1 \geq 0$ *ou* $t_2 \leq 0$, *respectivamente. Além disso, para* $|\mu| > 0$ *suficientemente pequeno as soluções da Proposição* 2.2 *são ordenadas, veja o Teorema* 2.17 *e Proposição* 2.18.

Observe que a hipóteses $(h_\nabla)_2$ é uma condição do tipo Landesman-Lazer, veja [35]. Em [46] (veja também [18, 40]) Shaw considero o caso onde a não-linearidade h é da forma $h(x,s,\xi) = f(x) + g(s) + \Gamma(x,s,\xi)$, onde $f \in L^\sigma(\Omega)$, $g : \mathbb{R} \to \mathbb{R}$ é uma função contínua limitada, com limites finitos $g^- := \lim_{s\to-\infty} g(s)$ e $g^+ := \lim_{s\to\infty} g(s)$, e $\Gamma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ é uma função contínua limitada. Shaw provou a existência de uma solução para o Problema (2.1) assumindo o seguinte versão da condição Landesman-Lazer:

$$g^- \int_\Omega \varphi_1 dx + \int_\Omega f\varphi_1 dx + \alpha \int_\Omega \varphi_1 dx < 0 < g^+ \int_\Omega \varphi_1 dx + \int_\Omega f\varphi_1 dx - \alpha \int_\Omega \varphi_1 dx,$$

onde $\alpha = \sup\{|\Gamma|\}$. Consequentemente, existem números reais t_1 e t_2, com $t_1 < 0 < t_2$, tais que a condição $(h_\nabla)_2$ é valida para t_1 e t_2. Em outras palavras, a versão da condição de Landesman-Lazer usada por Shaw [46] implica na hipótese $(h_\nabla)_2$.

É importante mencionar que em muitas situações a condição $(h_\nabla)_2$ é suficiente para a existência de uma solução $u_\mu = t\varphi_1 + v$, $t \in (t_1, t_2)$, $v \in \langle\varphi_1\rangle^\perp$. A seguir estabelecemos a não-existência de soluções para o Problema (2.1) quando a hipótese $(h_\nabla)_2$ não é valida. De fato, se assumimos que h satisfaz

$(h_\nabla)_3$ existe $f \in L^\sigma(\Omega)$, com $\sigma > N$ se $N \geq 3$ e $\sigma > 2$ se $N = 1, 2$, tal que

$$|h(x,s,\xi)| \leq f(x)(1 + |s| + |\xi|), \text{ para todos } s \in \mathbb{R},\ \xi \in \mathbb{R}^N, \text{ para quase todo } x \in \overline{\Omega},$$

e

$(h_\nabla)_4$ existem números reais t_1 e t_2, com $t_1 < t_2$, tais que

$$\int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \neq 0, \quad \text{para todo } t \in [t_1, t_2],$$

podemos estabelecer:

Teorema 2.4. *Suponha que h satisfaz* $(h_\nabla)_3$ *e* $(h_\nabla)_4$*. Então, existem constantes positivas* μ^* *e* ν^* *tais que, para cada* $0 < |\mu| < \mu^*$ *e* $|\lambda - \lambda_1| < |\mu|\nu^*$*, o Problema* (2.1) *não possui solução fraca* $u_\mu = t\varphi_1 + v$*, com* $t \in [t_1, t_2]$ *e* $v \in \langle\varphi_1\rangle^\perp$.

Como uma primeira aplicação do Teorema 2.1 consideramos a existência de uma solução para o seguinte problema:

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x)u^{q_1}|\nabla u|^{q_2} + b_2(x)u^{p_1}|\nabla u|^{p_2} \text{ em } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{sobre } \partial\Omega, \end{cases} \tag{2.2}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 é o segundo autovalor do operador $-\Delta$ em $H_0^1(\Omega)$; $\beta > 0$ é um parâmetro real e b_1, $b_2 \in L^\sigma(\Omega)$, com $\sigma > N$ se $N \geq 3$ e $\sigma > 2$ se $N = 1, 2$.

Considerando

$$r_1 := \int_\Omega b_1\varphi_1^{q_1+1}|\nabla\varphi_1|^{q_2}dx \quad \text{e} \quad r_2 := \int_\Omega b_2\varphi_1^{p_1+1}|\nabla\varphi_1|^{p_2}dx,$$

estabelecemos o resultado:

Proposição 2.5. *Suponha que* $p = p_1 + p_2$*,* $q = q_1 + q_2$*,* p_1*,* p_2*,* q_1*,* $q_2 \geq 1$*,* $p > q$ *e* $r_1 r_2 < 0$*. Então, existem constantes positivas* β_1^* *e* ν_1^* *tais que o Problema* (2.2) *possui uma solução de classe* $C^{1,\gamma}(\overline{\Omega})$*, para todos* $\beta \in (0, \beta_1^*)$ *e* $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}}\nu_1^*$.

Outra aplicação do Teorema 2.1 é dada pelo seguinte problema:

$$\begin{cases} -\Delta u = & \lambda u + b(x)u^{p_1}|\nabla u|^{p_2} \text{ em } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\quad \text{sobre } \partial\Omega, \end{cases} \tag{2.3}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; $\lambda < \lambda_1$, λ_1 é o primeiro autovalor do operador $-\Delta$ em $H_0^1(\Omega)$; p_1, $p_2 > 0$ e $b \in L^\sigma(\Omega)$, com $\sigma > N/2$ se $N \geq 3$ e $\sigma > 1$ se $N = 1, 2$, satisfazendo

$$\int_\Omega b(x)\varphi_1^{p_1+1}|\nabla\varphi_1|^{p_2}dx > 0. \tag{2.4}$$

Proposição 2.6. *Suponha que b satisfaz* (2.4)*, com* p_1*,* $p_2 \geq 1$*, então existe* $\underline{\lambda}$ *tal que o Problema* (2.3) *possui uma solução positiva para todo* $\underline{\lambda} < \lambda < \lambda_1$.

Motivados por Landesman e Lazer [35] e Shaw [46], apresentamos outra aplicação do Teorema 2.1. Mais precisamente, estamos interessados na solubilidade do seguinte problema:

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u) + \Gamma(x, u, \nabla u)) \text{ em } \Omega, \\ u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{sobre } \partial\Omega, \end{cases} \tag{2.5}$$

onde Ω é um domínio suave limitado de $\mathbb{R}^N$, $N \geq 1$; λ e μ são como em (2.1) e $f \in L^\sigma(\Omega)$, com $\sigma > N$ se $N \geq 3$ e $\sigma > 2$ se $N = 1, 2$. Para garantir a existência de uma solução para o Problema (2.5), via o Teorema 2.1, consideramos que $g : \mathbb{R} \to \mathbb{R}$ é uma função localmente Lipschitz e que $\Gamma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ é uma função localmente Lipschitz em relação à segunda e terceira variáveis satisfazendo

(g_1) existe $M > 0$ tal que

$$g(s) \geq -M \text{ se } s \leq 0 \text{ e } g(s) \leq M \text{ se } s \geq 0;$$

e

(Γ_1) existe $\alpha > 0$ tal que, para todo $x \in \Omega$ e $\xi \in \mathbb{R}$,

$$\Gamma(x, s, \xi) \geq -\alpha \text{ se } s \leq 0 \text{ e } \Gamma(x, s, \xi) \leq \alpha \text{ se } s \geq 0.$$

Além disso, denotando por $g_i^- := \liminf_{s\to-\infty} g(s)$ e $g_s^+ := \limsup_{s\to+\infty} g(s)$, assumimos que

(LL_∇)
$$\int_\Omega (f + g_i^- - \alpha)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+ + \alpha)\varphi_1 dx.$$

Proposição 2.7. *Suponha que* (g_1), (Γ_1) *e* (LL_∇) *são satisfeitas. Então, existem constantes positivas* μ^* *e* ν^* *tais que, para todos* $\mu \in (0, \mu^*)$ *e* $|\lambda - \lambda_1| < \mu\nu^*$, *o Problema* (2.5) *possui uma solução* $u_\mu = t\varphi_1 + v$ *de classe* $C^{1,\gamma}(\overline{\Omega})$, *com* $t \in \mathbb{R}$ *e* $v \in \langle\varphi_1\rangle^\perp$.

Ressaltamos que, como uma aplicação da Proposição 2.2, fornecemos um resultado, veja Proposição 2.20, sobre a existência de múltiplas soluções para o Problema (2.1) quando h é dada por

$$h(x, t, \xi) = \sum_{i,j=0}^{m} \alpha_{ij}(x) t^i |\xi|^j, \text{ onde } \alpha_{ij} \in L^\sigma(\Omega), \text{ com } \sigma > N \text{ se } N \geq 3 \text{ e } \sigma > 2 \text{ se } N = 1, 2.$$

As equações elípticas com não-linearidade dependendo do gradiente desempenham um papel importante nas equações diferenciais parciais. Na literatura, há muitos artigos relacionados a este tópico, veja, por exemplo, [6, 8, 9, 12, 13, 17, 29, 33, 42, 25, 44, 45, 49, 50]. Nestes trabalhos, os autores usam diferentes métodos para estudar esse tipo de problemas, como o grau topológico, Teoremas do ponto fixo e métodos de sub e supersolução.

Em [25], De Figueiredo, Girardi e Matzeu desenvolveram um novo método para estudar problemas com dependência do gradiente usando a teoria do minimax. Mais especificamente, os autores consideraram a solubilidade do problema:

$$\begin{cases} -\Delta u = & f(x,u,\nabla u) \text{ em } \Omega, \\ u = & 0 \qquad \text{sobre } \partial\Omega. \end{cases} \tag{2.6}$$

Para aplicar o método minimax para o Problema (2.6), que não é variacional devido à presença do termo gradiente, em [25] os autores associaram a (2.6) uma família de problemas variacionais, fixando o termo ∇u, em seguida, aplicando o Teorema do Passo da Montanha, os autores obtiveram uma família de soluções para uma família de problemas elípticos semilineares que não depende do gradiente da solução. Posteriormente, estabelecendo estimativas sobre as normas dessas soluções e utilizando uma técnica iterativa, De Figueiredo, Girardi e Matzeu, estabeleceram a existência de uma solução não trivial para o Problema (2.6). Veja [28, 31, 32, 37] para o uso desta técnica em outras classes de problemas.

Como em (2.1) a não-linearidade depende do gradiente da solução, a técnica de congelamento do termo gradiente introduzida por De Figueiredo, Girardi e Matzeu, combinada com o método utilizado no Capítulo 1 nos permite estabelecer a existência de solução para o Problema (2.1). Mais especificamente, para provar o Teorema 2.1, primeiro assumimos que h é limitado a com respeito a $L^\sigma(\Omega)$, i.e., existe $f \in L^\sigma(\Omega)$ tal que $|h(x,s,\xi)| \leq f(x)$, para todos $s \in \mathbb{R}$, $\xi \in \mathbb{R}^{\mathbb{N}}$, para quase todo $x \in \overline{\Omega}$. Em seguida, considerando uma função truncamento adequada de h e usando a solubilidade de (2.1), com h limitada em relação a $L^\sigma(\Omega)$, garantimos a existência de uma solução $u_\mu = t\varphi_1 + v$ de classe $C^{1,\gamma}(\overline{\Omega})$. Posteriormente, via um argumento bootstrap, provamos que $\|v\|_{C^1(\overline{\Omega})} \to 0$, quando $\mu \to 0$, isto permite encontrar $\mu^* > 0$ tal que u_μ é solução do Problema (2.1), para todo $\mu \in (0,\mu^*)$ e $|\lambda - \lambda_1| < \mu\nu^*$.

Introduction

In this work we use variational methods, the Lyapunov-Schmidt Reduction Method and the term gradient freeze technique to study existence, multiplicity and non existence of solutions for nonlinear elliptic problems depending on a parameter under Landesman-Lazer type hypotheses. Our main objective is to consider problems in which there is no global growth restriction on the nonlinear term and settings where this term may change sign.

In Chapter 1 of this work we consider the existence and non existence of weak solutions for the following problem

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x,u) \text{ in } \Omega, \\ u = & 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\mu \neq 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

(h_0) for every $A > 0$, there exists $f_A \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, such that

$$|h(x,s)| \leq f_A(x), \text{ for every } |s| \leq A, \quad \text{for almost every } x \in \overline{\Omega};$$

and

(h_1) given $A_1 > 0$ there exists $\zeta := \zeta(A_1) \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, such that

$$|h(z,s_1) - h(z,s_2)| \leq \zeta(z)|s_1 - s_2|, \text{ for every } z \in \overline{\Omega}, \ |s_1|, \ |s_2| \leq A_1.$$

Our main objective is to establish the existence of multiple solutions for Problem (1.1). For this result, we consider that

(h_2) there exist $k \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$ such that

$$\left[\int_\Omega h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1)\varphi_1 dx\right] < 0,$$

where φ_1 is a positive eigenfunction associated to λ_1.

In this case our main result is:

Theorem 1.1. *Suppose h satisfies (h_0), (h_1) and (h_2). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{0,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.*

It is important to note that in Theorem 1.1 the nonlinearity has no global growth restriction and the projections of the solutions u_i on the direction of φ_1 are located between $t_i\varphi_1$ and $t_{i+1}\varphi_1$.

Theorem 1.1 is a direct consequence of two results that guarantee the existence of a solution for Problem (1.1) under one of the following particular case of (h_2):

(h_2^+) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t_1\varphi_1)\varphi_1 dx > 0 > \int_\Omega h(x, t_2\varphi_1)\varphi_1 dx,$$

or

(h_2^-) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t_1\varphi_1)\varphi_1 dx < 0 < \int_\Omega h(x, t_2\varphi_1)\varphi_1 dx.$$

In our first result on existence, we use variational methods to establish the existence of a weak solution for Problem (1.1), considering the hypothesis (h_2^+) and less regularity on the function h. More precisely we have:

Theorem 1.2. *Suppose h satisfies (h_0) and (h_2^+). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

In order to establish the existence of solutions under the hypotheses (h_2^-) we need to suppose more regularity on the function h.

Theorem 1.3. *Suppose h satisfies (h_0), (h_1) and (h_2^-). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.1) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in (t_1, t_2)$ and $v \in \langle \varphi_1 \rangle^\perp$.*

Remark 1.4. *If we assume in (h_0) and (h_1) that $\sigma > N$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, the solution u_μ of Problem* (1.1)*, given in Theorem* 1.2 *or* 1.3*, is of class $C^{1,\gamma}(\overline{\Omega})$. Using this fact we may prove that u_μ is positive or negative in Ω provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively. Moreover, for $|\mu| > 0$ sufficiently small the solutions of Theorem* 1.1 *are ordered, see Theorems* 1.20 *and* 1.21*.*

We observe that the hypothesis (h_2) is a Landesman-Lazer type condition, see [35]. Indeed, in the case where nonlinearity h, given in (1.1), is of the form $h(x, s) = f(x) - g(s)$, where $f \in L^\sigma(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, with finite limits $g^- := \lim_{s\to-\infty} g(s)$ and $g^+ := \lim_{s\to\infty} g(s)$, the Landesman-Lazer Condition

$$g^\mp \int_\Omega \varphi_1 dx < \int_\Omega f\varphi_1 dx < g^\pm \int_\Omega \varphi_1 dx$$

assures us that

$$\lim_{t\to-\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \int_\Omega (f - g^-)\varphi_1 dx > (<)0$$

and

$$\lim_{t\to+\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \int_\Omega (f - g^+)\varphi_1 dx < (>)0.$$

Consequently, there exist real numbers t_1 and t_2, with $t_1 < 0 < t_2$, such that the condition (h_2^+) (or (h_2^-)) is valid for t_1 and t_2. In other words, when $h(x, s) = f(x) - g(s)$, where f and g are as above, we obtain that the Landesman-Lazer Condition implies (h_2^+) (or (h_2^-)).

It is important to note that under our hypotheses h may change sign in Ω, this characterizes the Problem (1.1) as indefinite. This class of problems has been the object of an intense research in the last three decades since the works of Alama and Tarantello [2], Berestycki, Capuzzo and Nirenberg [11] and Ouyang [41]. See [3, 4, 5, 21, 23, 26, 27, 38, 47] and its references for more details on this type of problems.

The proofs of Theorems 1.2 and 1.3 are inspired by the Lyapunov-Schmidt Reduction Method, as presented in the articles [19, 20, 22, 36]. However, under the hypotheses of those theorems, that method can not be applied directly since we do not impose any global growth restriction on the term nonlinear h. To overcome this difficulty we combine the Lyapunov-Schmidt Reduction Method with truncation and approximation arguments via bootstrap methods.

It is worthwhile mentioning that in several situations the conditions (h_2^+) or (h_2^-) are sufficient to the existence of a solution $u_\mu = t\varphi_1 + v$, $t \in (t_1, t_2)$, $v \in \langle\varphi_1\rangle^\perp$. Next we establish the non existence of solutions to the Problem (1.1) when the hypothesis (h_2) is not valid. Indeed, if we suppose

(h_3) there exists $f \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, such that

$$|h(x,s)| \leq f(x)(1+|s|), \text{ for every } s \in \mathbb{R}, \quad \text{for almost every } x \in \overline{\Omega};$$

and

(h_4) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t\varphi_1)\varphi_1 dx \neq 0, \quad \text{for every } t \in [t_1, t_2],$$

we may state the following non existence result.

Theorem 1.5. *Suppose h satisfies (h_3) and (h_4). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$.*

It is important to note that Theorem 1.5 is a consequence of a more geral result that establishes that, under the hypothesis (h_4), Problem (1.1) has no bounded solutions in $L^\infty(\Omega)$.

As an application of Theorems 1.2 and 1.3 we consider the existence of a solution to the following problem

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x) u^q + b_2(x) u^p \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\beta > 0$ is a real parameter; $p > q$, with $p \neq 1$, and b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$. We also set:

$$r_1 := \int_\Omega b_1 \varphi_1^{q+1} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p+1} dx.$$

Considering the nomenclature for elliptic problems used in the literature, Problem (1.2) is superlinear or sublinear at infinity if $p > 1$ or $0 < p < 1$ and it is superlinear, linear or sublinear at the origin if $q > 1$, $q = 1$ or $0 < q < 1$. For the case linear or superlinear at the origin and superlinear at infinity we can establish the following result.

Proposition 1.6. *Suppose $p > q \geq 1$ and $r_1 r_2 < 0$. Then there exist positive constants β^* and ν^* such that Problem (1.2) has a solution of class $C^{0,\gamma}(\overline{\Omega})$, for every $\beta \in (0, \beta^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu^*$.*

For the case sublinear at the origin and superlinear at infinity we may applyging the Theorem 1.2.

Proposition 1.7. *Suppose $r_1 > 0 > r_2$. Then*

(*i*) *if $0 < q < 1 < p$, there exist positive constants β_1^* and ν_1^* such that Problem (1.2) has a solution of class $C^{0,\gamma}(\overline{\Omega})$, for every $\beta \in (0, \beta_1^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_1^*$.*

(*ii*) *if $0 < q < p < 1$, there exist positive constants β_2^* and ν_2^* such that Problem (1.2) has a solution of class $C^{0,\gamma}(\overline{\Omega})$, for every $\beta \in (\beta_2^*, \infty)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_2^*$.*

When in (1.2) we consider $b_1, b_2 \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, the solution given by Proposition 1.6 or 1.7 is of class $C^{1,\gamma}(\overline{\Omega})$ and it is positive in Ω, see Remark 1.27. Furthermore, as noted in the beginning of this introduction, the weight functions b_1 and b_2 may change signal. In this case, Problem (1.2) is indefinite. We emphasize that in Proposition 1.6 and item (*i*) of Proposition 1.7, we do not assume the restriction $p < (N + 2)/(N - 2)$ to ensure the existence of a solution for (1.2).

For the case linear or superlinear at the origin and superlinear at infinity we refer the reader to the papers by Ouyang [41], Alama and Tarantello [3] and Medeiros, Severo and Silva [38]. For the case sublinear at the origin see the papers by Ambrozetti, Brezis and Cerami [7] and De Figueiredo, Gossez and Ubilla [26].

Inspired by the paper of Brezis and Nirenberg, see [15], we consider an application of Proposition 1.6 to following problem:

$$\begin{cases} -\Delta u = & \lambda u + b(x) u^p \text{ in } \Omega, \\ u = & 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda < \lambda_1$, λ_1 is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $p > 0$ and $b \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, satisfies

$$\int_\Omega b(x) \varphi_1^{p+1} dx > 0. \tag{1.4}$$

Proposition 1.8. *Suppose b satisfies* (1.4)*, with $p > 0$ and $p \neq 1$, then there exists $\underline{\lambda}$ such that Problem* (1.3) *has a positive solution for every $\underline{\lambda} < \lambda < \lambda_1$.*

Motivated by Landesman and Lazer, see [35], we also present another application of Theorem 1.2 or 1.3. Namely, we are interested in the solubility of the following problem:

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u)) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; λ and μ are as in (1.1) and $f \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$.

To ensure the existence of a solution for Problem (1.5) we consider

(g_1) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists $M > 0$ such that

$$g(s) \geq -M \text{ if } s \leq 0 \text{ and } g(s) \leq M \text{ if } s \geq 0.$$

Denoting by $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g_s^+ := \limsup_{s\to+\infty} g(s)$, we assume:

(LL^+)
$$\int_\Omega (f + g_i^-)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+)\varphi_1 dx.$$

Proposition 1.9. *Suppose g satisfies* (g_1) *and* (LL^+)*. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.5) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in \mathbb{R}$ and $v \in \langle\varphi_1\rangle^\perp$.*

Denoting $g_s^- := \limsup_{s\to-\infty} g(s)$ and $g_i^+ := \liminf_{s\to+\infty} g(s)$ we provide an analogous result when

($\hat{g}_1$) $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and there exists $M > 0$ such that

$$g(s) \leq M \text{ if } s \leq 0 \text{ and } g(s) \geq -M \text{ if } s \geq 0,$$

and

(LL^-)
$$\int_\Omega (f + g_s^-)\varphi_1 dx < 0 < \int_\Omega (f + g_i^+)\varphi_1 dx.$$

Proposition 1.10. *Suppose g satisfies* ($\hat{g}_1$) *and* (LL^-)*. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.5) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in \mathbb{R}$ and $v \in \langle\varphi_1\rangle^\perp$.*

Remark 1.11. *The above results allow us to consider g such that $g_i^- = +\infty$ and $g_s^+ = -\infty$ or $g_s^- = -\infty$ and $g_i^+ = +\infty$, respectively. Moreover g may have unbounded oscillatory behavior, which does not occur, for example, in [35].*

We point out that an application of Theorem 1.1 is given when h is a polynomial function in the second variable. See Proposition 1.28.

As mentioned previously we are not suppose that h has a global growth restriction, which prevents the direct application of variational methods, since the associated functional is not well defined. To prove Theorem 1.2, firstly we assume that h is bounded with respect to $L^\sigma(\Omega)$.

In this case, using a minimization argument, we prove that there exist a positive constant ν^* such that, for $\mu > 0$ sufficiently small and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$. Next, considering a suitable truncation function of h and using the solubility of (1.1), with h bounded with respect to $L^\sigma(\Omega)$ we guarantee the existence of a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$. Subsequently, by a bootstrap argument, we prove that $\|v\|_\infty \to 0$, when $\mu \to 0$, this allows us to find $\mu^* > 0$ such that u_μ is a solution of Problem (1.1), for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$.

We call the reader's attention to the fact that, under the condition (h_2^-), we do not apply the minimization method used in the proof of Theorem 1.2. The solution, in this case, is a minimax critical point. To prove Theorem 1.3, firstly we assume that h is bounded with respect to $L^\sigma(\Omega)$ and Lipschitz with respect to $L^\sigma(\Omega)$ in the second variable.

Under these hypotheses, we can apply Lyapunov-Schmidt Reduction Method to find a critical point for the functional associated with the truncated problem. Next we argunent as in the proof of Theorem 1.2 to show that there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$, is a solution of Problem (1.1).

In Chapter 2 we consider the existence and non existence of weak solutions for the following problem:

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x, u, \nabla u) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\quad \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\mu \neq 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$(h_\nabla)_0$ for every $A > 0$, there exists $f_A \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if

$N = 1, 2$, such that

$$|h(x, s, \xi)| \leq f_A(x), \text{ for every } |s| \leq A, \ |\xi| \leq A, \text{ for almost every } x \in \overline{\Omega};$$

and

$(h_\nabla)_1$ for every A_1, $A_2 > 0$, there exist $\zeta_1 = \zeta_1(A_1)$, $\zeta_2 = \zeta_2(A_2) \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$, such that

$$|h(z, s_1, \xi) - h(z, s_2, \xi)| \leq \zeta_1(z)|s_1 - s_2|, \text{ for every } z \in \overline{\Omega}, \ |s_1|, |s_1| \leq A_1, \ |\xi| \leq A_2$$

and

$$|h(z, s, \xi_1) - h(z, s, \xi_2)| \leq \zeta_2(z)|\xi_1 - \xi_2|, \text{ for every } z \in \overline{\Omega}, \ |s| \leq A_1, \ |\xi_1|, |\xi_2| \leq A_2.$$

Our main result establishes the existence of a solution for Problem (2.1). For this result we assume

$(h_\nabla)_2$ there exist t_1 and $t_2 \in \mathbb{R}$, $t_1 < t_2$, such that

$$\left[\int_\Omega h(x, t_1\varphi_1, t_1\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_2\varphi_1, t_2\nabla\varphi_1)\varphi_1 dx\right] < 0,$$

where φ_1 is a positive eigenfunction associated to λ_1.

Theorem 2.1. *Suppose h satisfies $(h_\nabla)_0$, $(h_\nabla)_1$ and $(h_\nabla)_2$. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (2.1) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{1,\gamma}(\overline{\Omega})$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

It is important to note that in Theorem 2.1 we do not impose any global growth restriction on the nonlinear term with respect to the second and third variable. We also observe that the projection of the solution u_μ on the direction of φ_1 is located between $t_1\varphi_1$ and $t_2\varphi_1$.

As a direct consequence of Theorem 2.1 we derive the existence of multiple solutions for Problem (2.1). Indeed, assuming

$(\hat{h}_\nabla)_2$ there exist $k \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$, such that

$$\left[\int_\Omega h(x, t_i\varphi_1, t_i\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1, t_{i+1}\nabla\varphi_1)\varphi_1 dx\right] < 0,$$

we may state

Proposition 2.2. *Suppose h satisfies $(h_\nabla)_0$, $(h_\nabla)_1$ and $(\hat{h}_\nabla)_2$. Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (2.1) *has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{1,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.*

Remark 2.3. *The solution u_μ, given by Theorem* 2.1, *is positive or negative in Ω provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively. Moreover, for $|\mu| > 0$ sufficiently small the solutions of Proposition* 2.2 *are ordered, see Theorems* 2.17 *and Proposition* 2.18.

We observe that hypothesis $(h_\nabla)_2$ is a Landesman-Lazer type condition, see [35]. In [46] Shaw, see too [18, 40], considered the case where nonlinearity h is of the form $h(x, s, \xi) = f(x) + g(s) + \Gamma(x, s, \xi)$, where $f \in L^\sigma(\Omega)$, $g : \mathbb{R} \to \mathbb{R}$ is bounded continuous function, with finite limits $g^- := \lim_{s\to-\infty} g(s)$ and $g^+ := \lim_{s\to\infty} g(s)$, and $\Gamma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is bounded continuous function. Shaw has proved the existence of a solution for Problem (2.1) by assuming the following verssion of the Landesman-Lazer condition:

$$g^- \int_\Omega \varphi_1 dx + \int_\Omega f\varphi_1 dx + \alpha \int_\Omega \varphi_1 dx < 0 < g^+ \int_\Omega \varphi_1 dx + \int_\Omega f\varphi_1 dx - \alpha \int_\Omega \varphi_1 dx,$$

where $\alpha = \sup\{|\Gamma|\}$. Consequently there exist real numbers t_1 and t_2, with $t_1 < 0 < t_2$, such that the condition $(h_\nabla)_2$ is valid for t_1 and t_2. In other words the version of the Landesman-Lazer condition used by Shaw [46] implies the hypothesis $(h_\nabla)_2$.

It is worthwhile mentioning that in several situations the condition $(h_\nabla)_2$ is necessary to the existence of a solution $u_\mu = t\varphi_1 + v$, $t \in (t_1, t_2)$, $v \in \langle\varphi_1\rangle^\perp$. Next we establish the non existence of solutions to the Problem (2.1) when the hypothesis $(h_\nabla)_2$ is not valid. Indeed, if h satisfies

$(h_\nabla)_3$ there exists $f \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$, such that

$$|h(x, s, \xi)| \leq f(x)(1 + |s| + |\xi|), \text{ for every } s \in \mathbb{R},\ \xi \in \mathbb{R}^N, \text{ for almost every } x \in \overline{\Omega},$$

and

$(h_\nabla)_4$ there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \neq 0, \quad \text{for every } t \in [t_1, t_2],$$

we may state the following non existence result.

Theorem 2.4. *Suppose h satisfies $(h_\nabla)_3$ and $(h_\nabla)_4$. Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem* (2.1) *has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$.*

As a first application of Theorem 2.1 we consider the existence of a solution for the following problem

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x) u^{q_1} |\nabla u|^{q_2} + b_2(x) u^{p_1} |\nabla u|^{p_2} \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\beta > 0$ is a real parameter and b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$.

Setting

$$r_1 := \int_\Omega b_1 \varphi_1^{q_1+1} |\nabla \varphi_1|^{q_2} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p_1+1} |\nabla \varphi_1|^{p_2} dx,$$

we state the following result:

Proposition 2.5. *Suppose $p = p_1 + p_2$, $q = q_1 + q_2$, p_1, p_2, q_1, $q_2 \geq 1$, $p > q$ and $r_1 r_2 < 0$. Then there exist positive constants β_1^* and ν_1^* such that Problem* (2.2) *has a solution of class $C^{1,\gamma}(\overline{\Omega})$, for every $\beta \in (0, \beta_1^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_1^*$.*

Another application of Theorem 2.1 is given by the following problem:

$$\begin{cases} -\Delta u = & \lambda u + b(x) u^{p_1} |\nabla u|^{p_2} \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda < \lambda_1$, λ_1 is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; p_1, $p_2 > 0$ and $b \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, satisfying

$$\int_\Omega b(x) \varphi_1^{p_1+1} |\nabla \varphi_1|^{p_2} dx > 0. \tag{2.4}$$

Proposition 2.6. *Suppose b satisfies* (2.4)*, with p_1, $p_2 \geq 1$, then there exists $\underline{\lambda}$ such that Problem* (2.3) *has a positive solution for every $\underline{\lambda} < \lambda < \lambda_1$.*

Motivated by Landesman and Lazer [35] and Shaw [46], we also present another application of Theorem 2.1. Namely, we are interested in the solubility of the following problem:

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u) + \Gamma(x, u, \nabla u)) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; λ and μ are as in (2.1) and $f \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$. To ensure the existence of a solution for Problem (2.5) we consider $g : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function and $\Gamma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a locally Lipschitz function with respect to the second and third variable satisfiying

(g_1) there exists $M > 0$ such that

$$g(s) \geq -M \text{ if } s \leq 0 \text{ and } g(s) \leq M \text{ if } s \geq 0;$$

and

(Γ_1) there exists $\alpha > 0$ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}$,

$$\Gamma(x, s, \xi) \geq -\alpha \text{ if } s \leq 0 \text{ and } \Gamma(x, s, \xi) \leq \alpha \text{ if } s \geq 0.$$

Denoting by $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g_s^+ := \limsup_{s\to+\infty} g(s)$, we assume that

$$(LL_\nabla) \qquad \int_\Omega (f + g_i^- - \alpha)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+ + \alpha)\varphi_1 dx.$$

Proposition 2.7. *Suppose (g_1), (Γ_1) and (LL_∇) are satisfied. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (2.5) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{1,\gamma}(\overline{\Omega})$, with $t \in \mathbb{R}$ and $v \in \langle\varphi_1\rangle^\perp$.*

We point out that, as an application of Proposition 2.2, we provide a result, see Proposition 2.20, on the existence of multiple solution for Problem (2.1) when h is given by

$$h(x, t, \xi) = \sum_{i,j=0}^{m} \alpha_{ij}(x) t^i |\xi|^j, \text{ where } \alpha_{ij} \in L^\sigma(\Omega), \text{ with } \sigma > N \text{ if } N \geq 3 \text{ and } \sigma > 2 \text{ if } N = 1, 2.$$

Elliptic equations with nonlinearity depending on the gradient play an important role in partial differential equations. In the literature, there are many papers related to this topic, See, for example, [6, 8, 9, 12, 13, 17, 29, 33, 42, 25, 44, 45, 49, 50]. In these works, the authors use different methods to study this type of problems such as Topological Degree, Fixed Point Theorems and Sub and Supersolution Method.

In [25], De Figueiredo, Girardi and Matzeu developed a new method to study problems with gradient dependency using minimax theory. More specifically, the authors considered the solubility of the problem

$$\begin{cases} -\Delta u = & f(x,u,\nabla u) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

In order to apply the minimax method for Problem (2.6), which is not variational due to the presence of the term gradient, in [25] the authors associated with (2.6) a family of variational problems, fixing the term ∇u, then, applying Mountain Pass Theorem, they obtained a family of solutions for the family of semilinear elliptic problems that does not depend on the gradient of the solution. Subsequently, establishing appropriate estimates on the norms of these solutions and using an iterative technique, De Figueiredo, Girardi and Matzeu, established the existence of a non-trivial solution for Problem (2.6), see [28, 31, 32, 37] for the use of this technique in other classes of problems.

As in (2.1) the nonlinearity depends on the gradient of the solution, the term gradient freeze technique introduced by De Figueiredo, Girardi and Matzeu, combined with the method used in Chapter 1 allows us to establish the existence of solution for Problem (2.1). More specifically, in order to prove Theorem 2.1, first we assume that h is bounded with respect to $L^{\sigma}(\Omega)$. Next, considering a suitable truncation function of h and using the solubility of (2.1), with h bounded with respect to $L^{\sigma}(\Omega)$ we guarantee the existence of a solution $u_{\mu} = t\varphi_1 + v$ of class $C^{1,\gamma}(\overline{\Omega})$. Posteriorly, by a bootstrap argument, we prove that $\|v\|_{C^1(\overline{\Omega})} \to 0$, when $\mu \to 0$, this allows us to find $\mu^* > 0$ such that u_{μ} is solution of Problem (2.1), for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$.

Chapter 1

A Landesman-Lazer local condition for semilinear elliptic problem depending on a parameter

1.1 Main results

In this chapter we are interested in the existence and non existence of weak solutions for the following problem

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x,u) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\mu > 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(h_0) for every $A > 0$, there exists $f_A \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, such that

$$|h(x,t)| \leq f_A(x), \text{ for every } |t| \leq A, \text{ for almost every } x \in \overline{\Omega};$$

and

(h_1) given $A_1 > 0$ there exists $\zeta := \zeta(A_1) \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if

$N = 1, 2$, such that

$$|h(z, s_1) - h(z, s_2)| \leq \zeta(z)|s_1 - s_2|, \text{ for every } z \in \overline{\Omega}, \ |s_1|, \ |s_2| \leq A_1.$$

Our main result establishes multiplicity of solutions for Problem (1.1) under a local Landesman-Lazer condition. More specifically, supposing

(h_2) there exist $k \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$ such that

$$\left[\int_{\Omega} h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_{\Omega} h(x, t_{i+1}\varphi_1)\varphi_1 dx\right] < 0,$$

where φ_1 is a positive eigenfunction associated to λ_1, we establish

Theorem 1.1. *Suppose h satisfies (h_0), (h_1) and (h_2). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{0,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^{\perp}$, $i = 1, \cdots, k$.*

Theorem 1.1 is a direct consequence of two results that guarantee the existence of a solution for Problem (1.1) under one of the following particular case of (h_2):

(h_2^+) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_{\Omega} h(x, t_1\varphi_1)\varphi_1 dx > 0 > \int_{\Omega} h(x, t_2\varphi_1)\varphi_1 dx,$$

or

(h_2^-) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_{\Omega} h(x, t_1\varphi_1)\varphi_1 dx < 0 < \int_{\Omega} h(x, t_2\varphi_1)\varphi_1 dx.$$

In our first result on existence we use variational methods to establish the existence of a weak solution for Problem (1.1) considering the hypothesis (h_2^+) and less regularity in the function h. More precisely we have:

Theorem 1.2. *Suppose h satisfies (h_0) and (h_2^+). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^{\perp}$.*

In order to establish the existence of solutions under the hypotheses (h_2^-) we need to suppose more regularity on the function h.

Theorem 1.3. *Suppose h satisfies (h_0), (h_1) and (h_2^-). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.1) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

Remark 1.4. *If we assume in (h_0) and (h_1) that $\sigma > N$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, the solution u_μ of Problem* (1.1)*, given in Theorem* 1.2 *or* 1.3*, is of class $C^{1,\gamma}(\overline{\Omega})$. Using this fact we may prove that u_μ is positive or negative in Ω provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively. Moreover, for $|\mu| > 0$ sufficiently small the solutions of Theorem* 1.1 *are ordered, see Theorems* 1.20 *and* 1.21.

It is worthwhile mentioning that in several situations the conditions (h_2^+) or (h_2^-) are sufficient to the existence of a solution $u_\mu = t\varphi_1 + v$, $t \in (t_1, t_2)$, $v \in \langle\varphi_1\rangle^\perp$. Next we establish the non existence of solutions to the Problem (1.1) when the hypothesis (h_2) is not valid. Indeed, if we suppose h satisfiying

(h_3) there exists $f \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, such that

$$|h(x,t)| \leq f(x)(1 + |t|), \text{ for every } t \in \mathbb{R}, \text{ for almost every } x \in \overline{\Omega};$$

and

(h_4) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t\varphi_1)\varphi_1 dx \neq 0, \quad \text{for every } t \in [t_1, t_2],$$

we may state the following non existence result.

Theorem 1.5. *Suppose h satisfies (h_3) and (h_4). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem* (1.1) *has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$.*

As an application of Theorems 1.2 and 1.3 we consider the existence of a solution to the following problem

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x)u^q + b_2(x)u^p \quad \text{in } \Omega, \\ \quad u = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\beta > 0$ is a real parameter; $p > q$ and b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ e $\sigma > 1$ if $N = 1, 2$.

Setting

$$r_1 := \int_\Omega b_1 \varphi_1^{q+1} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p+1} dx,$$

we establish the following results:

Proposition 1.6. *Suppose $p > q \geq 1$ and $r_1 r_2 < 0$. Then there exist positive constants β^* and ν^* such that Problem* (1.2) *has a solution of class $C^{0,\gamma}(\overline{\Omega})$, for every $\beta \in (0, \beta^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu^*$.*

Proposition 1.7. *Suppose $r_1 > 0 > r_2$. Then*

(*i*) *if $p > 1 > q > 0$, there exist positive constants β_1^* and ν_1^* such that Problem* (1.2) *has a solution of class $C^{0,\gamma}(\overline{\Omega})$, for every $\beta \in (0, \beta_1^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_1^*$.*

(*ii*) *if $0 < q < p < 1$, there exist positive constants β_2^* and ν_2^* such that Problem* (1.2) *has a solution of class $C^{0,\gamma}(\overline{\Omega})$, for every $\beta \in (\beta_2^*, \infty)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_2^*$.*

When in (1.2) we consider $b_1, b_2 \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, the solution given by Proposition 1.6 or 1.7 is of class $C^{1,\gamma}(\overline{\Omega})$ and it is positive in Ω, see Remark 1.27.

Inspired by the paper of Brezis and Nirenberg, see [15], we can give an application of Proposition 1.6. More specifically, we consider the following problem

$$\begin{cases} -\Delta u = & \lambda u + b(x) u^p \text{ in } \Omega, \\ u = & 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda < \lambda_1$, λ_1 is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $p > 0$ and $b \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ e $\sigma > 1$ if $N = 1, 2$, satisfying

$$\int_\Omega b(x) \varphi_1^{p+1} dx > 0. \tag{1.4}$$

Proposition 1.8. *Suppose b satisfies* (1.4)*, with $p > 0$ and $p \neq 1$. Then there exists $\underline{\lambda}$ such that Problem* (1.3) *has a positive solution for every $\underline{\lambda} < \lambda < \lambda_1$.*

Motivated by Landesman and Lazer, see [35], we also present another application of Theorems 1.2 or 1.3. Namely, we are interested in the solubility of the following problem:

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u)) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; λ and μ are as in (1.1) and $f \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ e $\sigma > 1$ if $N = 1, 2$.

To ensure the existence of a solution for Problem (1.5), by Theorem 1.2, we consider

(g_1) $g : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying there exists $M > 0$ such that

$$g(s) \geq -M \text{ if } s \leq 0 \text{ and } g(s) \leq M \text{ if } s \geq 0.$$

Still, denoting by $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g_s^+ := \limsup_{s\to+\infty} g(s)$, we assume that

$$(LL^+) \qquad \int_\Omega (f + g_i^-)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+)\varphi_1 dx.$$

Proposition 1.9. *Suppose g satisfies (g_1) and (LL^+). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.5) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in \mathbb{R}$ and $v \in \langle\varphi_1\rangle^\perp$.*

Denoting $g_s^- := \limsup_{s\to-\infty} g(s)$ and $g_i^+ := \liminf_{s\to+\infty} g(s)$, by Theorem 1.3, we establish an analogous result when

$(\hat{g}_1)$ $g : \mathbb{R} \longrightarrow \mathbb{R}$ is locally Lipschitz function satisfying there exists $M > 0$ such taht

$$g(s) \leq M \text{ if } s \leq 0 \text{ and } g(s) \geq -M \text{ if } s \geq 0.$$

and

$$(LL^-) \qquad \int_\Omega (f + g_s^-)\varphi_1 dx < 0 < \int_\Omega (f + g_i^+)\varphi_1 dx.$$

Proposition 1.10. *Suppose g satisfies $(\hat{g}_1)$ and (LL^-). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.5) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{0,\gamma}(\overline{\Omega})$, with $t \in \mathbb{R}$ and $v \in \langle\varphi_1\rangle^\perp$.*

Remark 1.11. *The above results allow us to consider g such that $g_i^- = +\infty$ and $g_s^+ = -\infty$ or $g_s^- = -\infty$ and $g_i^+ = +\infty$, respectively. Moreover g may have unbounded oscillatory behavior, which does not occur in [35].*

We point out that an application of Theorem 1.1 is given when h is a polynomial function in the second variable. See Proposition 1.28.

This chapter is organized as follows: in the Section 1.2, we show the existence of a solution for Problem (1.1) with h Carathéodory, bounded with respect to $L^\sigma(\Omega)$ and satisfying (h_2^+). In the Section 1.3 we show that Problem (1.1) has a solution supposing that h is also Lipschitz with respect to $L^\sigma(\Omega)$ in the second variable and satisfying (h_2^-), instead of (h_2^+). The Section 1.4 is dedicated to the proof of Theorems 1.1, 1.2 and 1.3. The Section 1.5, we study the nonsolubility of Problem (1.1). Finally, in the Section 1.6 we give applications of Theorems 1.1, 1.2 and 1.3.

1.2 Proof of Theorem 1.2 for h bounded on $\mathbf{L}^\sigma(\boldsymbol{\Omega})$

In this section, we prove a version of Theorem 1.2 with h being a Carathéodory function satisfying (h_2^+) and bounded with respect to $L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$. More specifically, there exists $f \in L^\sigma(\Omega)$ such that

$$|h(x,s)| \leq f(x), \text{ for every } s \in \mathbb{R}, \text{ for almost every } x \in \overline{\Omega}. \tag{1.6}$$

In this case the associated functional $I_\mu : H_0^1(\Omega) \to \mathbb{R}$, is defined by

$$I_\mu(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \mu \int_\Omega H(x,u)dx, \tag{1.7}$$

where $H(x,t) = \displaystyle\int_0^t h(x,s)ds$.

Under the hypothesis (1.6), the functional I_μ is well defined and is of class C^1 in $H_0^1(\Omega)$. In addition, the critical points of I_μ are weak solutions of Problem (1.1).

Theorem 1.12. *Suppose h satisfies (h_2^+) and* (1.6)*. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.1) *has a solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

As mentioned in the introduction of this work, to prove Theorem 1.12 we use a minimization argument. For this purpose, firstly we show that functional I_μ has a local minimum in the orthogonal space to φ_1, for every $\tau \in [t_1, t_2]$. To this end, we use the following auxiliary lemma:

Lemma 1.13. *Given $\rho > 0$, there exist positive numbers α and μ_1 such that, for every $\tau \in [t_1, t_2]$ and for $\mu \in (0, \mu_1)$,*

$$I_\mu(\tau\varphi_1 + v) \geq \alpha + I_\mu(\tau\varphi_1), \quad \textit{for every } v \in \langle\varphi_1\rangle^\perp, \|v\| = \rho.$$

Proof. From (1.6), the Mean Value Theorem, the Hölder Inequality and the Sobolev Embedding Theorem, it follows that

$$\left| \int_\Omega [\ H(x, \tau\varphi_1 + v) - H(x, \tau\varphi_1)\ ]dx \right| \leq \int_\Omega f|v|dx \leq \|f\|_\sigma \|v\|_{\sigma'} \leq d_{\sigma'}\|f\|_\sigma\|v\|,$$

where we use the fact that

$$H_0^1(\Omega) \text{ is continuously embedded in } L^{2\sigma'}(\Omega), \text{ where } \sigma' = \frac{\sigma}{\sigma - 1}. \tag{1.8}$$

Therefore, since we are considering $\lambda < \overline{\lambda} < \lambda_2$, for every $v \in \langle \varphi_1 \rangle^\perp$, with $\|v\| = \rho$,

$$\begin{aligned} I_\mu(\tau\varphi_1 + v) - I_\mu(\tau\varphi_1) &\geq \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\|v\|^2 - \mu \int_\Omega [\ H(x, \tau\varphi_1 + v) - H(x, \tau\varphi_1)\ ]dx \\ &\geq \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\|v\|^2 - \mu d_{\sigma'}\|f\|_\sigma \|v\| = \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\rho^2 - \mu d_{\sigma'}\|f\|_\sigma \rho. \end{aligned}$$

To complete the proof of this lemma, is sufficient to choose $\alpha = (\lambda_2 - \overline{\lambda})\rho^2/4\lambda_2$ and $\mu_1 < \alpha/d_{\sigma'}\|f\|_\sigma\rho$. □

Fixing $\rho > 0$, for every $\tau \in [t_1, t_2]$, we define

$$m_\tau := \inf\{I_\mu(\tau\varphi_1 + v) : \|v\| \leq \rho, v \in \langle\varphi_1\rangle^\perp\} \tag{1.9}$$

and

$$\beta_\tau := \inf\{I_\mu(\tau\varphi_1 + v) : \|v\| = \rho, v \in \langle\varphi_1\rangle^\perp\}. \tag{1.10}$$

Lemma 1.13 and (1.10) allows us to conclude that

$$m_\tau \leq I_\mu(\tau\varphi_1) < \alpha + I_\mu(\tau\varphi_1) \leq \beta_\tau, \text{ for every } \tau \in [t_1, t_2], \text{ and } \mu \in (0, \mu_1). \tag{1.11}$$

We are ready to guarantee the existence of a local minimum in the orthogonal space to φ_1, for each $\tau \in [t_1, t_2]$.

Lemma 1.14. *Suppose $\mu \in (0, \mu_1)$. Then, for every $\tau \in [t_1, t_2]$, there exists $v_\mu^\tau \in \langle\varphi_1\rangle^\perp$, with $\|v_\mu^\tau\| < \rho$, such that $m_\tau = I_\mu(\tau\varphi_1 + v_\mu^\tau)$.*

Proof. Let $(v_n) \subset \overline{B_\rho(0)} \cap \langle\varphi_1\rangle^\perp$ such that $I_\mu(\tau\varphi_1 + v_n) \to m_\tau$. Then, taking a subsequence if necessary, there exists $v_\mu^\tau \in \overline{B_\rho(0)} \cap \langle\varphi_1\rangle^\perp$ such that $v_n \to v_\mu^\tau$ weakly in $H_0^1(\Omega)$. Considering $w_n = \tau\varphi_1 + v_n$ and $w = \tau\varphi_1 + v_\mu^\tau$ we may also suppose that

$$\begin{cases} w_n \to w \text{ weakly in } H_0^1(\Omega), \\ w_n \to w \text{ strongly in } L^2(\Omega), \\ w_n \to w \quad \text{a.e. in } \Omega, \\ |w_n| \leq \eta \in L^{\sigma'}(\Omega) \text{ a.e. in } \Omega. \end{cases} \tag{1.12}$$

From (1.6) and (1.12), we may apply the Lebesgue Dominated Convergence Theorem to get

$$\int_\Omega H(x, w_n)dx \to \int_\Omega H(x, w)dx. \tag{1.13}$$

Since $v_\mu^\tau \in \overline{B_\rho(0)} \cap \langle\varphi_1\rangle^\perp$, from (1.9), (1.11), (1.12) and (1.13), it follows that

$$\begin{aligned}\beta_\tau &> m_\tau = \lim_{n\to\infty} I_\mu(w_n) = \lim_{n\to\infty}\left\{\frac{1}{2}\|w_n\|^2 - \frac{\lambda}{2}\|w_n\|_2^2 - \mu\int_\Omega H(x,w_n)dx\right\}\\ &\geq \frac{1}{2}\|w\|^2 - \frac{\lambda}{2}\|w\|_2^2 - \mu\int_\Omega H(x,w)dx = I_\mu(w) = I_\mu(\tau\varphi_1 + v_\mu^\tau) \geq m_\tau.\end{aligned}$$

This implies that $I_\mu(\tau\varphi_1 + v_\mu^\tau) = m_\tau$, with $\|v_\mu^\tau\| < \rho$. □

We now define, for each $\tau \in [t_1, t_2]$, the set

$$S_\tau := \{v \in \langle\varphi_1\rangle^\perp : \|v\| \leq \rho,\ I_\mu(\tau\varphi_1 + v) = m_\tau\}. \tag{1.14}$$

Remark 1.15. *(i) For $\mu \in (0,\mu_1)$, it follows directly of Lemma 1.14 that the set $S_\tau \neq \emptyset$, for every $\tau \in [t_1,t_2]$. However, we can say nothing about the number of elements of S_τ.*

(ii) For every $\tau \in [t_1,t_2]$ and $\mu \in (0,\mu_1)$, Lemma 1.14 and (1.9) assure us that

$$\langle I_\mu'(\tau\varphi_1 + v), z\rangle = 0, \text{ for every } v \in S_\tau, z \in \langle\varphi_1\rangle^\perp.$$

Lemma 1.16. *Given $\delta > 0$, there exists $\mu_2 \in (0,\mu_1)$ such that, for every $\mu \in (0,\mu_2)$,*

$$\|v\| < \delta, \text{ for every } v \in S_{t_1} \cup S_{t_2}.$$

Proof. Suppose, without loss of generality, that $v \in S_{t_1}$. Using (1.6), the Hölder Inequality and (1.8), we obtain

$$\left|\int_\Omega h(x,t_1\varphi_1 + v)v dx\right| \leq \int_\Omega f|v|dx \leq \|f\|_\sigma\|v\|_{\sigma'} \leq d_{\sigma'}\|f\|_\sigma\|v\|.$$

Thus, from Remark 1.15-(ii) and the above inequality it follows that

$$\|v\|^2 = \lambda\|v\|_2^2 + \mu\int_\Omega h(x,t_1\varphi_1+v)vdx \leq \frac{\overline{\lambda}}{\lambda_2}\|v\|^2 + \mu d_{\sigma'}\|f\|_\sigma\|v\|.$$

Hence,

$$\|v\| \leq \frac{\lambda_2}{\lambda_2 - \overline{\lambda}}\mu d_{\sigma'}\|f\|_\sigma.$$

Therefore, taking $\mu_2 < \min\{\mu_1, \delta(\lambda_2 - \overline{\lambda})/\lambda_2 d_{\sigma'}\|f\|_\sigma\}$, the proof is complete. □

Now we are in conditions of proving Theorem 1.12.

Proof of Theorem 1.12. For this proof we will consider the convex set

$$C := \{\tau\varphi_1 + v : \tau \in [t_1, t_2], v \in \langle\varphi_1\rangle^{\perp}, \|v\| \leq \rho\},$$

and we denote by $m_C := \inf\{I_\mu(u) : u \in C\}$.

In order to prove Theorem 1.12 firstly we claim that there exists $u_\mu \in C$ such that $I_\mu(u_\mu) = m_C$, for every $\mu \in (0, \mu_1)$, where μ_1 is given by Lemma 1.13. Indeed, from the definition of C, there exists $M > 0$ such that $\|u\| \leq M$, for every $u \in C$. Consequently, from $\lambda < \lambda_2$, (1.6), the Hölder Inequality and (1.8), it follows that

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda_2}{2}\|u\|_2^2 - \mu\int_\Omega H(x,u)dx \geq \frac{\lambda_1 - \lambda_2}{2\lambda_1}\|u\|^2 - \mu\int_\Omega f|u|dx \\ &\geq \frac{\lambda_1 - \lambda_2}{2\lambda_1}\|u\|^2 - \mu\|f\|_\sigma\|u\|_{\sigma'} \geq -\left(\frac{\lambda_2 - \lambda_1}{2\lambda_1}\right)\|u\|^2 - \mu d_{\sigma'}\|f\|_\sigma\|u\| \\ &\geq -\left(\frac{\lambda_2 - \lambda_1}{2\lambda_1}\right)M^2 - \mu d_{\sigma'}\|f\|_\sigma M, \end{aligned}$$

for every $u \in C$. Therefore, $m_C > -\infty$.

Let $u_n = \tau_n\varphi_1 + v_n \in C$, for each $n \in \mathbb{N}$, such that $I_\mu(u_n) \to m_C$. As C is convex and closed in the strong topology we can assume that there exists $u_\mu \in C$ such that $u_n \to u_\mu$ weakly in $H_0^1(\Omega)$, up to a subsequence. Arguing as in Lemma 1.14, we conclude that $I_\mu(u_\mu) = m_C$, which verifies the claim.

In view of the above statement, to ensure the existence of a weak solution for Problem (1.1) is sufficient to find $\mu^* \in (0, \mu_1)$ and $\nu^* > 0$ such that $u_\mu \in int(C)$, whenever $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. Write

$$u_\mu = t\varphi_1 + v, \text{ with } t \in [t_1, t_2] \text{ and } v \in \langle\varphi_1\rangle^{\perp}, \|v\| \leq \rho.$$

Immediately, we have $m_C = m_t$. Consequently, $v \in S_t$, $\mu \in (0, \mu_1)$. Therefore, of according to Lemma 1.14, $\|v\| < \rho$. This shows that $u_\mu \in int(C)$ when $t \in (t_1, t_2)$.

It remains to verify that we can not have $t = t_1$ or $t = t_2$. In this regard, from (1.6), the continuity of h with respect to the second variable and of the fact that $H_0^1(\Omega)$ is continuously embedded in $L^1(\Omega)$, we can apply the Lebesgue Dominated Convergence Theorem to ensure that the functional $T_i : \langle\varphi_1\rangle^{\perp} \to \mathbb{R}$, $i = 1, 2$, given by

$$T_i(z) = \int_\Omega h(x, t_i\varphi_1 + z)\varphi_1 dx,$$

is continuous. Therefore, of the condition (h_2^+), there exists $\delta > 0$ such that, for every

$z \in \langle \varphi_1 \rangle^\perp$, with $\|z\| < \delta$,

$$\int_\Omega h(x, t_1\varphi_1 + z)\varphi_1 dx > \frac{T_1(0)}{2} > 0 > \frac{T_2(0)}{2} > \int_\Omega h(x, t_2\varphi_1 + z)\varphi_1 dx. \tag{1.15}$$

Then applying Lemma 1.16, there exists $\mu^* \in (0, \mu_1)$ such that, for every $\mu \in (0, \mu^*)$, $\|w\| < \delta$, for every $w \in S_{t_1} \cup S_{t_2}$. Thus, as a consequence from (1.15) we have, for every $\mu \in (0, \mu^*)$,

$$\int_\Omega h(x, t_1\varphi_1 + v_1)\varphi_1 dx > \frac{T_1(0)}{2} > 0 > \frac{T_2(0)}{2} > \int_\Omega h(x, t_2\varphi_1 + v_2)\varphi_1 dx,$$

for every $v_1 \in S_{t_1}, v_2 \in S_{t_2}$. This implies that

$$\frac{\lambda - \lambda_1}{\lambda_1 \mu}\|\varphi_1\|^2 t_1 + \int_\Omega h(x, t_1\varphi_1 + v_1)\varphi_1 dx \;\; > \;\; -\frac{|\lambda - \lambda_1|}{\lambda_1 \mu}\|\varphi_1\|^2 |t_1| + \frac{T_1(0)}{2},$$

for every $v_1 \in S_{t_1}$, and

$$\frac{\lambda - \lambda_1}{\lambda_1 \mu}\|\varphi_1\|^2 t_2 + \int_\Omega h(x, t_2\varphi_1 + v_2)\varphi_1 dx \;\; < \;\; \frac{|\lambda - \lambda_1|}{\lambda_1 \mu}\|\varphi_1\|^2 |t_2| + \frac{T_2(0)}{2},$$

for every $v_2 \in S_{t_2}$. Thus the above inequalities assure us that there exists $\nu^* < (\lambda_1/2\|\varphi_1\|^2)\min\{T_1(0)/(|t_1|+1), -T_2(0)/(|t_2|+1)\}$ such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, we have that

$$\langle I'_\mu(t_1\varphi_1 + v_1), \varphi_1 \rangle = -\mu\left[\frac{\lambda - \lambda_1}{\lambda_1 \mu}\|\varphi_1\|^2 t_1 + \int_\Omega h(x, t_1\varphi_1 + v_1)\varphi_1 dx\right] < 0, \tag{1.16}$$

for every $v_1 \in S_{t_1}$, and

$$\langle I'_\mu(t_2\varphi_1 + v_2), \varphi_1 \rangle = -\mu\left[\frac{\lambda - \lambda_1}{\lambda_1 \mu}\|\varphi_1\|^2 t_2 + \int_\Omega h(x, t_2\varphi_1 + v_2)\varphi_1 dx\right] > 0, \tag{1.17}$$

for every $v_2 \in S_{t_2}$.

If $u_\mu = t_1\varphi_1 + v$, as $v \in S_{t_1}$, follow from (1.16) that there is $\varepsilon > 0$ such that, for very $\tau \in (t_1, t_1 + \varepsilon)$,

$$m_C \le m_\tau \le I_\mu(\tau\varphi_1 + v) < I_\mu(t_1\varphi_1 + v) = I_\mu(u_\mu) = m_C.$$

This implies that we can not have $t = t_1$.

If $u_\mu = t_2\varphi_1 + v$, using (1.17) and arguing as above we conclude that we can not have $t = t_2$. This implies that $t \neq t_1$ and $t \neq t_2$. □

1.3 Proof of Theorem 1.3 for h bounded and Lipschitz with respect to the second variable on $\mathbf{L}^{\sigma}(\mathbf{\Omega})$

In this section we investigate the existence of a solution of (1.1) with h being a Carathéodory function satisfying (h_2^-), bounded with respect to $L^{\sigma}(\Omega)$ and Lipschitz with respect to $L^{\sigma}(\Omega)$ in the second variable, where $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, i.e., there exists $\zeta \in L^{\sigma}(\Omega)$ such that

$$|h(x, s_1) - h(x, s_2)| \leq \zeta(x)|s_1 - s_2|, \text{ for every } x \in \overline{\Omega}, \; s_1, s_2 \in \mathbb{R}. \tag{1.18}$$

As in the Section 1.2, we are interested in the search of weak solutions of Problem (1.1) that are critical points of the functional I_{μ}, given in (1.7), which is well defined and it is of class C^1 in $H_0^1(\Omega)$. The goal of this section is to prove the following theorem:

Theorem 1.17. *Suppose h satisfies (h_2^-), (1.6) and (1.18). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a solution $u_{\mu} = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^{\perp}$.*

In the proof of Theorem 1.17 we use the Lyapunov-Schmidt reduction method, which reduces the study of critical points of a functional Φ defined in a Hilbert space X of infinite dimension to the study of the critical points of a functional $\hat{\Phi}$ defined in a closed subspace of X, generally, of finite dimension. In our proof of Theorem 1.17 we apply the following result, which is stated and proved in [20].

Theorem 1.18. *Let Y and Z be closed subspaces of a real Hilbert X such that $X = Y \oplus Z$. Let $\Phi : X \to \mathbb{R}$ be a functional of class C^1. If there exists an increasing function $\phi : (0, \infty) \to (0, \infty)$ such that $\phi(s) \to \infty$ as $s \to \infty$ and*

$$\langle \Phi'(y + z_1) - \Phi'(y + z_2), z_1 - z_2 \rangle \geq \|z_1 - z_2\|\phi(\|z_1 - z_2\|), \text{ for every } y \in Y, z_1, z_2 \in Z.$$

Then,

(i) There exists a continuous function $\psi : Y \to Z$ such that $\Phi(y + \psi(y)) = \min_{z \in Z} \Phi(y + z)$. Moreover, $\psi(y)$ is the only element of Z such that $\langle \Phi'(y + \psi(y)), z \rangle = 0$, for every $z \in Z$.

(ii) The function $\hat{\Phi} : Y \to \mathbb{R}$ defined by $\hat{\Phi}(y) = \Phi(y + \psi(y))$ is of class C^1 and

$$\langle \hat{\Phi}'(y_1), y_2 \rangle = \langle \Phi'(y_1 + \psi(y_1)), y_2 \rangle, \text{ for every } y_1, y_2 \in Y.$$

(iii) $y \in Y$ *is a critical point of* $\hat{\Phi}$ *if and only if* $y + \psi(y)$ *is a critical point of* Φ.

We are ready to prove Theorem 1.17.

Proof of Theorem 1.17. From (1.18), the Hölder Inequality and (1.8), for every $\tau \in \mathbb{R}$ and $v_1,\ v_2 \in \langle \varphi_1 \rangle^{\perp}$ we have that

$$\begin{aligned}
&\langle I_{\mu}^{'}(\tau\varphi_1 + v_1) - I_{\mu}^{'}(\tau\varphi_1 + v_2), v_1 - v_2 \rangle \\
\geq\ & \|v_1 - v_2\|^2 - \frac{\overline{\lambda}}{\lambda_2}\|v_1 - v_2\|^2 - \mu \int_{\Omega} [\ h(x, \tau\varphi_1 + v_1) - h(x, \tau\varphi_1 + v_2)\ ](v_1 - v_2)dx \\
\geq\ & \frac{\lambda_2 - \overline{\lambda}}{\lambda_2}\|v_1 - v_2\|^2 - \mu \int_{\Omega} \zeta |v_1 - v_2|^2 dx \\
\geq\ & \frac{\lambda_2 - \overline{\lambda}}{\lambda_2}\|v_1 - v_2\|^2 - \mu \|\zeta\|_{\sigma} \Big(\int_{\Omega} |v_1 - v_2|^{2\sigma'} dx \Big)^{\frac{1}{\sigma'}} \\
\geq\ & \frac{\lambda_2 - \overline{\lambda}}{\lambda_2}\|v_1 - v_2\|^2 - \mu \|\zeta\|_{\sigma} d_{2\sigma'} \|v_1 - v_2\|^2 = \left[\frac{\lambda_2 - \overline{\lambda}}{\lambda_2} - \mu \|\zeta\|_{\sigma} d_{2\sigma'} \right] \|v_1 - v_2\|^2.
\end{aligned}$$

Therefore, taking $\mu_1 < (\lambda_2 - \overline{\lambda})/2\lambda_2 \|\zeta\|_{\sigma} d_{2\sigma'}$ we obtain, for every $\mu \in (0, \mu_1)$, $\tau \in \mathbb{R}$ and $v_1,\ v_2 \in \langle \varphi_1 \rangle^{\perp}$,

$$\langle I_{\mu}^{'}(\tau\varphi_1 + v_1) - I_{\mu}^{'}(\tau\varphi_1 + v_2), v_1 - v_2 \rangle \geq \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\|v_1 - v_2\|^2.$$

Thus, by Theorem 1.18–(i), there exists a continuous function $\psi : \langle \varphi_1 \rangle \to \langle \varphi_1 \rangle^{\perp}$ such that $\psi(\tau\varphi_1)$, with $\tau \in \mathbb{R}$, is the only element of the space $\langle \varphi_1 \rangle^{\perp}$ that satisfies

$$\langle I_{\mu}^{'}(\tau\varphi_1 + \psi(\tau\varphi_1)), v \rangle = 0, \text{ for every } \tau \in \mathbb{R},\ v \in \langle \varphi_1 \rangle^{\perp}. \tag{1.19}$$

Moreover, from Theorem 1.18–(ii), the functional $\hat{I} : \langle \varphi_1 \rangle \to \mathbb{R}$ given by $\hat{I}_{\mu}(\tau\varphi_1) = I_{\mu}(\tau\varphi_1 + \psi(\tau\varphi_1))$ is of class C^1 and

$$\begin{aligned}
\langle \hat{I}_{\mu}^{'}(\tau\varphi_1), \varphi_1 \rangle &= \langle I_{\mu}^{'}(\tau\varphi_1 + \psi(\tau\varphi_1)), \varphi_1 \rangle \\
&= -\mu \left[\frac{\lambda - \lambda_1}{\lambda_1 \mu} \|\varphi_1\|^2 \tau + \int_{\Omega} h(x, \tau\varphi_1 + \psi(\tau\varphi_1))\varphi_1 dx \right].
\end{aligned} \tag{1.20}$$

On the other hand, from (1.6) and (h_2^-), instead of (h_2^+), we can work as in (1.15) to conclude that there exists $\delta > 0$ such that, for every $v \in \langle \varphi_1 \rangle^{\perp}$, with $\|v\| < \delta$, we have

$$\begin{aligned}
\int_{\Omega} h(x, t_1\varphi_1 + v)\varphi_1 dx &< \frac{1}{2} \int_{\Omega} h(x, t_1\varphi_1)\varphi_1 dx \\
&< 0 < \frac{1}{2} \int_{\Omega} h(x, t_2\varphi_1)\varphi < \int_{\Omega} h(x, t_2\varphi_1 + v)\varphi_1 dx.
\end{aligned} \tag{1.21}$$

It follows from (1.19), (1.6), the Hölder Inequality and (1.8) that

$$\begin{aligned}\|\psi(\tau\varphi_1)\|^2 &= \lambda\|\psi(\tau\varphi_1)\|_2^2 + \mu\int_\Omega h(x,\tau\varphi_1+\psi(\tau\varphi_1))\psi(\tau\varphi_1)dx\\ &\le \overline{\lambda}\|\psi(\tau\varphi_1)\|_2^2 + \mu\int_\Omega f|\psi(\tau\varphi_1)|dx \le \frac{\overline{\lambda}}{\lambda_2}\|\psi(\tau\varphi_1)\|^2 + \mu d_{\sigma'}\|f\|_\sigma\|\psi(\tau\varphi_1)\|.\end{aligned}$$

Hence

$$\|\psi(\tau\varphi_1)\| \le \frac{\lambda_2}{(\lambda_2-\overline{\lambda})}\mu d_{\sigma'}\|f\|_\sigma, \text{ for every } \mu\in(0,\mu_1),\ \tau\in\mathbb{R}. \tag{1.22}$$

Then, taking $\mu^* < \min\{\mu_1, (\lambda_2-\overline{\lambda})\delta/\lambda_2 d_{\sigma'}\|f\|_\sigma\}$, where $\delta > 0$ is given in (1.21), we obtain that $\|\psi(\tau\varphi_1)\| < \delta$, for every $\tau\in\mathbb{R}$ and $\mu\in(0,\mu^*)$. Consequently, from (1.20) and (1.21),

$$\langle \hat{I}'_\mu(t_1\varphi_1),\varphi_1\rangle > -\mu\left[\frac{|\lambda-\lambda_1|}{\lambda_1\mu}\|\varphi_1\|^2|t_1| + \frac{1}{2}\int_\Omega h(x,t_1\varphi_1)\varphi_1 dx\right]$$

and

$$\langle \hat{I}'_\mu(t_2\varphi_1),\varphi_1\rangle < -\mu\left[-\frac{|\lambda-\lambda_1|}{\lambda_1\mu}\|\varphi_1\|^2|t_2| + \frac{1}{2}\int_\Omega h(x,t_2\varphi_1)\varphi_1 dx\right].$$

Therefore, there exists $\nu^* > 0$ such that $\langle \hat{I}'_\mu(t_1\varphi_1),\varphi_1\rangle > 0$ and $\langle \hat{I}'_\mu(t_2\varphi_1),\varphi_1\rangle < 0$, for every $\mu\in(0,\mu^*)$ and $|\lambda-\lambda_1| < \mu\nu^*$. Then, by the Intermediate Value Theorem, there exists $t\in(t_1,t_2)$ such that $\langle \hat{I}'_\mu(t\varphi_1),\varphi_1\rangle = 0$. Thus, of Theorem 1.18–(*iii*), $u_\mu = t\varphi_1+\psi(t\varphi_1)$ is a critical point of the functional I_μ. This concludes the proof of Theorem 1.17. □

1.4 Proofs of the main results

In order to apply Theorems 1.12 and 1.17, we consider the truncated function h_R defined by

$$h_R(x,s) = \chi(s)h(x,s), \text{ for every } x\in\overline{\Omega},\ s\in\mathbb{R}, \tag{1.23}$$

where $R > \max\{|t_1|,|t_2|\}\|\varphi_1\|_\infty > 0$ and $\chi\in C^\infty(\mathbb{R},[0,1])$ is a function satisfiying

$$\chi(s) = \begin{cases} 1 \text{ if } |s|\le R+1,\\ 0 \text{ if } |s|\ge R+2.\end{cases}$$

Associated with h_R, we consider the truncaded problem

$$\begin{cases} -\Delta u = & \lambda u + \mu h_R(x,u) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

As $\|t_i\varphi_1\|_\infty < R$, $i = 1, 2$, it follows from (1.23) and (h_2^+) (or (h_2^-)), that h_R satisfies the hypothesis (h_2^+) ou (h_2^-). Moreover, from (h_0), there exists $f_{R+2} \in L^\sigma(\Omega)$ such that $|h(x,s)| \leq f_{R+2}(x)$, for every $|s| \leq R+2$ and for almost every $x \in \overline{\Omega}$. Therefore, from (1.23),

$$|h_R(x,s)| \leq f_{R+2}(x), \text{ for every } s \in \mathbb{R}, \text{ for almost every } x \in \overline{\Omega}. \tag{1.25}$$

Before proving Theorems 1.2 and 1.3, we state the following lemma (see Appendix for a proof).

Lemma 1.19. *Suppose h_R satisfies* (1.25) *and that there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, $u_\mu = t\varphi_1 + v$ is a weak solution of Problem* (1.24)*, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$. Then there exist $q > \frac{N}{2}$ and $b_q > 0$ such that $\|v\|_{2,q} \leq b_q\mu$.*

Proof of Theorem 1.2. Since h_R satisfies (h_2^+), from (1.25) and Theorem 1.12, there exist positive constants $\hat{\mu}$ and ν^* such that, for every $\mu \in (0, \hat{\mu})$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.24) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$. Since $q > N/2$ we may apply the Sobolev Embedding Theorem to conclude that there exists K such that $\|v\|_\infty \leq K\|v\|_{2,q} \leq Kb_q\mu$. Consequently, $\|v\|_\infty \to 0$, as $\mu \to 0$. This implies that there exists $\mu^* \in (0, \hat{\mu})$ such that $\|v\|_\infty < 1$, for every $\mu \in (0, \mu^*)$. Thus, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$,

$$\|u_\mu\|_\infty = \|t\varphi_1 + v\|_\infty \leq \|t\varphi_1\|_\infty + \|v\|_\infty < R + 1.$$

Hence $\chi(u_\mu) = 1$ and $h_R(x, u_\mu) = h(x, u_\mu)$, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. Consequently, u_μ is a solution from (1.1). Noting that $q > N/2$ in Lemma 1.19, we have that $u_\mu \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma \in (0, 1)$. This concludes the proof of Theorem 1.2. □

Proof of Theorem 1.3. To verify that h_R satisfies (1.18) it suffices to consider $|s_1|$, $|s_2| \leq R + 2$, with $s_1 < s_2$, because the other cases follow from the definition of h_R, (h_0) and the fact of χ' is bounded. Indeed, from (h_0) and (h_1) there exist f_{R+2} and $\zeta \in L^\sigma(\Omega)$ such that

$$\begin{aligned} |h_R(x,s_1) - h_R(x,s_2)| &\leq |\chi(s_1)|\ |h(x,s_1) - h(x,s_2)| + |h(x,s_2)|\ |\chi(s_1) - \chi(s_2)| \\ &\leq \zeta(x)|s_1 - s_2| + f_{R+2}(x)\ |\chi'(\theta)|\ |s_1 - s_2|, \end{aligned}$$

where $\theta \in (s_1, s_2)$. Thus, as χ' is bounded it follows that h_R satisfies (1.18).

Therefore, since h_R satisfies (h_2^-), from (1.25) and Theorem 1.17, there exist positive constants $\hat{\mu}$ and ν^* such that, for every $\mu \in (0, \hat{\mu})$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.24) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.

From now on, using Lemma 1.19, we may argue as in the proof of Theorem 1.2 to conclude that there exists $\mu^* \in (0, \hat{\mu})$ such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, u_μ is solution of (1.1), which is of class $C^{0,\gamma}(\overline{\Omega})$. □

Finally, we are in conditions to prove our main result.

Proof of Theorem 1.1. From Theorem 1.2 or 1.3, for each $i = 1, \ldots, k$, there exist positive constants μ_i and ν_i such that, for every $0 < |\mu| < \mu_i$ and $|\lambda - \lambda_1| < |\mu|\nu_i$, Problem (1.1) has a solution $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{0,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$. Therefore, taking $0 < \mu^* < \min\{\mu_i : i = 1, \ldots, k\}$ and $0 < \nu^* < \min\{\nu_i : i = 1, \ldots, k\}$, Problem (1.1) has k solutions, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$. □

To justify Remark 1.4 we apply a well known result that can be found in [14, 24, 39]. There exists $K > 0$ such that

$$\varphi_1(x) \geq Kd(x, \partial\Omega), \text{ for every } x \in \overline{\Omega}, \tag{1.26}$$

where $d(x, \partial\Omega)$ denote the distance from x to $\partial\Omega$.

In the following we denote by $(\hat{h}_0)$ and $(\hat{h}_1)$ the hypothesis (h_0) and (h_1) with $\sigma > N$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$. In addition we consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(t) = \int_\Omega h(x, t\varphi_1)\varphi_1 dx, \quad \text{for every } t \in \mathbb{R}. \tag{1.27}$$

Theorem 1.20. *Suppose h satisfies $(\hat{h}_0)$, $(\hat{h}_1)$ and (h_2^+) or (h_2^-). Consider the positive constants μ^* and ν^* given by Theorem 1.2 or 1.3. Then there exists $\mu^{**} \in (0, \mu^*)$ such that, if $\mu \in (0, \mu^{**})$ and $|\lambda - \lambda_1| < \mu\nu^*$, the solution of Problem (1.1), $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$, is of class $C^{1,\gamma}(\overline{\Omega})$ and positive or negative provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively.*

Proof. Firstly we prove the theorem when h satisfies (h_2^+), com $t_1 \geq 0$. Note that we may assume without loss of generality that $t_1 > 0$. Indeed, if $t_1 = 0$, from (h_2^+) and the continuity of Φ in $[t_1, t_2]$, there exists $\tilde{t}_1 > t_1$ satisfying (h_2^+).

Arguing as in the proof of Lemma 1.19 there exist positive constants $q > N$ and b_q such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, the solution of Problem (1.1), $u_\mu = t\varphi_1 + v$, with $t = t(\mu) \in (t_1, t_2)$, $v = v(\mu) \in \langle\varphi_1\rangle^\perp$, satisfies $\|v\|_{2,q} \leq b_q\mu$. Noting that $q > N$ we have that $u_\mu \in C^{1,\gamma}(\overline{\Omega})$, with $\gamma \in (0, 1)$, and $v \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$.

We claim that
$$\lim_{\mu\to 0}\frac{|v(x)|}{d(x,\partial\Omega)}=0,\quad \text{for every } x\in\Omega.$$
Suppose the claim is true. As $t\geq t_1>0$, from (1.26) we have that
$$\frac{u_\mu(x)}{d(x,\partial\Omega)}=\frac{t\varphi_1(x)+v(x)}{d(x,\partial\Omega)}\geq\frac{t_1Kd(x,\partial\Omega)+v(x)}{d(x,\partial\Omega)},\ \text{for every } x\in\Omega.$$
So, from the above claim, there exists $\mu^{**}\in(0,\mu^*)$ such that, for $\mu\in(0,\mu^{**})$,
$$u_\mu(x)\geq\frac{t_1K}{2}d(x,\partial\Omega),\ \text{for every } x\in\Omega.$$
Therefore u_μ is positive in Ω. We note that under the hypothesis (h_2^-), with $t_2\leq 0$, the proof of the theorem is similar.

Finally, to complete the proof of the theorem it remains to prove the claim. Indeed, suppose there exist $\varepsilon>0$, $(x_n)\subset\Omega$ and $(\mu_n)\subset\mathbb{R}^+\setminus\{0\}$, with $\mu_n\to 0$, such that $(|v_{\mu_n}(x_n)|/d(x_n,\partial\Omega))\geq\varepsilon$, for every $n\in\mathbb{N}$. As $(x_n)\subset\overline{\Omega}$ there exists $x_0\in\overline{\Omega}$ such that $x_n\to x_0$, up to subsequences.

If $x_0\in\Omega$, as $d(x_n,\partial\Omega)\to d(x_0,\partial\Omega)$ and $\|v\|_\infty\to 0$, as $\mu\to 0$, there exists $n_0\in\mathbb{N}$ such that
$$\frac{\varepsilon}{2}d(x_0,\partial\Omega)\leq\varepsilon d(x_{n_0},\partial\Omega)\leq|v_{\mu_{n_0}}(x_{n_0})|\leq\|v_{\mu_{n_0}}\|_\infty\leq\frac{\varepsilon}{4}d(x_0,\partial\Omega),$$
which is an absurd.

If $x_0\in\partial\Omega$, since $x_n\to x_0$ and $\|\nabla v\|_\infty\to 0$, as $\mu\to 0$, for n_0 sufficiently large there exists $y_{n_0}\in\partial\Omega$ such that $d(x_{n_0},\partial\Omega)=|x_{n_0}-y_{n_0}|$ and $\|\nabla v_{n_0}\|\leq\varepsilon/2$. Consequently, from $v_{n_0}(y_{n_0})=0$ and by the Mean Value Theorem, there exists $\theta\in(0,1)$ such
$$\begin{aligned}\varepsilon\leq\frac{|v_{n_0}(x_{n_0})|}{d(x_{n_0},\partial\Omega)} &= \frac{|v_{n_0}(x_{n_0})-v_{n_0}(y_{n_0})|}{|x_{n_0}-y_{n_0}|}\\ &= |\langle\nabla v_{\mu_{n_0}}(x_{n_0}+\theta(y_{n_0}-x_{n_0})),\frac{x_{n_0}-y_{n_0}}{|x_{n_0}-y_{n_0}|}\rangle|\leq\|v_{\mu_{n_0}}\|_\infty\leq\frac{\varepsilon}{2},\end{aligned}$$
which is a contradition. This completes the proof of theorem. □

Arguing as in proof of Theorem 1.20, we may establish the ordering of the solutions provided by Theorem 1.1.

Theorem 1.21. *Suppose h satisfies $(\hat{h}_0)$, $(\hat{h}_1)$ and (h_2). Consider the positive constants μ^* and ν^* given by Theorem 1.1. Then there exists $\mu^{**}\in(0,\mu^*)$ such that, if $0<|\mu|<\mu^{**}$*

and $|\lambda - \lambda_1| < |\mu|\nu^*$*, the solutions of Problem* (1.1)*,* $u_i = \hat{t}_i\varphi_1 + v_i$*, with* $\hat{t}_i \in (t_i, t_{i+1})$ *and* $v_i \in \langle\varphi_1\rangle^\perp$*, are of class* $C^{1,\gamma}(\overline{\Omega})$ *and* $u_i < u_{i+1}$ *in* Ω*,* $i = 1, \cdots, k$*.*

Proof. From (h_2) and the continuity of Φ in $[t_i, t_{i+1}]$, where Φ is given by (1.27), there exist t_i^i, $t_{i+1}^i \in (t_i, t_{i+1})$, with $t_i^i < t_{i+1}^i$, satisfying (h_2). As $\sigma > N$ the argument used in Lemma 1.19 proves that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, the solutions of Problem (1.1), $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i = \hat{t}_i(\mu) \in (t_i^i, t_{i+1}^i)$ and $v_i = v_i(\mu) \in \langle\varphi_1\rangle^\perp$, are of class $C^{1,\gamma}(\overline{\Omega})$ and $v_i \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$. Consequently, $u_{i+1} - u_i = (\hat{t}_{i+1} - \hat{t}_i)\varphi_1 + (v_{i+1} - v_i)$, with $\hat{t}_{i+1} - \hat{t}_i > t_{i+1}^{i+1} - t_{i+1}^i > 0$ and $(v_{i+1} - v_i) \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$, Therefore, arguing as in Theorem 1.20, there exists $\mu^{**} \in (0, \mu^*)$ such that $u_{i+1} - u_i > 0$ in Ω, for every $0 < |\mu| < \mu^{**}$. □

1.5 No solubility of Problem (1.1)

The goal of this section is to establish the non existence of solutions for Problem (1.1) when the conditions (h_2^+) or (h_2^-) are not satisfied, more specifically, we prove Theorem 1.5. To achieve this goal, we need some preliminary results. We start proving that under suitable hypotheses Problem (1.1) has no bounded weak solution in $L^\infty(\Omega)$.

Proposition 1.22. *Suppose* h *satisfies* (h_0) *and* (h_4)*. Then, given* $M > 0$*, there exist positive constants* μ^* *and* ν^* *such that, for every* $0 < |\mu| < \mu^*$ *and* $|\lambda - \lambda_1| < |\mu|\nu^*$*, Problem* (1.1) *has no weak solution* $u_\mu = t\varphi_1 + v$*,* $t \in [t_1, t_2]$ *and* $v \in \langle\varphi_1\rangle^\perp$*, with* $\|u_\mu\|_\infty \le M$*.*

Proof. To prove that Problem (1.1) has no bounded weak solution, assume by contradiction that there exist $M > 0$, $(\mu_k) \subset \mathbb{R} \setminus \{0\}$, with $|\mu_k| \to 0$, and $(\hat{\lambda}_k) \subset \mathbb{R}$ such that $|\hat{\lambda}_k - \lambda_1| < |\mu_k|/k$, $\|u_k\|_\infty \le M$ and $u_k = \tau_k\varphi_1 + v_k$, with $\tau_k \in [t_1, t_2]$ and $v_k \in \langle\varphi_1\rangle^\perp$, is solution of (1.1). Then, from the boundedness of u_k and from the fact that $\tau_k \in [t_1, t_2]$, there exists K such that $\|v_k\|_\infty \le K$. Furthermore, $\hat{\lambda}_k < |\mu_k|/k + \lambda_1 \le |\mu_k| + \lambda_1$.

From (1.1) and (h_0) there exists $f_M \in L^\sigma(\Omega)$ such that

$$\begin{aligned} \frac{\lambda_2 - \hat{\lambda}_k}{\lambda_2}\|v_k\|^2 &\le \|v_k\|^2 - \hat{\lambda}_k\|v_k\|_2^2 = \mu_k \int_\Omega h(x, \tau_k\varphi_1 + v_k)v_k dx \\ &\le |\mu_k| \int_\Omega f_M|v_k| dx \le |\mu_k| K \|f_M\|_1. \end{aligned}$$

Therefore, as $\hat{\lambda}_k \to \lambda_1$, making $k \to \infty$ it follows that $\|v_k\| \to 0$. Hence, from the compactness of $[t_1, t_2]$, there exists $\tau_0 \in [t_1, t_2]$ such that $\tau_k\varphi_1 + v_k \to \tau_0\varphi_1$ a.e. in Ω, up to subsequences. Thus, invoking (h_0), we may use the Lebesgue Dominated Convergence

Theorem to obtain

$$\int_\Omega h(x,\tau_k\varphi_1+v_k)\varphi_1 dx \to \int_\Omega h(x,\tau_0\varphi_1)\varphi_1 dx.$$

Consequently, using that $|\hat{\lambda}_k-\lambda_1||\tau_k|/|\mu_k| \leq \max\{|t_1|,|t_2|\}/k$, we have that

$$\lim_{k\to\infty}\left[\frac{\hat{\lambda}_k-\lambda_1}{\mu_k\lambda_1}\|\varphi_1\|^2\tau_k+\int_\Omega h(x,\tau_k\varphi_1+v_k)\varphi_1 dx\right]=\int_\Omega h(x,\tau_0\varphi_1)\varphi_1 dx.$$

Hence, from (h_4), there exists $k\in\mathbb{N}$ sufficiently large such that

$$\begin{aligned}\int_\Omega \nabla u_k\nabla\varphi_1 dx-\hat{\lambda}_k\int_\Omega u_k\varphi_1 dx-\mu_k\int_\Omega h(x,\tau_k\varphi_1+v_k)\varphi_1 dx&\\ =-\mu_k\left[\frac{\hat{\lambda}_k-\lambda_1}{\mu_k\lambda_1}\|\varphi_1\|^2\tau_k+\int_\Omega h(x,\tau_k\varphi_1+v_k)\varphi_1 dx\right]&\neq 0.\end{aligned}$$

But this contradicts the fact that u_k is solution of Problem (1.1). □

Next, we prove that under hypothesis (h_3) every solution de (1.1) is bounded in $H_0^1(\Omega)$, for μ sufficiently small.

Lemma 1.23. *Suppose h satisfies (h_3) and $u_\mu = t\varphi_1+v$, with $t\in[t_1,t_2]$ and $v\in\langle\varphi_1\rangle^\perp$, is a weak solution of Problem (1.1). Then there exist positive constants μ^{**}, ν^{**} and $M_1=M_1(\mu^{**},\nu^{**},\|f\|_\sigma)$ such that $\|u_\mu\|\leq M_1$, for every $0<|\mu|<\mu^{**}$ and $|\lambda-\lambda_1|<|\mu|\nu^{**}$.*

Proof. If $N\geq 3$, and u_μ is a solution of Problem (1.1), from (h_3), (1.8) and the Hölder Inequality it follows that

$$\begin{aligned}\|v\|^2 &= \lambda\|v\|_2^2+\mu\int_\Omega h(x,t\varphi_1+v)v dx \leq \frac{|\mu|\nu^{**}+\lambda_1}{\lambda_2}\|v\|^2+|\mu|K\int_\Omega f(1+|v|)|v|dx\\ &\leq \frac{|\mu|\nu^{**}+\lambda_1}{\lambda_2}\|v\|^2+|\mu|K(\|f\|_\sigma d_{\sigma'}\|v\|+\|f\|_\sigma d_{2\sigma'}^2\|v\|^2)\\ &\leq \left[\frac{|\mu|\nu^{**}+\lambda_1}{\lambda_2}+|\mu|K\|f\|_\sigma d_{2\sigma'}^2\right]\|v\|^2+|\mu|K\|f\|_\sigma d_{\sigma'}\|v\|,\end{aligned}$$

where $K=K(\max\{|t_1|,|t_2|\},\|\varphi_1\|_\infty)$. This implies that

$$\left[1-\frac{|\mu|\nu^{**}+\lambda_1}{\lambda_2}-|\mu|K\|f\|_\sigma d_{2\sigma'}^2\right]\|v\|\leq|\mu|K\|f\|_\sigma d_{\sigma'}.$$

Thus, for $\nu^{**}>0$ fixed and $0<\mu^{**}<(\lambda_2-\lambda_1)/(\nu^{**}+\lambda_2K\|f\|_\sigma d_{2\sigma'}^2)$ we have that $\|v\|$

is bounded, for every $0 < |\mu| < \mu^{**}$ and $|\lambda - \lambda_1| < |\mu|\nu^{**}$. Therefore, as $t \in [t_1, t_2]$, there exists $M_1 > 0$ such that $\|u_\mu\| \leq M_1$, for every $0 < |\mu| < \mu^{**}$ and $|\lambda - \lambda_1| < |\mu|\nu^{**}$.

If $N = 1$ or $N = 2$, the result it follows arguing as in the previous case and from the fact that $H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$, for every $1 \leq q < \infty$. This concludes the proof of the lemma. □

If $N = 1$, $H_0^1(\Omega)$ is continuously embedded in $L^\infty(\Omega)$. Therefore, of Lemma 1.23, the solution u_μ from (1.1) is bounded in $L^\infty(\Omega)$. Using the argument presented by Struwe, see Lemma B.3 in [48], and Agmon-Douglis-Nirenberg Theorem, see [1], we may prove that under hypothesis (h_3) the same occurs if $N \geq 2$.

Lemma 1.24. *Suppose h satisfies (h_3). Consider the positive constants M_1, μ^{**} and ν^{**} given by Lemma 1.23. Then, for every $0 < |\mu| < \mu^{**}$ and $|\lambda - \lambda_1| < |\mu|\nu^{**}$, the solution of Problem (1.1), $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$, is bounded in $L^\infty(\Omega)$ by a constant $M_2 = M_2(\mu^{**}, \nu^{**}, \|f\|_\sigma)$.*

Proof. From Lemma 1.23 and $t \in [t_1, t_2]$, there exists $K_1 = K_1(M_1)$ tal que $\|v\| \leq K_1$. For each $x \in \Omega$, we define $w(x) = s\varphi_1(x) + v(x)$ where

$$s = \frac{\mu}{\|\varphi_1\|^2} \int_\Omega h(x, t\varphi_1 + v)\varphi_1 dx.$$

As u_μ is solution from (1.1), for every $z \in \langle\varphi_1\rangle^\perp$,

$$\int_\Omega \nabla w \nabla z dx = \int_\Omega \nabla v \nabla z dx = \lambda \int_\Omega vz dx + \mu \int_\Omega h(x, t\varphi_1 + v) z dx.$$

Furthermore,

$$\int_\Omega \nabla w \nabla \varphi_1 dx = s\|\varphi_1\|^2 = \mu \int_\Omega h(x, t\varphi_1 + v)\varphi_1 dx.$$

Defining $\hat{h}(x, w) := \lambda(w - s\varphi_1) + \mu h(x, t\varphi_1 + w - s\varphi_1) = \lambda v + \mu h(x, t\varphi_1 + v)$ the above equalities imply that

$$\int_\Omega \nabla w \nabla \psi = \int_\Omega \hat{h}(x, w)\psi dx, \text{ for every } \psi \in H_0^1(\Omega). \tag{1.28}$$

From (h_3), Hölder Inequality, (1.8) and $\|v\| \leq K_1$, we have that

$$\begin{aligned} |s| &\leq \frac{|\mu|}{\|\varphi_1\|^2} \int_\Omega f(1 + |t\varphi_1 + v|)|\varphi_1| dx \leq \frac{\mu^{**}}{\|\varphi_1\|^2} K_2 \int_\Omega f(1 + |v|) dx \\ &\leq \frac{\mu^{**}}{\|\varphi_1\|^2} K_2 \|f\|_\sigma (|\Omega|^{\frac{1}{\sigma'}} + d_{\sigma'} K_1) := K_3, \end{aligned} \tag{1.29}$$

where $K_2 = K_2(\max\{|t_2|,|t_1|\}, \|\varphi_1\|_\infty)$.

We claim that $w \in L^p(\Omega)$, for every $1 \le p < \infty$. Indeed, If $N = 2$, then the claim it follows from the fact that $H_0^1(\Omega)$ is continuously embedded in $L^p(\Omega)$, for every $1 \le p < \infty$.

If $N \ge 3$, from $|\mu| < \mu^{**}$, $|\lambda - \lambda_1| < |\mu|\nu^{**}$, (h_3) and (1.29),

$$\begin{aligned}
|\hat{h}(x,w)| &\le |\lambda|(|w| + |s||\varphi_1|) + |\mu| f(1 + |t\varphi_1 + w - s\varphi_1|)\\
&\le (\mu^{**}\nu^{**} + \lambda_1)(|w| + K_3\|\varphi_1\|_\infty) + \mu^{**} f(1 + \max\{|t_1|,|t_2|\}\|\varphi_1\|_\infty + |w| + K_3\|\varphi_1\|_\infty)\\
&\le a(x)(1 + |w|), \qquad (1.30)
\end{aligned}$$

where $a(x) := \max\{(\mu^{**}\nu^{**} + \lambda_1)\max\{1, K_3\|\varphi_1\|_\infty\}, \mu^{**}\max\{1 + (\max\{|t_1|,|t_2|\} + K_3)\|\varphi_1\|_\infty, 1\} f\} \in L^{\frac{N}{2}}(\Omega)$, because $f \in L^\sigma(\Omega)$. Thus, of Lemma B.3 in [48] it follows that $w \in L^p(\Omega)$, for every $1 \le p < \infty$. This concludes the proof of the claim.

We fix $q < \sigma$ such that $q > N/2$ if $N \ge 3$ and $q > 1$ if $N = 2$. Thus, from (h_3), the Hölder Inequality (with σ/q, $\sigma/\sigma - q > 1$) and the fact that $w \in L^p(\Omega)$, for every $1 \le p < \infty$,

$$\begin{aligned}
\int_\Omega |h(x, t\varphi_1 + w - s\varphi_1)|^q dx &\le 2^{q-1}[1 + (\max\{|t_1|,|t_2|\} + K_3)\|\varphi_1\|_\infty]^q \|f\|_q^q + 2^{q-1}\int_\Omega f^q |w|^q dx\\
&\le K_4\|f\|_\sigma^q + 2^{q-1}\|f\|_\sigma^q \|w\|^q_{\frac{q\sigma}{\sigma - q}}, \qquad (1.31)
\end{aligned}$$

where $K_4 = K_4(\max\{|t_1|,|t_2|\}, K_3, \|\varphi_1\|_\infty, q, |\Omega|)$.

We claim that if $N \ge 3$ there exists $K_5 = K_5(\mu^{**}, \nu^{**}, \|f\|_\sigma)$ such that

$$\left(\int_\Omega |w|^{2(n+1)} dx\right)^{\frac{N-2}{N}} \le K_5, \text{ for every } n \in \mathbb{N}. \qquad (1.32)$$

Assuming that (1.32) is true, we may prove that

$$\hat{h} \in L^q(\Omega) \text{ and } \|\hat{h}\|_q \le K_6, \qquad (1.33)$$

Where $K_6 = K_6(\mu^{**}, \nu^{**}, \|f\|_\sigma)$. Indeed, if $N \ge 3$, we fix $n \in \mathbb{N}$ such that $2(n+1) > q\sigma/(\sigma - q)$. Thus, from (1.31) and the continuous embedding of $L^{2(n+1)}(\Omega)$ on $L^{\frac{q\sigma}{\sigma-q}}(\Omega)$,

$$\begin{aligned}
\int_\Omega |h(x, t\varphi_1 + w - s\varphi_1)|^q dx &\le \|f\|_\sigma^q (K_4 + 2^{q-1} K_7 \|w\|^q_{2(n+1)})\\
&\le \|f\|_\sigma^q (K_4 + 2^{q-1} K_7 K_5^{\frac{N}{N-2}\frac{q}{2(n+1)}})\\
&\le \|f\|_\sigma^q [K_4 + 2^{q-1} K_7 (K_5^{\frac{Nq}{2(N-2)}} + 1)] := K_8.
\end{aligned}$$

Therefore, from $|\mu| < \mu^{**}$, $|\lambda - \lambda_1| < |\mu|\nu^{**}$, the continuous embedding of $L^{2(n+1)}(\Omega)$ on $L^q(\Omega)$, (1.29) and (1.32),

$$\begin{aligned}\|\hat{h}\|_q &\leq (\mu^{**}\nu^{**} + \lambda_1)(K_9\|w\|_{2(n+1)} + K_3\|\varphi_1\|_q) + \mu^{**}K_8^{\frac{1}{q}} \\ &\leq (\mu^{**}\nu^{**} + \lambda_1)(K_9 K_5^{\frac{N}{N-2}\frac{1}{2(n+1)}} + K_3\|\varphi_1\|_q) + \mu^{**}K_8^{\frac{1}{q}} \\ &\leq (\mu^{**}\nu^{**} + \lambda_1)[K_9(K_5^{\frac{N}{2(N-2)}} + 1) + K_3\|\varphi_1\|_q] + \mu^{**}K_8^{\frac{1}{q}}.\end{aligned}$$

If $N = 2$, arguing as in the previous case, (1.33) it follows from (1.31), the fact that $H_0^1(\Omega)$ is continuously embedded in $L^{\frac{q\sigma}{\sigma-q}}(\Omega)$, $\|v\| \leq K_1$ and (1.29). This concludes the proof of the claim.

Consequently, from (1.28), (1.33) and the fact that $q > N/2$ if $N \geq 3$ and $q > 1$ if $N = 2$, we may use Agmon-Douglis-Nirenberg Theorem and Sobolev Embedding Theorem to ensure that there exist K_{10} and K_{11} such that

$$\|w\|_\infty \leq K_{10}\|w\|_{2,p} \leq K_{10}K_{11}\|\hat{h}\|_q \leq K_{10}K_{11}K_6.$$

Therefore, from (1.29),

$$\begin{aligned}\|u_\mu\|_\infty &\leq \|t\varphi_1\|_\infty + \|v\|_\infty \leq \max\{|t_1|, |t_2|\}\|\varphi_1\|_\infty + \|w\|_\infty + |s|\|\varphi_1\|_\infty \\ &\leq \max\{|t_1|, |t_2|\}\|\varphi_1\|_\infty + K_{10}K_{11}K_6 + K_3\|\varphi_1\|_\infty.\end{aligned}$$

This concludes the proof that u_μ is bounded in $L^\infty(\Omega)$.

To prove (1.32), for $n \in \mathbb{N}$ and $R > 1$, we consider

$$m_{n,R}(x) := \min\{|w(x)|^n, R\} = \begin{cases} |w(x)|^n \text{ if } |w(x)|^n < R, \\ R \text{ if } |w(x)|^n \geq R \end{cases}$$

and

$$m_{2n,R^2}(x) := (m_{n,R}(x))^2 = \min\{|w(x)|^{2n}, R^2\} = \begin{cases} |w(x)|^{2n} \text{ if } |w(x)|^n < R, \\ R^2 \text{ if } |w(x)|^n \geq R. \end{cases}$$

So, for each $x \in \Omega$,

$$|m_{n,R}| \leq R, \; |m_{2n,R^2}| \leq R^2 \text{ and } (1 + |w|)|w|m_{2n,R^2} \leq 2 + 2|w|^2 m_{2n,R^2}. \tag{1.34}$$

For each $x \in \Omega$, we define $z(x) := m_{2n,R^2}(x)w(x)$. Thus,

$$\nabla z = m_{2n,R^2}\nabla w + 2n|w|^{2n}\nabla w \ \chi_{\{|w|^n<R\}}.$$

Consequently, from (1.28) and (1.30),

$$\begin{aligned}\int_\Omega m_{2n,R^2}|\nabla w|^2 dx + 2n\int_{\{|w|^n<R\}} |w|^{2n}|\nabla w|^2 dx = \int_\Omega \nabla w \nabla z dx = \int_\Omega \hat{h}(x,w)z dx \\ \leq \int_\Omega a(x)(1+|w|)m_{2n,R^2}|w|dx. \quad (1.35)\end{aligned}$$

On the other hand, as $\nabla(m_{n,R}w) = m_{n,R}\nabla w + n|w|^n\nabla w \ \chi_{\{|w|^n<R\}}$, we have that

$$\begin{aligned}&\int_\Omega |\nabla(m_{n,R}w)|^2 dx \leq 2\int_\Omega m_{2n,R^2}|\nabla w|^2 dx + 2n^2\int_{\{|w|^n<R\}} |w|^{2n}|\nabla w|^2 dx \\ &\leq 2\int_\Omega m_{2n,R^2}|\nabla w|^2 dx + n\int_\Omega m_{2n,R^2}|\nabla w|^2 dx + 4n\int_{\{|w|^n<R\}} |w|^{2n}|\nabla w|^2 dx + 2n^2\int_{\{|w|^n<R\}} |w|^{2n}|\nabla w|^2 dx \\ &= 2(1+\frac{n}{2})\int_\Omega m_{2n,R^2}|\nabla w|^2 dx + 4n(1+\frac{n}{2})\int_{\{|w|^n<R\}} |w|^{2n}|\nabla w|^2 dx \\ &= 2(1+\frac{n}{2})\Big[\int_\Omega m_{2n,R^2}|\nabla w|^2 dx + 2n\int_{\{|w|^n<R\}} |w|^{2n}|\nabla w|^2 dx\Big].\end{aligned}$$

Thus, from (1.35), (1.34) and the Hölder Inequality,

$$\begin{aligned}\int_\Omega |\nabla(m_{n,R}w)|^2 dx &\leq 2(1+\frac{n}{2})\int_\Omega a(x)(1+|w|)m_{2n,R^2}|w|dx \\ &\leq 2(1+\frac{n}{2})\int_\Omega a(x)(2+2|w|^2 m_{2n,R^2})dx \\ &\leq 4(1+\frac{n}{2})|\Omega|^{\frac{N-2}{N}}\|a\|_{\frac{N}{2}} + 4(1+\frac{n}{2})\int_\Omega a(x)|w|^2 m_{2n,R^2}dx\end{aligned}$$

Therefore, for $A > 0$,

$$\begin{aligned}\int_\Omega |\nabla(m_{n,R}w)|^2 dx &\leq K + 4(1+\frac{n}{2})\Big[A\int_{\{a(x)<A\}} |w|^2 m_{2n,R^2}dx + \int_{\{a(x)\geq A\}} a(x)|w|^2 m_{2n,R^2}dx\Big] \\ &\leq K + 4(1+\frac{n}{2})\Big[A\int_{\{a(x)<A\}} |w|^{2(n+1)}dx + \int_{\{a(x)\geq A\}} a(x)|w|^2 m_{2n,R^2}dx\Big], \quad (1.36)\end{aligned}$$

where $K = 4(1+n/2)|\Omega|^{\frac{N-2}{N}}\|a\|_{\frac{N}{2}}$. Since

$$\int_{\{a(x)\geq A\}} a(x)^{\frac{N}{2}} dx \to 0, \text{ as } A \to \infty,$$

we can choose $A > 0$, sufficiently large, such that

$$4(1+\frac{n}{2})d_{2^*}\left(\int_{\{a(x)\geq A\}} a(x)^{\frac{N}{2}}dx\right)^{\frac{2}{N}} < \frac{1}{2}.$$

Thus, from (1.36) and the Hölder Inequality,

$$\begin{aligned}\int_\Omega |\nabla(m_{n,R}w)|^2dx &\leq K+4(1+\frac{n}{2})\left[A\|w\|_{2(n+1)}^{2(n+1)}+\left(\int_{\{a(x)\geq A\}} a(x)^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\left(\int_\Omega |m_{n,R}w|^{2^*}dx\right)^{\frac{N-2}{N}}\right]\\ &\leq K+4(1+\frac{n}{2})\left[A\|w\|_{2(n+1)}^{2(n+1)}+d_{2^*}\left(\int_{\{a(x)\geq A\}} a(x)^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\int_\Omega|\nabla(m_{n,R}w)|^2dx\right]\\ &\leq K+4(1+\frac{n}{2})A\|w\|_{2(n+1)}^{2(n+1)}+\frac{1}{2}\int_\Omega|\nabla(m_{n,R}w)|^2dx,\end{aligned}$$

which implies that

$$\left(\int_\Omega |m_{n,R}w|^{2^*}dx\right)^{\frac{N-2}{N}} \leq d_{2^*}\int_\Omega |\nabla(m_{n,R}w)|^2dx \leq 2Kd_{2^*}+8(1+\frac{n}{2})Ad_{2^*}\|w\|_{2(n+1)}^{2(n+1)}.$$

Letting $R\to\infty$, the Fatou Lemma we deduce

$$\left(\int_\Omega |w|^{2(n+1)\frac{N}{N-2}}dx\right)^{\frac{N-2}{N}} \leq 2Kd_{2^*}+8(1+\frac{n}{2})Ad_{2^*}\|w\|_{2(n+1)}^{2(n+1)}. \tag{1.37}$$

Considering $n_0 = 0$, $n_i+1 = (n_{i-1}+1)\frac{N}{N-2}$, for $i\in\mathbb{N}$, and using the previous inequality, from (1.29) and $\|v\|\leq K_1$, we have that

$$\begin{aligned}\left(\int_\Omega |w|^{2(n_1+1)}dx\right)^{\frac{N-2}{N}} &= \left(\int_\Omega |w|^{2^*}dx\right)^{\frac{N-2}{N}} \leq 2d_{2^*}K+8d_{2^*}Ad_2\|w\|^2\\ &\leq 2d_{2^*}K+8d_{2^*}Ad_2(|s|\|\varphi_1\|+\|v\|)^2\\ &\leq 2d_{2^*}K+8d_{2^*}Ad_2(K_3\|\varphi_1\|+K_1)^2.\end{aligned}$$

Analogously, from (1.37),

$$\begin{aligned}\left(\int_\Omega |w|^{2(n_2+1)}dx\right)^{\frac{N-2}{N}} &= \left(\int_\Omega |w|^{2(n_1+1)\frac{N}{N-2}}dx\right)^{\frac{N-2}{N}} \leq 2Kd_{2^*}+8(1+\frac{n_1}{2})Ad_{2^*}\|w\|_{2(n_1+1)}^{2(n_1+1)}\\ &= 2Kd_{2^*}+8(1+\frac{1}{N-2})Ad_{2^*}\|w\|_{2^*}^{2^*}.\end{aligned}$$

From now on, (1.32) continues to argue by induction on n_i. □

Remark 1.25. *Arguing as in the proof of Lemma* 1.24 *we can prove that this lemma is valid for* h *satisfying: there exists a constant* $a > 0$*, such that*

$$|h(x,t)| \leq a(1+|t|^{2^*-1}), \text{ for every } t \in \mathbb{R}, \text{ for almost every } x \in \overline{\Omega},$$

instead of (h_3)*.*

We are now in conditions to prove the main goal of this section.

Proof of Theorem 1.5. Suppose that there exist $(\mu_k) \subset \mathbb{R} \setminus \{0\}$, with $|\mu_k| \to 0$, and $(\hat{\lambda}_k) \subset \mathbb{R}$ such that $|\hat{\lambda}_k - \lambda_1| < |\mu_k|/k$ and $u_k = \tau_k\varphi_1 + v_k$, with $\tau_k \in [t_1, t_2]$ and $v_k \in \langle\varphi_1\rangle^\perp$, is solution of Problem (1.1). Let $0 < \mu_0 < \min\{\mu^*, \mu^{**}\}$ and $0 < \nu_0 < \min\{\nu^*, \nu^{**}\}$, where μ^*, μ^{**}, ν^* and μ^{**} are given in Proposition 1.22 and Lemma 1.23, respectively. As $|\mu_k| \to 0$, there exists $k_0 \in \mathbb{N}$ such that $|u_k| \leq \mu_0$, for every $k \geq k_0$. Moreover, fixing $k \geq \max\{k_0, 1/\nu_0\}$ we have that $|\hat{\lambda}_k - \lambda_1| < |u_k|/k < |u_k|\nu_0$. Thus, of Lemma 1.23 and 1.24, u_k is bounded in $L^\infty(\Omega)$, which contradicts Proposition 1.22. □

The next result is another consequence of Proposition 1.22 and Lemmas 1.23 and 1.24.

Proposition 1.26. *Suppose* h *satisfies* (h_3) *and* (h_4)*. Given* $\Lambda \in (0, (\lambda_2 - \lambda_1)/\lambda_2)$ *and* $d > 0$*, there exist positive constants* μ^* *and* ν^* *such that, for every* $\mu \in (0, \mu^*)$ *and* $|\lambda - \lambda_1| < \mu\nu^*$*, Problem* (1.1) *has no weak solution* $u_\mu = t\varphi_1 + v$*, with* $t \in [t_1, t_2]$ *and* $v \in \langle\varphi_1\rangle^\perp$*, satisfying*

$$|I_\mu(u_\mu)| \leq \frac{\Lambda}{2}\|u_\mu\|^2 + d.$$

Proof. It is sufficient to prove that there exist positive constants μ^* and ν^* such that if $\mu \in (0, \mu^*)$, $|\lambda - \lambda_1| < \mu\nu^*$ and $u_\mu = \tau\varphi_1 + v$ is solution from (1.1), with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$, then $\|v\|$ is bounded. Indeed, from (h_3), the Hölder Inequality and (1.8) we get

$$\begin{aligned}
\int_\Omega |H(x,u_\mu)|dx &\leq \int_\Omega f\left[|t\varphi_1 + v| + \frac{1}{2}|t\varphi_1 + v|^2\right]dx \leq K\int_\Omega f(1 + |v| + |v|^2)dx \\
&\leq K\int_\Omega f[(1+|v|)^2 + |v|^2]dx \leq K\int_\Omega f[\ 2(1+|v|^2) + |v|^2]dx \\
&\leq 2K\int_\Omega f dx + 3K\int_\Omega f|v|^2 dx \leq 2K\|f\|_\sigma|\Omega|^{\frac{1}{\sigma'}} + 3K\|f\|_\sigma\|v\|^2_{2\sigma'} \\
&\leq 2K\|f\|_\sigma|\Omega|^{\frac{1}{\sigma'}} + 3K\|f\|_\sigma d^2_{2\sigma'}\|v\|^2,
\end{aligned}$$

where $K = K(\max\{|t_1|, |t_2|\}, \|\varphi\|_\infty)$. Consequently,

$$\begin{aligned}\frac{\lambda_1-\lambda}{2\lambda_1}\|\varphi_1\|^2t^2+\frac{1}{2}\|v\|^2-\frac{\lambda}{2}\|v\|_2^2 &= \frac{1}{2}\|u_\mu\|^2-\frac{\lambda}{2}\|u_\mu\|_2^2 = I_\mu(u_\mu)+\mu\int_\Omega H(x,u_\mu)dx\\ &\leq \frac{\Lambda}{2}\|u_\mu\|^2+d+\mu(2K\|f\|_\sigma|\Omega|^{\frac{1}{\sigma'}}+3K\|f\|_\sigma d_{2\sigma'}^2\|v\|^2).\end{aligned}$$

Thus, as $u_\mu = t\varphi_1 + v$,

$$\begin{aligned}\frac{\lambda_1-\lambda}{2\lambda_1}\|\varphi_1\|^2t^2+\frac{1}{2}\|v\|^2-\frac{\lambda}{2}\|v\|_2^2 &= \frac{\Lambda}{2}t^2\|\varphi_1\|^2+\frac{\Lambda}{2}\|v\|^2\\ &\quad +d+\mu(2K\|f\|_\sigma|\Omega|^{\frac{1}{\sigma'}}+3K\|f\|_\sigma d_{2\sigma'}^2\|v\|^2)\\ &\leq \frac{\Lambda}{2}\|v\|^2+3\mu K\|f\|_\sigma d_{2\sigma'}^2\|v\|^2+d+K_1(\Lambda+\mu),\end{aligned}$$

where $K_1 = K_1(\max\{|t_1|, |t_2|\}, \|\varphi_1\|_\infty, K, \|f\|_\sigma, |\Omega|)$. Hence,

$$\begin{aligned}\frac{1}{2}\left[\frac{\lambda_2-\lambda_1}{\lambda_2}-\Lambda-\frac{\mu\nu^*}{\lambda_2}-6\mu K\|f\|_\sigma d_{2\sigma'}^2\right]\|v\|^2 &= \frac{1}{2}\left[1-\frac{\mu\nu^*+\lambda_1}{\lambda_2}-\Lambda-6\mu K\|f\|_\sigma d_{2\sigma'}^2\right]\|v\|^2\\ &\leq \frac{1}{2}\left[1-\frac{\lambda}{\lambda_2}-\Lambda-6\mu K\|f\|_\sigma d_{2\sigma'}^2\right]\|v\|^2\\ &\leq \frac{\lambda-\lambda_1}{2\lambda_1}\|\varphi_1\|^2t^2+d+K_1(\Lambda+\mu).\end{aligned}$$

Therefore, for $\nu^* > 0$ fixed and $0 < \mu^* < \lambda_2\overline{\lambda}/(\nu^* + 6\mu K\|f\|_\sigma d_{2\sigma'}^2)$, where $\overline{\lambda} = [(\lambda_2 - \lambda_1)/\lambda_2 - \Lambda]$, the result follows. □

1.6 Applications

As mentioned in the introduction of this paper, applications of Theorems 1.2 and 1.3 are given by Problems (1.2) and (1.5). The goal of this section is to prove Propositions 1.6, 1.7, 1.9, 1.10 and give an application of Theorem 1.1.

Proof of Proposition 1.6. Considering $u = \beta^{\frac{1}{p-q}}w$ in (1.2) we obtain

$$-\beta^{\frac{1}{p-q}}\Delta w = \lambda\beta^{\frac{1}{p-q}}w + \beta b_1\beta^{\frac{q}{p-q}}w^q + b_2\beta^{\frac{p}{p-q}}w^p.$$

This implies that

$$\begin{cases} -\Delta w = & \lambda w + \mu(b_1w^q + b_2w^p) \text{ in } \Omega,\\ \quad w = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega,\end{cases} \tag{1.38}$$

where $\mu = \beta^{\frac{p-1}{p-q}}$. Defining $h : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ by $h(x, s) = b_1(x)s^q + b_2(x)s^p$ we have that h

is a Carathéodory function and it satisfies (h_0). Moreover, for p, $q \geq 1$, h satisfies (h_1). Considering the function Φ given in (1.27) we have that

$$\Phi(t) = \int_\Omega h(x, t\varphi_1)\varphi_1 dx = t^q \int_\Omega b_1\varphi_1^{q+1} dx + t^p \int_\Omega b_2\varphi_1^{p+1} dx = r_1 t^q + r_2 t^p,$$

for every $t \in \mathbb{R}^+$.

(i) As $p > q \geq 1$ and $r_1 r_2 < 0$ we have the following cases:

- (i_1) If $r_1 > 0 > r_2$, taking $\delta = \sqrt[p-q]{-r_1/r_2} > 0$ we have that $\Phi(t) > 0$, for every $t \in (0, \delta)$, and $\Phi(t) < 0$, for every $t > \delta$. Therefore, there exist $t_1 \in (0, \delta)$ and $t_2 \in (\delta, \infty)$ such that (h_2^+) is satisfied. Consequently, the result follows from Theorem 1.2, where $\beta_1^* = (\mu^*)^{\frac{p-q}{p-1}}$, with μ^* given by Theorem 1.2.
- (i_2) If $r_2 > 0 > r_1$, arguing as in item (i_1), the result follows by Theorem 1.3. $\square$

To justify the Proposition 1.7 it is sufficient to argue as in Proposition 1.6.

Remark 1.27. *When b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, the nonlinearity $h(x, t) = b_1(x)t^q + b_2(x)t^p$, given in* (1.2)*, satisfies $(\hat{h}_0)$ and $(\hat{h}_1)$, for p, $q \geq 1$. Therefore, arguing as in Proposition* 1.6 *and using Theorem* 1.20*, we may conclude that the solution given by Proposition* 1.6 *is of class $C^{1,\gamma}(\overline{\Omega})$ and it is positive in Ω.*

In the following, we present only the proof of Proposition 1.9 since the argument used in this proof may be easily adapted to prove Proposition 1.10.

Proof of Proposition 1.9. By Theorem 1.2, it is enough to prove that there exist real numbers t_1 and t_2, with $t_1 < t_2$, satisfying (h_2^+). Indeed, remembering that $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g(s) + M \geq 0$, for every $s \leq 0$, we can use Fatou's Lemma and (LL^+) to get

$$\begin{aligned} \liminf_{t\to-\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx &= \liminf_{t\to-\infty} \int_\Omega (f + g(t\varphi_1))\varphi_1 dx = \int_\Omega f\varphi_1 dx + \liminf_{t\to-\infty} \int_\Omega g(t\varphi_1)\varphi_1 dx \\ &\geq \int_\Omega (f + g_i^-)\varphi_1 dx > 0. \end{aligned}$$

Analogously, as $g_s^+ := \limsup_{s\to-\infty} g(s)$ and $M - g(s) \geq 0$, for every $s \geq 0$, Fatou's Lemma and (LL^+) assure us that

$$\begin{aligned} \limsup_{t\to\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx &= \limsup_{t\to\infty} \int_\Omega (f + g(t\varphi_1))\varphi_1 dx = \int_\Omega f\varphi_1 dx + \limsup_{t\to\infty} \int_\Omega g(t\varphi_1)\varphi_1 dx \\ &\leq \int_\Omega (f + g_s^+)\varphi_1 dx < 0. \end{aligned}$$

Consequently, there exist real numbers t_1 and t_2, with $t_1 < 0 < t_2$, such that the condition (h_2^+) is valid for t_1 and t_2. This concludes the proof of the proposition. □

To finish this section, we give an application of Theorem 1.1, with this purpose, we consider $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a polynomial function in the variable t, i.e., h is given by

$$h(x,t) = \sum_{i=0}^{m} \alpha_i(x)t^i, \text{ where } \alpha_i \in L^\sigma(\Omega), \text{ with } \sigma > N/2 \text{ if } N \geq 3 \text{ and } \sigma > 1 \text{ if } N = 1, 2. \tag{1.39}$$

Thus, the function Φ, given in (1.27), is a polynomial function in the variable t of same degree that h. More specifically , $\Phi(t) = \sum_{i=0}^{m} d_i t^i$, with $d_i = \displaystyle\int_\Omega \alpha_i(x)\varphi_1^{i+1}dx$.

In this case, the existence of solutions provided by Problem (1.1) depends on the multiplicity of the roots of Φ:

Proposition 1.28. *Suppose h is a polynomial function in the variable t. If the function Φ has $\tau_1, \dots, \tau_k$ roots of multiplicity odd then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{0,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in \mathbb{R}$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$. Furthermore, if $(\lambda - \lambda_1)/\mu \to 0$, as $\mu \to 0$, the solutions converge to $\tau_i\varphi_1$, as $\mu \to 0$, for $i = 1, \dots, k$.*

Proof. We consider only the case where Φ has k roots of odd multiplicity, with k odd. The case k even may be argued in a similar way.

Since h is a polynomial function of degree m, it is sufficient to prove that there exist $t_i \in \mathbb{R}$, with $t_i < t_{i+1}$, $i = 1, \cdots, k$, satisfying (h_2). Indeed, as observed above, $\Phi(t)$ is a polynomial function of degree m. Therefore, we can express Φ in the form

$$\Phi(t) = (t - \tau_1)^{2n_1+1} \cdots (t - \tau_k)^{2n_k+1}(t - c_1)^{2z_1} \cdots (t - c_l)^{2z_l}p(t),$$

where c_j are the roots of multiplicity even from $\Phi(t)$, with $j = 1, \cdots, l$ and the τ_i are ordered in an increasing manner, n_i, $z_i \in \mathbb{N}$ and $p(t)$ is the product of quadratic polynomials irreducible. Without loss of generality we can assume that $p(t) > 0$. Thus, as k is odd, for $t \notin \{c_1, \dots, c_l\}$, we have that

$$\begin{cases} \Phi(t) < 0 & \text{for every } \quad t < \tau_1, \\ \Phi(t) > 0 & \text{for every } \tau_1 < t < \tau_2, \\ & \vdots \\ \Phi(t) < 0 & \text{for every } \tau_{k-1} < t < \tau_k, \\ \Phi(t) > 0 & \text{for every } \quad t > \tau_k. \end{cases}$$

This implies that there exist t_i, $i = 1, \cdots, k+1$, with $t_1 \in (-\infty, \tau_1)$, $t_{k+1} \in (\tau_k, \infty)$ and $t_i \in (\tau_{i-1}, \tau_i)$, $i = 2, \cdots, k$, satisfying (h_2) such that $c_j \notin (t_i, t_{i+1})$, for every $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Consequently, by Theorem 1.1, there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{0,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.

Finally, we prove that the solutions u_i, $i = 1, \ldots, k$, converge to $\tau_i\varphi_1$, $i = 1, \ldots, k$. Fixing i, by Lemma 1.19, $v_i = v_i(\mu) \to 0$ in $C(\overline{\Omega})$, as $\mu \to 0$, and there exists $\hat{t}_0 \in [t_i, t_{i+1}]$ such that, up to a subsequence, $\hat{t}_i = \hat{t}_i(\mu) \to \hat{t}_0$ and $u_i = u_i(\mu) = \hat{t}_i(\mu)\varphi_1 + v_i(\mu) \to \hat{t}_0\varphi_1$ for every $x \in \Omega$, as $\mu \to 0$. Moreover,

$$0 = -\frac{1}{\mu}\langle I'(\hat{t}_i\varphi_1 + v_i), \varphi_1\rangle = \frac{\lambda - \lambda_1}{\mu}\|\varphi_1\|_2^2\hat{t}_i + \int_\Omega h(x, \hat{t}_i\varphi_1 + v_i)\varphi_1 dx.$$

Therefore, since $(\lambda - \lambda_1)/\mu \to 0$, as $\mu \to 0$, the Lebesgue Dominated Convergence Theorem assures us that

$$\Phi(\hat{t}_0) = \int_\Omega h(x, \hat{t}_0\varphi_1)\varphi_1 dx = \lim_{\mu \to 0}\left[\frac{\lambda - \lambda_1}{\mu}\|\varphi_1\|_2^2\hat{t}_i + \int_\Omega h(x, \hat{t}_i\varphi_1 + v_i)\varphi_1 dx\right] = 0.$$

Consequently, from de (h_2), $\hat{t}_0 \in (t_i, t_{i+1})$. Thus, as τ_i is the only root of Φ in (t_i, t_{i+1}) we conclude that $u_i = \hat{t}_i\varphi_1 + v_i \to \hat{t}_0\varphi_1 = \tau_i\varphi_1$. □

Remark 1.29. *(i) Considering in* (1.39)*,* $\alpha_i \in L^\sigma(\Omega)$*, with* $\sigma > N$ *if* $N \geq 3$ *and* $\sigma > 1$ *if* $N = 1, 2$ *, it follows from Theorem* 1.21 *that the solutions* u_i*, given by Proposition* 1.28*, are of class* $C^{1,\gamma}(\overline{\Omega})$ *and ordered.*

(ii) Given $p(t) = \sum_{i=0}^m d_i t^i$*, with* d_i*,* $i = 1, \ldots, m$*, constants, we may find* $h(x,t) = \sum_{i=0}^m \alpha_i t^i$ *such that* $p(t) = \Phi(t)$*. Indeed, for this it suffices to take* $d_i = \alpha_i \int_\Omega \varphi_1^{i+1} dx$*.*

1.7 Appendix

In this appendix we present the proof of Lemma 1.19 which was omitted in Section 1.4 of this chapter.

Proof of Lemma 1.19. We consider only the case $N \geq 3$ since the cases $N = 1$ and $N = 2$ can be easily adapted using the fact that $H_0^1(\Omega)$ is continuously embedded in $L^p(\Omega)$, for every $p \geq 1$.

As $u = t\varphi_1 + v$ is a weak solution of Problem (1.24) we have that v is weak solution of

$$\begin{cases} -\Delta v = & (\lambda - \lambda_1)t\varphi_1 + \lambda v + \mu h_R(x, t\varphi_1 + v) \text{ in } \Omega, \\ v = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$

Then, using (1.25), the Hölder Inequality and (1.8), we obtain that

$$\begin{aligned} \|v\|^2 &= \lambda\|v\|_2^2 + \mu \int_\Omega h_R(x, t\varphi_1 + v)v dx \\ &\leq \overline{\lambda}\|v\|_2^2 + \mu \int_\Omega f_{R+2}|v| dx \leq \frac{\overline{\lambda}}{\lambda_2}\|v\|^2 + \mu d_{\sigma'}\|f_{R+2}\|_\sigma\|v\|. \end{aligned}$$

This implies that

$$\|v\| \leq \frac{\lambda_2}{\lambda_2 - \overline{\lambda}} d_{\sigma'}\|f_{R+2}\|_\sigma \mu. \tag{1.40}$$

Next, we define $\hat{h}(x) = (\lambda - \lambda_1)t\varphi_1(x) + \lambda v(x) + \mu h_R(x, t\varphi_1(x) + v(x))$, for $x \in \Omega$. Therefore, for $p_0 = \min\{\sigma, 2^*\}$, from (1.25), (1.40) and the fact of $L^\sigma(\Omega)$ is continuously embedded in $L^{p_0}(\Omega)$ we have that

$$\begin{aligned} \|\hat{h}\|_{p_0} &\leq |\lambda - \lambda_1||t|\|\varphi_1\|_{p_0} + \lambda\|v\|_{p_0} + \mu\|h_R(x, t\varphi_1 + v)\|_{p_0} \\ &\leq \mu\nu^* \max\{|t_1|, |t_2|\}|\Omega|^{\frac{1}{p_0}}\|\varphi_1\|_\infty + \overline{\lambda} d_{p_0}\|v\| + \mu\|f_{R+2}\|_{p_0} \leq K_1\mu, \end{aligned} \tag{1.41}$$

where $K_1 = \nu^* \max\{|t_1|, |t_2|\}|\Omega|^{\frac{1}{p_0}}\|\varphi_1\|_\infty + \overline{\lambda} d_{p_0}\lambda_2 d_{\sigma'}\|f_{R+2}\|_\sigma/(\lambda_2 - \overline{\lambda}) + \|f_{R+2}\|_{p_0}$. Thus, of Agmon-Douglis-Nirenberg Theorem, see [1], there exists $K_2 = K_2(|\Omega|, p_0)$ such that

$$\|v\|_{2,p_0} \leq K_2\|\hat{h}\|_{p_0} \leq K_2K_1\mu. \tag{1.42}$$

From now on, to prove the lemma, we consider three cases:
Case 1: If $N/2 < p_0$, the lemma follows from (1.42), taking $q = p_0$.
Case 2: If $N/2 = p_0$, as $\sigma \geq p_0$ we have that $W^{2,p_0}(\Omega)$ is continuously embedded in $L^\sigma(\Omega)$. Therefore $v \in L^\sigma(\Omega)$ and there exists K_3 such that

$$\begin{aligned} \|\hat{h}\|_\sigma &\leq |\lambda - \lambda_1||t|\|\varphi_1\|_\sigma + \lambda\|v\|_\sigma + \mu\|h_R(x, t\varphi_1 + v)\|_\sigma \\ &\leq \mu\nu^* \max\{|t_1|, |t_2|\}|\Omega|^{\frac{1}{\sigma}}\|\varphi_1\|_\infty + \overline{\lambda}K_3\|v\|_{2,p_0} + \mu\|f_{R+2}\|_\sigma. \end{aligned} \tag{1.43}$$

Hence, using (1.42), there exists $K_4 = K_4(\nu^*, \max\{|t_1|, |t_2|\}, |\Omega|, \|\varphi_1\|_\infty, \overline{\lambda}, K_3, K_2, K_1, \sigma, \|f_{R+2}\|_\sigma)$ such that $\|\hat{h}\|_\sigma \leq K_4\mu$. Thus, by the Agmon-Douglis-Nirenberg Theorem, we can find $K_5 = K_5(|\Omega|, \sigma)$ such that

$$\|v\|_{2,\sigma} \leq K_5 \|\hat{h}\|_\sigma \leq K_5 K_4 \mu. \tag{1.44}$$

This proves the lemma in this case, taking $q = \sigma$.

Case 3: If $N/2 > p_0$, $W^{2,p_0}(\Omega)$ is continuously embedded in $L^{p_1}(\Omega)$, where $p_1 = Np_0/(N-2p_0)$. Thus, if $\min\{p_1, \sigma\} = \sigma$, $W^{2,p_0}(\Omega)$ is continuously embedded in $L^\sigma(\Omega)$. Therefore the lemma follows using the same arguments as (1.43) and (1.44).

If $\min\{p_1, \sigma\} = p_1$, working as in (1.43) and using (1.42), there exists $K_6 = K_6(\nu^*, \max\{|t_1|, |t_2|\}, |\Omega|, \|\varphi_1\|_\infty, \overline{\lambda}, d_{p_1}, K_2, K_1, \|f_{R+2}\|_{p_1})$ such that $\|v\|_{2,p_1} \leq K_6 \mu$. If $N/2 \leq p_1$, arguing as in the Cases 1 or 2, the lemma is proved. Otherwise, $W^{2,p_1}(\Omega)$ is continuously embedded in $L^{p_2}(\Omega)$, $p_2 = Np_1/(N-2p_1)$. Moreover,

$$\frac{p_2}{p_1} = \frac{Np_1}{N-2p_1} \frac{N-2p_0}{Np_0} = \frac{p_1}{p_0} \frac{N-2p_0}{N-2p_1} > 1. \tag{1.45}$$

If $\min\{p_2, \sigma\} = \sigma$, $W^{2,p_1}(\Omega)$ is continuously embedded in $L^\sigma(\Omega)$ consequently the lemma follows from (1.43) and (1.44).

If $\min\{p_2, \sigma\} = p_2$, working as in (1.43) and using (1.42), there exists $K_7 = K_7(\nu^*, \max\{|t_1|, |t_2|\}, |\Omega|, \|\varphi_1\|_\infty, \overline{\lambda}, d_{p_2}, K_2, K_1, \|f_{R+2}\|_{p_2})$ such that $\|v\|_{2,p_2} \leq K_7 \mu$. If $N/2 \leq p_2$, the lemma is proved as in the Cases 1 or 2. Otherwise, we can repeat the argument previous k times, for some $k \in \mathbb{N}$ sufficiently large, and obtain p_k such that $\min\{p_k, \sigma\} = \sigma > N/2$. Thus, the lemma follows from (1.43) and (1.44), taking $q = \sigma$.

To ensure the existence of p_k we define $p_m = Np_{m-1}/(N-2p_{m-1})$, with $m = 1, 2, \cdots$. Thus,

$$\frac{p_3}{p_2} = \frac{Np_2}{N-2p_2} \frac{N-2p_1}{Np_1} = \frac{p_2}{p_1} \frac{N-2p_1}{N-2p_2} > \frac{p_2}{p_1}.$$

Hence, from (1.45),

$$\frac{p_3}{p_1} = \frac{p_3}{p_2} \frac{p_2}{p_1} > \left(\frac{p_2}{p_1}\right)^2, \text{ with } \frac{p_2}{p_1} > 1.$$

Iterating this process, we concluded that $p_m > (p_2/p_1)^{m-1} p_1$, with $p_2/p_1 > 1$, for $m = 3, \cdots$. This implies that there exists k, sufficiently large, such that $\min\{p_k, \sigma\} = \sigma$.

□

Chapter

2

A Landesman-Lazer local condition for Semilinear elliptic equations with dependence on the gradient

2.1 Main results

In this chapter we consider the existence and non existence of weak solutions for the following problem

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x, u, \nabla u) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\mu \neq 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying:

$(h_\nabla)_0$ for every $A > 0$, there exists $f_A \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$, such that

$$|h(x, s, \xi)| \leq f_A(x), \text{ for every } |s| \leq A, \ |\xi| \leq A, \text{ for almost every } x \in \overline{\Omega};$$

and

$(h_\nabla)_1$ for every A_1, $A_2 > 0$, there exist $\zeta_1 = \zeta_1(A_1)$, $\zeta_2 = \zeta_2(A_2) \in L^\sigma(\Omega)$, with $\sigma > N$ if

$N \geq 3$ and $\sigma > 2$ if $N = 1, 2$, such that

$$|h(z,s_1,\xi) - h(z,s_2,\xi)| \leq \zeta_1(z)|s_1 - s_2|, \text{ for every } z \in \overline{\Omega}, \ |s_1|,|s_1| \leq A_1, \ |\xi| \leq A_2$$

and

$$|h(z,s,\xi_1) - h(z,s,\xi_2)| \leq \zeta_2(z)|\xi_1 - \xi_2|, \text{ for every } z \in \overline{\Omega}, \ |s| \leq A_1, \ |\xi_1|,|\xi_2| \leq A_2.$$

Our main result establishes the existence of a solution for Problem (2.1) under a local Landesman-Lazer condition. More specifically, considering that

$(h_\nabla)_2$ there exist t_1 and $t_2 \in \mathbb{R}$, $t_1 < t_2$, such that

$$\left[\int_\Omega h(x,t_1\varphi_1,t_1\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x,t_2\varphi_1,t_2\nabla\varphi_1)\varphi_1 dx\right] < 0,$$

where φ_1 is a positive eigenfunction associated to λ_1, we establish:

Theorem 2.1. *Suppose h satisfies $(h_\nabla)_0$, $(h_\nabla)_1$ and $(h_\nabla)_2$. Then there exist positive constants μ^* and ν^* such that, for each $\mu \in (0,\mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (2.1) *has a solution $u_\mu = t\varphi_1 + v$ of class $C^{1,\gamma}(\overline{\Omega})$, with $t \in (t_1,t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

As a direct consequence of Theorem 2.1 we derive the existence of multiple solutions for Problem (2.1). Indeed, assuming that

$(\hat{h}_\nabla)_2$ there exist $k \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1,\ldots,k$, such that

$$\left[\int_\Omega h(x,t_i\varphi_1,t_i\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x,t_{i+1}\varphi_1,t_{i+1}\nabla\varphi_1)\varphi_1 dx\right] < 0.$$

we may state:

Proposition 2.2. *Suppose h satisfies $(h_\nabla)_0$, $(h_\nabla)_1$ and $(\hat{h}_\nabla)_2$. Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (2.1) *has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{1,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i,t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1,\cdots,k$.*

Remark 2.3. *The solution u_μ, given by Theorem* 2.1, *is positive or negative in Ω provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively. Moreover, for $|\mu| > 0$ sufficiently small the solutions of Proposition* 2.2 *are ordered, see Theorems* 2.17 *and Proposition* 2.18.

It is worthwhile mentioning that in several situations the condition $(h_\nabla)_2$ is sufficient for the existence of a solution $u_\mu = t\varphi_1 + v$, $t \in (t_1, t_2)$, $v \in \langle\varphi_1\rangle^\perp$. Next we establish the non existence of solutions to the Problem (2.1) when the hypothesis $(h_\nabla)_2$ is not valid. Indeed, if h satisfies

$(h_\nabla)_3$ there exists $f \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$, such that

$$|h(x,s,\xi)| \leq f(x)(1+|s|+|\xi|), \text{ for every } s \in \mathbb{R},\ \xi \in \mathbb{R}^N, \text{ for almost every } x \in \overline{\Omega},$$

and

$(h_\nabla)_4$ there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \neq 0, \quad \text{for every } t \in [t_1, t_2],$$

we may state:

Theorem 2.4. *Suppose h satisfies $(h_\nabla)_3$ and $(h_\nabla)_4$. Then there exist positive constants μ^* and ν^* such that, for each $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem* (2.1) *has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$.*

As a first application of Theorems 2.1 we consider the existence of a solution for the following problem

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x) u^{q_1}|\nabla u|^{q_2} + b_2(x)u^{p_1}|\nabla u|^{p_2} \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda \in (0, \overline{\lambda})$, $\overline{\lambda} < \lambda_2$, λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; $\beta > 0$ is a real parameter and b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$.

Setting

$$r_1 := \int_\Omega b_1 \varphi_1^{q_1+1}|\nabla\varphi_1|^{q_2} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p_1+1}|\nabla\varphi_1|^{p_2} dx$$

we may present the following result:

Proposition 2.5. *Suppose $p = p_1 + p_2$, $q = q_1 + q_2$, p_1, p_2, q_1, $q_2 \geq 1$, $p > q$ and $r_1 r_2 < 0$. Then there exist positive constants β_1^* and ν_1^* such that Problem* (2.2) *has a solution of class $C^{1,\gamma}(\overline{\Omega})$, for every $\beta \in (0, \beta_1^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}}\nu_1^*$.*

As a second application of Theorem 2.1 is given by the following problem

$$\begin{cases} -\Delta u = & \lambda u + b(x)u^{p_1}|\nabla u|^{p_2} \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda < \lambda_1$, λ_1 is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$; p_1, $p_2 > 0$ and $b \in L^\sigma(\Omega)$, with $\sigma > N/2$ if $N \geq 3$ and $\sigma > 1$ if $N = 1, 2$, satisfying

$$\int_\Omega b(x)\varphi_1^{p_1+1}|\nabla\varphi_1|^{p_2}dx > 0. \tag{2.4}$$

Proposition 2.6. *Suppose b satisfies* (2.4)*, with p_1, $p_2 \geq 1$, then there exists $\underline{\lambda}$ such that Problem* (2.3) *has a positive solution, for every $\underline{\lambda} < \lambda < \lambda_1$.*

Motivated by Landesman and Lazer [35] and Shaw [46], we also consider the existence of solution for the following problem

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u) + \Gamma(x, u, \nabla u)) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; λ and μ are as in (2.1) and $f \in L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1, 2$. We also suppose that

(g_1) $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function Satisfying there exists $M > 0$ such taht

$$g(s) \geq -M \text{ if } s \leq 0 \text{ and } g(s) \leq M \text{ if } s \geq 0;$$

and

(Γ_1) $\Gamma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz function satisfiying there exists $\alpha > 0$ such taht, for every $x \in \Omega$ and $\xi \in \mathbb{R}$,

$$\Gamma(x, s, \xi) \geq -\alpha \text{ if } s \leq 0 \text{ and } \Gamma(x, s, \xi) \leq \alpha \text{ if } s \geq 0.$$

Considering by $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g_s^+ := \limsup_{s\to+\infty} g(s)$ and assuming

(LL_∇)
$$\int_\Omega (f + g_i^- - \alpha)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+ + \alpha)\varphi_1 dx.$$

We may state

Proposition 2.7. *Suppose* (g_1)*,* (Γ_1) *and* (LL_∇) *are satisfied. Then there exist positive constants* μ^* *and* ν^* *such that, for every* $\mu \in (0, \mu^*)$ *and* $|\lambda - \lambda_1| < \mu\nu^*$*, Problem* (2.5) *has a solution* $u_\mu = t\varphi_1 + v$ *of class* $C^{1,\gamma}(\overline{\Omega})$*, with* $t \in \mathbb{R}$ *and* $v \in \langle\varphi_1\rangle^\perp$*.*

Finally, we point out that, as an application of Proposition 2.2, we provide a result on the existence of multiple solution for Problem (2.1) when h, see Proposition 2.20, is given by

$$h(x,t,\xi) = \sum_{i,j=0}^{m} \alpha_{ij}(x)t^i|\xi|^j, \text{ where } \alpha_{ij} \in L^\sigma(\Omega), \text{ with } \sigma > N \text{ if } N \geq 3 \text{ and } \sigma > 2 \text{ if } N = 1,2.$$

This chapter is organized as follows: in the Section 2.2 we prove the existence of a solution for Problem (2.1) with h Carathéodory, bounded with respect to $L^\sigma(\Omega)$ and satisfying $(h_\nabla)_1$ and $(h_\nabla)_2$. The Section 2.3 is dedicated to the proof of Theorem 2.1 and Proposition 2.2. In the Section 2.4 we study the non-solubility of Problem (2.1). Finally, in the Section 2.5 we give application of Theorem 2.1 and Proposition 2.2.

2.2 A particular case of Problem (2.1)

In this section we prove a version of Theorem 2.1 when the function h satisfies $(h_\nabla)_1$, $(h_\nabla)_2$ and it is bounded with respect to $L^\sigma(\Omega)$, with $\sigma > N$ if $N \geq 3$ and $\sigma > 2$ if $N = 1,2$. More specifically we assume that there exists $f \in L^\sigma(\Omega)$ such that

$$|h(x,s,\xi)| \leq f(x), \text{ for every } s \in \mathbb{R},\ \xi \in \mathbb{R}^N, \text{ for almost every } x \in \overline{\Omega} \tag{2.6}$$

Such hypotheses allow us to prove that Problem (2.1) has a solution.

Theorem 2.8. *Suppose* h *satisfies* $(h_\nabla)_1$*,* $(h_\nabla)_2$ *and* (2.6)*. Then there exist positive constants* μ^* *and* ν^* *such that, for each* $0 < |\mu| < \mu^*$ *and* $|\lambda - \lambda_1| < |\mu|\nu^*$*, Problem* (2.1) *has a solution* $u_\mu = t\varphi_1 + v$ *of class* $C^{1,\gamma}(\overline{\Omega})$*, with* $t \in (t_1, t_2)$ *and* $v \in \langle\varphi_1\rangle^\perp$*.*

Inspired by the Lyapunov-Schmidt Reduction Method, in our proof of Theorem 2.8 we solve Problem (2.1) on $\langle\varphi_1\rangle^\perp$, i.e., for every $t \in [t_1, t_2]$ fixed, we consider the following problem

$$\begin{cases} -\Delta v = & \lambda v + \mu h(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla v) \text{ in } \Omega, \\ & v \in \langle\varphi_1\rangle^\perp. \end{cases} \tag{2.7}$$

Here we note that $v \in \langle\varphi_1\rangle^\perp$ is solution of Problem (2.7) if it satisfies

$$\int_\Omega \nabla v \nabla z dx = \lambda \int_\Omega vz dx + \mu \int_\Omega h(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla v) z dx, \text{ for every } z \in \langle\varphi_1\rangle^\perp.$$

Remark 2.9. *As h depends on ∇v, Problem (2.7) is not variational which does not allow us to treat it directly by the critical point theory.*

By above observation we associate with Problem (2.7), a family of semilinear elliptic problems that do not depend on the gradient of the solution, see [25]. More specifically, for each $w \in \langle \varphi_1 \rangle^\perp$, we consider the problem

$$\begin{cases} -\Delta v = & \lambda v + \mu h(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla w) \text{ in } \Omega, \\ \quad v \in \langle \varphi_1 \rangle^\perp. \end{cases} \tag{2.8}$$

The functional associated with Problem (2.8), $J^t_{\mu,w} : \langle \varphi_1 \rangle^\perp \to \mathbb{R}$, is given by

$$J^t_{\mu,w}(v) = \frac{1}{2}\|v\|^2 - \frac{\lambda}{2}\|v\|_2^2 - \mu \int_\Omega H(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla w)dx,$$

where $H(x, \tau, \xi) = \displaystyle\int_0^\tau h(x, s, \xi)ds$.

The following two lemmas ensure the existence of a solution for Problem (2.8). Their proofs are analogous to Lemmas 1.13 and 1.14 of Chapter 1.

Lemma 2.10. *Given $\rho > 0$, there exist positive numbers α and μ_1 such that, for every $t \in [t_1, t_2]$, $w \in \langle \varphi_1 \rangle^\perp$ and for $0 < |\mu| < \mu_1$,*

$$J^t_{\mu,w}(v) \geq \alpha + J^t_{\mu,w}(0), \text{ for every } v \in \langle \varphi_1 \rangle^\perp, \|v\| = \rho.$$

Proof. From (2.6), the Mean Value Theorem, the Hölder Inequality and the Sobolev Embedding Theorem, it follows that

$$\begin{aligned} \left| \int_\Omega [H(x, t\varphi_1 + v, t\nabla\varphi_1 + w) - H(x, t\varphi_1, t\nabla\varphi_1 + w)]dx \right| &\leq \int_\Omega f|v|dx \leq \|f\|_\sigma \|v\|_{\sigma'} \\ &\leq d_{\sigma'}\|f\|_\sigma \|v\|, \end{aligned}$$

where we use the fact that

$$H_0^1(\Omega) \text{ is continuously embedded in } L^{2\sigma'}(\Omega), \text{ where } \sigma' = \frac{\sigma}{\sigma - 1}. \tag{2.9}$$

Therefore, since we are considering $\lambda < \overline{\lambda} < \lambda_2$, for every $v \in \langle \varphi_1 \rangle^\perp$, with $\|v\| = \rho$,

$$\begin{aligned} J^t_{\mu,w}(v) - J^t_{\mu,w}(0) &\geq \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\|v\|^2 - \mu \int_\Omega [H(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla w) - H(x, t\varphi_1, t\nabla\varphi_1 + \nabla w)]dx \\ &\geq \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\|v\|^2 - |\mu| d_{\sigma'}\|f\|_\sigma \|v\| = \frac{\lambda_2 - \overline{\lambda}}{2\lambda_2}\rho^2 - |\mu| d_{\sigma'}\|f\|_\sigma \rho. \end{aligned}$$

To complete the proof of the lemma is sufficies to choose $\alpha = (\lambda_2 - \overline{\lambda})\rho^2/4\lambda_2$ and $\mu_1 < \alpha/d_{\sigma'}\|f\|_\sigma \rho$. □

Fixing $\rho > 0$, we define

$$m(t, w) := \inf\{J^t_{\mu,w}(v) : \|v\| \leq \rho, v \in \langle\varphi_1\rangle^\perp\} \tag{2.10}$$

and

$$\beta(t, w) := \inf\{J^t_{\mu,w}(v) : \|v\| \leq \rho, v \in \langle\varphi_1\rangle^\perp\}. \tag{2.11}$$

Lemma 2.10 and (2.11) allow us to conclude, for every $t \in [t_1, t_2]$, $w \in \langle\varphi_1\rangle^\perp$, that

$$m(t, w) \leq J^t_{\mu,w}(0) < \alpha + J^t_{\mu,w}(0) \leq \beta(t, w), \text{ for every } \mu \in (0, \mu_1). \tag{2.12}$$

Lemma 2.11. *Suppose* $0 < |\mu| < \mu_1$. *Then, for every* $t \in [t_1, t_2]$ *and* $w \in \langle\varphi_1\rangle^\perp$, *Problem* (2.8) *has a solution* $v^t_{\mu,w} \in \langle\varphi_1\rangle^\perp$, *with* $\|v^t_{\mu,w}\| < \rho$, *such that* $J^t_{\mu,w}(v^t_{\mu,w}) = m(t, w)$.

Proof. Let $(v_n) \subset \overline{B_\rho(0)} \cap \langle\varphi_1\rangle^\perp$ such that $J^t_{\mu,w}(v_n) \to m(t, w)$. Then, taking a subsequence if necessary, there exists $v^t_{\mu,w} \in \overline{B_\rho(0)} \cap \langle\varphi_1\rangle^\perp$ such that

$$\begin{cases} v_n \to v^t_{\mu,w} \text{ weakly in } H^1_0(\Omega), \\ v_n \to v^t_{\mu,w} \text{ weakly in } L^2(\Omega), \\ v_n \to v^t_{\mu,w} \quad \text{a.e. in } \Omega, \\ |v_n| \leq \eta \in L^{\sigma'}(\Omega) \text{ a.e. in } \Omega. \end{cases} \tag{2.13}$$

From (2.6), (2.13) and the Lebesgue Dominated Convergence Theorem we get

$$\int_\Omega H(x, t\varphi_1 + v_n, t\nabla\varphi_1 + \nabla w)dx \to \int_\Omega H(x, t\varphi_1 + v^t_{\mu,w}, t\nabla\varphi_1 + \nabla w)dx. \tag{2.14}$$

Since $v^\tau_\mu \in \overline{B_\rho(0)} \cap \langle\varphi_1\rangle^\perp$, from (2.10), (2.12), (2.13) and (2.14), it follows that

$$\begin{aligned} \beta(t, w) &> m(t, w) = \lim_{n\to\infty} J^t_{\mu,w}(v_n) \\ &= \lim_{n\to\infty} \left\{ \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2}\|v_n\|_2^2 - \mu \int_\Omega H(x, t\varphi_1 + v_n, t\nabla\varphi_1 + \nabla w)dx \right\} \\ &\geq \frac{1}{2}\|v^t_{\mu,w}\|^2 - \frac{\lambda}{2}\|v^t_{\mu,w}\|_2^2 - \mu \int_\Omega H(x, t\varphi_1 + v^t_{\mu,w}, t\nabla\varphi_1 + \nabla w)dx \\ &= J^t_{\mu,w}(v^t_{\mu,w}) \geq m(t, w). \end{aligned}$$

This implies that $J^t_{\mu,w}(v^t_{\mu,w}) = m_\tau$, with $\|v^t_{\mu,w}\| < \rho$. □

Next we verify that $v^t_{\mu,w}$ is bounded in $C^{1,\gamma}(\overline{\Omega})$ by a positive constant independent of t, w and μ.

Lemma 2.12. *Given $w \in \langle \varphi_1 \rangle^{\perp}$ the solution $v^t_{\mu,w}$ of Problem (2.8) is of class $C^{1,\gamma}(\overline{\Omega})$. Moreover there exists a positive constant D, independent of $t \in [t_1, t_2]$, $w \in \langle \varphi_1 \rangle^{\perp}$ and $0 < |\mu| < \mu_1$, such that $\|v^t_{\mu,w}\|_{C^{1,\gamma}} \leq D$.*

Proof. We present only the proof for the case $N = 3$. The cases $N = 1$ or $N = 2$ follows from the fact that $H^1_0(\Omega)$ is continuously embedded in $L^p(\Omega)$, for every $p \geq 1$.

We define $\hat{u}(x) := \hat{u}^t_{\mu,w}(x) = s\varphi_1(x) + v^t_{\mu,w}(x)$ where

$$s = \frac{\mu}{\|\varphi_1\|^2} \int_\Omega h(x, t\varphi_1 + v^t_{\mu,w}, t\nabla\varphi_1 + \nabla w)\varphi_1 dx. \tag{2.15}$$

As $v^t_{\mu,w}$ is solution from, (2.8), for every $v \in \langle \varphi_1 \rangle^{\perp}$, we have that

$$\int_\Omega \nabla\hat{u}\nabla v dx = \lambda \int_\Omega v^t_{\mu,w} v dx + \mu \int_\Omega h(x, t\varphi_1 + v^t_{\mu,w}, t\nabla\varphi_1 + \nabla w) v dx. \tag{2.16}$$

Moreover, from (2.15) we obtain

$$\int_\Omega \nabla\hat{u}\nabla\varphi_1 dx = s\|\varphi_1\|^2 = \mu \int_\Omega h(x, t\varphi_1 + v^t_{\mu,w}, t\nabla\varphi_1 + \nabla w)\varphi_1 dx. \tag{2.17}$$

Considering that $\hat{h}(x, \hat{u}) = \lambda(\hat{u} - s\varphi_1) + \mu h(x, t\varphi_1 + \hat{u} - s\varphi_1, t\nabla\varphi_1 + \nabla w)$, we also have

$$\hat{h}(x, \hat{u}) = \lambda v^t_{\mu,w} + \mu h(x, t\varphi_1 + v^t_{\mu,w}, t\nabla\varphi_1 + \nabla w). \tag{2.18}$$

From (2.16), (2.17) and (2.18), we conclude that

$$\int_\Omega \nabla\hat{u}\nabla z dx = \int_\Omega \hat{h}(x, \hat{u}) z dx, \text{ for every } z \in H^1_0(\Omega). \tag{2.19}$$

On the other hand, in view of (2.15) and (2.6), we have that

$$|s| \leq \frac{|\mu|}{\|\varphi_1\|^2} \int_\Omega f\varphi_1 dx \leq \frac{\mu_1}{\|\varphi_1\|^2} |\Omega|^{\frac{1}{\sigma'}} \|\varphi_1\|^{\sigma'}_\infty \|f\|_\sigma. \tag{2.20}$$

Therefore, from (2.6), (2.18) and (2.20), it follows that

$$\begin{aligned} |\hat{h}(x, \hat{u})| &\leq \lambda(|\hat{u}| + |s||\varphi_1|) + |\mu| f \leq \overline{\lambda}(|\hat{u}| + \frac{\mu_1}{\|\varphi_1\|^2} |\Omega|^{\frac{1}{\sigma'}} \|\varphi_1\|^{\sigma'}_\infty \|f\|_\sigma \|\varphi_1\|_\infty) + \mu_1 f \\ &\leq a(x)(1 + |\hat{u}|). \end{aligned} \tag{2.21}$$

where $a(x) := \max\{\overline{\lambda}, (\overline{\lambda}\mu_1|\Omega|^{\frac{1}{\sigma'}}\|f\|_\sigma\|\varphi_1\|_\infty^{\sigma'+1}/\|\varphi_1\|^2) + \mu_1 f(x)\}$ belongs to $L^{\frac{N}{2}}(\Omega)$, since $f \in L^\sigma(\Omega)$. Thus, from (2.19), (2.21) and of Lemma B.3 in [48], it follows that $\hat{u} \in L^q(\Omega)$, for every $q \in [1, \infty)$.

We claim that, given $k \in \mathbb{N}$, there exists $K = K(k) > 0$, independent of $t \in [t_1, t_2]$, w and μ, such that

$$\left(\int_\Omega |\hat{u}|^{2(k+1)} dx\right)^{\frac{N-2}{N}} \leq K. \tag{2.22}$$

Assuming that the claim is true, we fix $k \in \mathbb{N}$ such that $2(k+1) > \sigma$. So, From (2.6), (2.18), (2.20) and the continuously embedded of $L^{2(k+1)}(\Omega)$ on $L^\sigma(\Omega)$, there exists K_1 such that

$$\begin{aligned}
\|\hat{h}\|_\sigma &\leq \overline{\lambda}(\|\hat{u}\|_\sigma + |s|\|\varphi_1\|_\sigma) + \mu\|h\|_\sigma \leq \overline{\lambda}[K_1\|\hat{u}\|_{2(k+1)} + \frac{\mu_1}{\|\varphi_1\|^2}|\Omega|^{\frac{1}{\sigma'}}\|\varphi_1\|_\infty^{\sigma'}\|f\|_\sigma\|\varphi_1\|_\sigma] + \mu_1\|f\|_\sigma \\
&\leq \overline{\lambda}[K_1 K^{\frac{N}{2(N-2)(k+1)}} + \frac{\mu_1}{\|\varphi_1\|^2}|\Omega|^{\frac{1}{\sigma'}}\|\varphi_1\|_\infty^{\sigma'}\|f\|_\sigma\|\varphi_1\|_\sigma] + \mu_1\|f\|_\sigma \\
&\leq \overline{\lambda}K_1(K^{\frac{N}{2(N-2)}} + 1) + \overline{\lambda}\frac{\mu_1}{\|\varphi_1\|^2}|\Omega|^{\frac{1}{\sigma'}}\|\varphi_1\|_\infty^{\sigma'}\|f\|_\sigma\|\varphi_1\|_\sigma + \mu_1\|f\|_\sigma.
\end{aligned}$$

Therefore, by Agmon-Douglis-Nirenberg Theorem and the Sobolev Embedding Theorem, there exist K_2 and K_3 such that

$$\begin{aligned}
\|\hat{u}\|_{C^{1,\gamma}} &\leq K_3\|\hat{u}\|_{2,\sigma} \leq K_3K_2\|\hat{h}\|_\sigma \\
&\leq K_3K_2\left[\overline{\lambda}K_1(K^{\frac{N}{2(N-2)}} + 1) + \overline{\lambda}\frac{\mu_1}{\|\varphi_1\|^2}|\Omega|^{\frac{1}{\sigma'}}\|\varphi_1\|_\infty^{\sigma'}\|f\|_\sigma\|\varphi_1\|_\sigma + \mu_1\|f\|_\sigma\right].
\end{aligned}$$

Consequently, from (2.20), there exists K_4 such that

$$\|v_{\mu,w}^t\|_{C^{1,\gamma}} \leq \|\hat{u}\|_{C^{1,\gamma}} + |s|\|\varphi_1\|_{C^{1,\gamma}} \leq K_4.$$

In order, to conclude the proof of this lemma it remains to verify (2.22). With this objective, for $k \in \mathbb{N}$ and $R > 1$, we consider, for each $x \in \Omega$,

$$m_{k,R}(x) := \min\{|\hat{u}(x)|^k, R\} = \begin{cases} |\hat{u}(x)|^k \text{ if } |\hat{u}|^k < R, \\ R \text{ if } |\hat{u}(x)|^k \geq R, \end{cases}$$

and

$$m_{2k,R^2}(x) := (m_{k,R}(x))^2 = \min\{|\hat{u}(x)|^{2k}, R^2\} = \begin{cases} |\hat{u}|^{2k} \text{ if } |\hat{u}(x)|^k < R, \\ R^2 \text{ if } |\hat{u}(x)|^k \geq R. \end{cases}$$

Thus, for each $x \in \Omega$,

$$|m_{k,R}(x)| \leq R,\ |m_{2k,R^2}(x)| \leq R^2 \text{ and } (1+|\hat{u}|)|\hat{u}|m_{2k,R^2} \leq 2 + 2|\hat{u}|^2 m_{2k,R^2}. \tag{2.23}$$

Defining $z(x) := m_{2k,R^2}(x)\hat{u}(x)$, for each $x \in \Omega$, we obtain

$$\nabla z = m_{2k,R^2}\nabla\hat{u} + 2k|\hat{u}|^{2k}\nabla\hat{u}\ \chi_{\{|\hat{u}|^k<R\}}.$$

Consequently, from (2.19) and (2.21),

$$\begin{aligned}\int_\Omega m_{2k,R^2}|\nabla\hat{u}|^2dx + 2k\int_{\{|\hat{u}|^k<R\}}|\hat{u}|^{2k}|\nabla\hat{u}|^2dx &= \int_\Omega \nabla\hat{u}\nabla z dx = \int_\Omega \hat{h}(x,\hat{u})z dx\\ &\leq \int_\Omega a(x)(1+|\hat{u}|)m_{2k,R^2}|\hat{u}|dx. \end{aligned}\tag{2.24}$$

On the other hand, as $\nabla(m_{k,R}\hat{u}) = m_{k,R}\nabla\hat{u} + k|\hat{u}|^k\nabla\hat{u}\ \chi_{\{|\hat{u}|^k<R\}}$,

$$\begin{aligned}&\int_\Omega |\nabla(m_{k,R}\hat{u})|^2dx \leq 2\int_\Omega m_{2k,R^2}|\nabla\hat{u}|^2dx + 2k^2\int_{\{|\hat{u}|^k<R\}}|\hat{u}|^{2k}|\nabla\hat{u}|^2dx\\ &\leq 2\int_\Omega m_{2k,R^2}|\nabla\hat{u}|^2dx + k\int_\Omega m_{2k,R^2}|\nabla\hat{u}|^2dx + 4k\int_{\{|\hat{u}|^k<R\}}|\hat{u}|^{2k}|\nabla\hat{u}|^2dx + 2k^2\int_{\{|\hat{u}|^k<R\}}|\hat{u}|^{2k}|\nabla\hat{u}|^2dx\\ &= 2(1+\frac{k}{2})\int_\Omega m_{2k,R^2}|\nabla\hat{u}|^2dx + 4k(1+\frac{k}{2})\int_{\{|\hat{u}|^k<R\}}|\hat{u}|^{2k}|\nabla\hat{u}|^2dx\\ &= 2(1+\frac{k}{2})\Big[\int_\Omega m_{2k,R^2}|\nabla\hat{u}|^2dx + 2k\int_{\{|\hat{u}|^k<R\}}|\hat{u}|^{2k}|\nabla\hat{u}|^2dx\Big].\end{aligned}$$

Therefore, from (2.24) and (2.23), for every $M > 0$ we have that

$$\begin{aligned}\int_\Omega |\nabla(m_{k,R}\hat{u})|^2dx &\leq 2(1+\frac{k}{2})\int_\Omega a(x)(1+|\hat{u}|)m_{2k,R^2}|\hat{u}|dx \leq 2(1+\frac{k}{2})\int_\Omega a(x)(2+2|\hat{u}|^2m_{2k,R^2})dx\\ &\leq 4(1+\frac{k}{2})|\Omega|^{\frac{N-2}{N}}\|a\|_{\frac{N}{2}} + 4(1+\frac{k}{2})\int_\Omega a(x)|\hat{u}|^2m_{2k,R^2}dx\\ &\leq K_5 + 4(1+\frac{k}{2})\Big[M\int_{\{a(x)<M\}}|\hat{u}|^2m_{2k,R^2}dx + \int_{\{a(x)\geq M\}}a(x)|\hat{u}|^2m_{2k,R^2}dx\Big]\\ &\leq K_5 + 4(1+\frac{k}{2})\Big[M\int_{\{a(x)<M\}}|\hat{u}|^{2(k+1)}dx + \int_{\{a(x)\geq M\}}a(x)|\hat{u}|^2m_{2k,R^2}dx\Big],\end{aligned}\tag{2.25}$$

where $K_5 = 4(1+k/2)|\Omega|^{\frac{N-2}{N}}\|a\|_{\frac{N}{2}}$. Since

$$\int_{\{a(x)\geq M\}} a(x)^{\frac{N}{2}}dx \to 0,\ \text{ as } M\to\infty,$$

We may find $M > 0$ such that

$$4(1+\frac{k}{2})d_{2^*}\Big(\int_{\{a(x)\geq M\}} a(x)^{\frac{N}{2}}dx\Big)^{\frac{2}{N}} < \frac{1}{2}.$$

Thus, from (2.25) and the Hölder Inequality,

$$\begin{aligned}\int_\Omega |\nabla(m_{k,R}\hat{u})|^2 dx &\leq K_5 + 4(1+\frac{k}{2})\Big[M\|\hat{u}\|_{2(k+1)}^{2(k+1)} + \Big(\int_{\{a(x)\geq M\}} a(x)^{\frac{N}{2}}dx\Big)^{\frac{2}{N}}\Big(\int_\Omega |m_{k,R}\hat{u}|^{2^*}dx\Big)^{\frac{N-2}{N}}\Big]\\ &\leq K_5 + 4(1+\frac{k}{2})\Big[M\|\hat{u}\|_{2(k+1)}^{2(k+1)} + d_{2^*}\Big(\int_{\{a(x)\geq M\}} a(x)^{\frac{N}{2}}dx\Big)^{\frac{2}{N}}\int_\Omega |\nabla(m_{k,R}\hat{u})|^2 dx\Big]\\ &\leq K_5 + 4(1+\frac{k}{2})M\|\hat{u}\|_{2(k+1)}^{2(k+1)} + \frac{1}{2}\int_\Omega |\nabla(m_{k,R}\hat{u})|^2 dx.\end{aligned}$$

This implies that

$$\Big(\int_\Omega |m_{k,R}\hat{u}|^{2^*}dx\Big)^{\frac{N-2}{N}} \leq d_{2^*}\int_\Omega |\nabla(m_{k,R}\hat{u})|^2 dx \leq 2K_5 d_{2^*} + 8(1+\frac{k}{2})Md_{2^*}\|\hat{u}\|_{2(k+1)}^{2(k+1)}.$$

Next, letting $R \to \infty$ and invoking the Fatou Lemma, we deduce

$$\Big(\int_\Omega |\hat{u}|^{2(k+1)\frac{N}{N-2}}dx\Big)^{\frac{N-2}{N}} \leq 2K_5 d_{2^*} + 8(1+\frac{k}{2})Md_{2^*}\|\hat{u}\|_{2(k+1)}^{2(k+1)}. \tag{2.26}$$

Considering $k_0 = 0$, $k_i + 1 = (k_{i-1}+1)\frac{N}{N-2}$, para $i \in \mathbb{N}$, and using the previous inequality, from (2.20) and Lemma 2.11, we have that

$$\begin{aligned}\Big(\int_\Omega |\hat{u}|^{2(k_1+1)}dx\Big)^{\frac{N-2}{N}} &= \Big(\int_\Omega |\hat{u}|^{2^*}dx\Big)^{\frac{N-2}{N}} \leq 2d_{2^*}K_5 + 8d_{2^*}Md_2\|\hat{u}\|^2\\ &\leq 2d_{2^*}K_5 + 8d_{2^*}Md_2(|s|\|\varphi_1\| + \|v_{\mu,w}^t\|)^2\\ &\leq 2d_{2^*}K_5 + 8d_{2^*}Md_2 2\Big[\frac{\mu_1^2}{\|\varphi_1\|^2}|\Omega|^{\frac{2}{\sigma'}}\|\varphi_1\|_\infty^{2\sigma'}\|f\|_\sigma^2 + \rho^2\Big].\end{aligned}$$

Next, using (2.26), one more time, get

$$\begin{aligned}\Big(\int_\Omega |\hat{u}|^{2(k_2+1)}dx\Big)^{\frac{N-2}{N}} &= \Big(\int_\Omega |\hat{u}|^{2(k_1+1)\frac{N}{N-2}}dx\Big)^{\frac{N-2}{N}} \leq 2K_5 d_{2^*} + 8(1+\frac{k_1}{2})Md_{2^*}\|\hat{u}\|_{2(k_1+1)}^{2(k_1+1)}\\ &= 2K_5 d_{2^*} + 8(1+\frac{1}{N-2})Md_{2^*}\|\hat{u}\|_{2^*}^{2^*}.\end{aligned}$$

From now on, noting that $K_i + 1 = (k_0+1)[N/(N-2)]^i \to \infty$, as $i \to \infty$, we have that

(2.22) continues to argue by induction on k_i. □

Now we may establish the existence of a solution for Problem (2.7). The proof of the next result uses the iterative technique given by [25].

Theorem 2.13. *Suppose h satisfies $(h_\nabla)_1$ and (2.6). Given $t \in [t_1, t_2]$, there exists $\mu_2 \in (0, \mu_1)$, independent of $t \in [t_1, t_2]$, such that Problem (2.7) has a unique solution $v_\mu^t \in \langle \varphi_1 \rangle^\perp$, for every $0 < |\mu| < \mu_2$. Moreover the solution is of class $C^{1,\gamma}(\overline{\Omega})$ and there exists a positive constant D, independent of $t \in [t_1, t_2]$ and μ, such that $\|v_\mu^t\|_{C^{1,\gamma}} \leq D$.*

Proof. First we prove that Problem (2.7) has at least a solution. Indeed, given $v_0 \in \langle \varphi_1 \rangle^\perp$, by Lemma 2.11, there exists $v_{\mu,1}^t \in \langle \varphi_1 \rangle^\perp$ solution of the problem

$$\begin{cases} -\Delta v_1^t = & \lambda v_1^t + \mu h(x, t\varphi_1 + v_1^t, t\nabla \varphi_1 + \nabla v_0) \text{ in } \Omega, \\ \quad v_1^t \in \langle \varphi_1 \rangle^\perp. \end{cases}$$

Similarly, Lemma 2.11 also ensures the existence of $v_{\mu,2}^t \in \langle \varphi_1 \rangle^\perp$ solution of the problem

$$\begin{cases} -\Delta v_2^t = & \lambda v_2^t + \mu h(x, t\varphi_1 + v_2^t, t\nabla \varphi_1 + \nabla v_1^t) \text{ in } \Omega, \\ \quad v_2^t \in \langle \varphi_1 \rangle^\perp. \end{cases}$$

Arguing recursively, we obtain a sequence $(v_{\mu,n}^t) \subset \langle \varphi_1 \rangle^\perp$ such that every $v_{\mu,n}^t$, $n \in \mathbb{N}$, is a solution of the problem

$$\begin{cases} -\Delta v_n^t = & \lambda v_n^t + \mu h(x, t\varphi_1 + v_n^t, t\nabla \varphi_1 + \nabla v_{n-1}^t) \text{ in } \Omega, \\ \quad v_n^t \in \langle \varphi_1 \rangle^\perp. \end{cases} \tag{2.27}$$

By Lemma 2.12 there exist D_1 and $D_2 > 0$, both independent of $t \in [t_1, t_2]$ and $\mu \in (0, \mu_1)$, such that

$$\|v_{\mu,n}^t\|_\infty \leq D_1 \text{ and } \|\nabla u_{\mu,n}^t\|_\infty \leq D_2, \text{ for every } n \geq 1.$$

Therefore, from $(h_\nabla)_1$, there exist ζ_1 and $\zeta_2 \in L^\sigma(\Omega)$ such that, for every $t \in [t_1, t_2]$, $\mu \in (0, \mu_1)$ and $n \geq 1$,

$$\begin{aligned} |h(x, t\varphi_1 + v_{\mu,n+1}^t, t\nabla\varphi_1 + \nabla v_{\mu,n}^t) - h(x, t\varphi_1 + v_{\mu,n}^t, t\nabla\varphi_1 + \nabla v_{\mu,n}^t)| &\leq \zeta_1 |v_{\mu,n+1}^t - v_{\mu,n}^t| \\ |h(x, t\varphi_1 + v_{\mu,n}^t, t\nabla\varphi_1 + \nabla v_{\mu,n}^t) - h(x, t\varphi_1 + v_{\mu,n}^t, t\nabla\varphi_1 + \nabla v_{\mu,n-1}^t)| &\leq \zeta_2 |\nabla v_{\mu,n}^t - \nabla v_{\mu,n-1}^t| \end{aligned} \tag{2.28}$$

As $v_{\mu,n+1}^t$ is a solution of (2.27), taking $v_{\mu,n+1}^t$ and $v_{\mu,n}^t$ as test functions, we obtain, respectively,

$$\int_\Omega \nabla v^t_{\mu,n+1}\nabla v^t_{\mu,n+1}dx = \lambda\int_\Omega v^t_{\mu,n+1}v^t_{\mu,n+1}dx + \mu\int_\Omega h(x,t\varphi_1+v^t_{\mu,n+1},t\nabla\varphi_1+\nabla v^t_{\mu,n})v^t_{\mu,n+1}dx$$

and

$$\int_\Omega \nabla v^t_{\mu,n+1}\nabla v^t_{\mu,n}dx = \lambda\int_\Omega v^t_{\mu,n+1}v^t_{\mu,n}dx + \mu\int_\Omega h(x,t\varphi_1+v^t_{\mu,n+1},t\nabla\varphi_1+\nabla v^t_{\mu,n})v^t_{\mu,n}dx.$$

Subtracting the previous inequalities, it follows that

$$\begin{aligned}\int_\Omega \nabla v^t_{\mu,n+1}(\nabla v^t_{\mu,n+1}-\nabla v^t_{\mu,n})dx &= \lambda\int_\Omega v^t_{\mu,n+1}(v^t_{\mu,n+1}-v^t_{\mu,n})dx\\ &+ \mu\int_\Omega h(x,t\varphi_1+v^t_{\mu,n+1},t\nabla\varphi_1+\nabla v^t_{\mu,n})(v^t_{\mu,n+1}-v^t_{\mu,n})dx.\end{aligned}$$

Arguing analogously for $v^t_{\mu,n}$, we obtain

$$\begin{aligned}\int_\Omega \nabla v^t_{\mu,n}(\nabla v^t_{\mu,n+1}-\nabla v^t_{\mu,n})dx &= \lambda\int_\Omega v^t_{\mu,n}(v^t_{\mu,n+1}-v^t_{\mu,n})dx\\ &+ \mu\int_\Omega h(x,t\varphi_1+v^t_{\mu,n},t\nabla\varphi_1+\nabla v^t_{\mu,n-1})(v^t_{\mu,n+1}-v^t_{\mu,n})dx.\end{aligned}$$

From these relations and from (2.28), we have

$$\begin{aligned}&\|v^t_{\mu,n+1}-v^t_{\mu,n}\|^2 = \lambda\|v^t_{\mu,n+1}-v^t_{\mu,n}\|_2^2\\ &+\mu\int_\Omega[h(x,t\varphi_1+v^t_{\mu,n+1},t\nabla\varphi_1+\nabla v^t_{\mu,n})-h(x,t\varphi_1+v^t_{\mu,n},t\nabla\varphi_1+\nabla v^t_{\mu,n})](v^t_{\mu,n+1}-v^t_{\mu,n})dx\\ &+\mu\int_\Omega[h(x,t\varphi_1+v^t_{\mu,n},t\nabla\varphi_1+\nabla v^t_{\mu,n})-h(x,t\varphi_1+v^t_{\mu,n},t\nabla\varphi_1+\nabla v^t_{\mu,n-1})](v^t_{\mu,n+1}-v^t_{\mu,n})dx\\ &\le \frac{\overline{\lambda}}{\lambda_2}\|v^t_{\mu,n+1}-v^t_{\mu,n}\|^2+|\mu|\int_\Omega\zeta_1|v^t_{\mu,n+1}-v^t_{\mu,n}|^2dx+|\mu|\int_\Omega\zeta_2|\nabla v^t_{\mu,n}-\nabla v^t_{\mu,n-1}||v^t_{\mu,n+1}-v^t_{\mu,n}|dx.\end{aligned}$$

Thus, by Hölder Inequality, (2.9) and the fact that $H^1_0(\Omega)$ is continuously embedded in $L^{\frac{2\sigma}{\sigma-2}}(\Omega)$, we obtain

$$\begin{aligned}\left[\frac{\lambda_2-\overline{\lambda}}{\lambda_2}\right]\|v^t_{\mu,n+1}-v^t_{\mu,n}\|^2 &\le |\mu|[\ d^2_{2\sigma'}\|\zeta_1\|_\sigma\|v^t_{\mu,n+1}-v^t_{\mu,n}\|^2\\ &+ d_{\frac{2\sigma}{\sigma-2}}\|\zeta_2\|_\sigma\|v^t_{\mu,n+1}-v^t_{\mu,n}\|\|v^t_{\mu,n}-v^t_{\mu,n-1}\|\ ].\end{aligned}$$

Consequently,

$$\left[\frac{\lambda_2-\overline{\lambda}}{\lambda_2}-|\mu|d^2_{2\sigma'}\|\zeta_1\|_\sigma\right]\|v^t_{\mu,n+1}-v^t_{\mu,n}\|\leq|\mu|d_{\frac{2\sigma}{\sigma-2}}\|\zeta_2\|_\sigma\|v^t_{\mu,n}-v^t_{\mu,n-1}\|. \tag{2.29}$$

Taking $\overline{\mu}_2<\min\{(\lambda_2-\overline{\lambda})/2\lambda_2d^2_{2\sigma'}\|\zeta_1\|_\sigma,(\lambda_2-\overline{\lambda})/4\lambda_2d_{\frac{2\sigma}{\sigma-2}}\|\zeta_2\|_\sigma,\mu_1\}$, it follows that, for every $n\in\mathbb{N}$,

$$\|v^t_{\mu,n+1}-v^t_{\mu,n}\|\leq\frac{1}{2}\|v^t_{\mu,n}-v^t_{\mu,n-1}\|,\ \text{for each } |\mu|\in(0,\overline{\mu}_2).$$

We claim that $(v^t_{\mu,n})\subset\langle\varphi_1\rangle^\perp$ is a Cauchy sequence. Indeed, for n, $m\in\mathbb{N}$, with $m>n$, considering $s\in\mathbb{N}$, we have that

$$\begin{aligned}\|v^t_{\mu,m}-v^t_{\mu,n}\| &\leq \|v^t_{\mu,n+s}-v^t_{\mu,n+s-1}\|+\cdots+\|v^t_{\mu,n+1}-v^t_{\mu,n}\|\\ &\leq \left(\frac{1}{2}\right)^n\left[\left(\frac{1}{2}\right)^{s-1}+\cdots+1\right]\|v^t_{\mu,1}-v_0\|\\ &\leq \left(\frac{1}{2}\right)^n\sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s\|v^t_{\mu,1}-v_0\|\leq\frac{1}{2^{n-1}}\|v^t_{\mu,1}-v_0\|.\end{aligned}$$

Therefore, by the above $\|v^t_{\mu,m}-v^t_{\mu,n}\|\to0$, as n, $m\to\infty$. The claim is proved.

Since $\langle\varphi_1\rangle^\perp$ is a Banach space, there exists $v^t_\mu\in\langle\varphi_1\rangle^\perp$ such that $v^t_{\mu,n}\to v^t_\mu$ strongly in $H^1_0(\Omega)$, as $n\to\infty$. Moreover, taking a subsequence if necessary,

$$\begin{cases} v^t_{\mu,n}\to v^t_\mu \text{ strongly in } L^2(\Omega),\\ v^t_{\mu,n}\to v^t_\mu \text{ a.e. in } \Omega,\\ \nabla v^t_n\to\nabla v^t_\mu \text{ a.e. in } \Omega.\end{cases}$$

Thus, taking $n\to\infty$ in

$$\int_\Omega\nabla v^t_{\mu,n}\nabla zdx=\lambda\int_\Omega v^t_{\mu,n}zdx+\mu\int_\Omega h(x,t\varphi_1+v^t_{\mu,n},t\nabla\varphi_1+\nabla v^t_{\mu,n-1})zdx,\ z\in\langle\varphi_1\rangle^\perp,$$

it follows from (2.6) and of Lebesgue Dominated Convergence Theorem that

$$\int_\Omega\nabla v^t_\mu\nabla zdx=\lambda\int_\Omega v^t_\mu zdx+\mu\int_\Omega h(x,t\varphi_1+v^t_\mu,t\nabla\varphi_1+\nabla v^t_\mu)zdx,\ \text{for every } z\in\langle\varphi_1\rangle^\perp.$$

Consequently, v^t_μ is solution of Problem (2.7). This concludes the proof of the existence of a solution for Problem (2.7).

Considering $w = v_\mu^t$ in Lemma 2.12, we obtain that $v_\mu^t \in C^{1,\gamma}(\overline{\Omega})$ and

$$\|v_\mu^t\|_{C^{1,\gamma}} \leq D, \text{ for every } t \in [t_1, t_2], \ \mu \in (0, \overline{\mu}_2).$$

Finally, in order to prove that Problem (2.7) has a unique solution we suppose that v_μ^t and w_μ^t are solution of that Problem. Then using an argument similar to the prove of inequality (2.29) we obtain

$$\left[\frac{\lambda_2 - \overline{\lambda}}{\lambda_2} - |\mu| d_{2\sigma'}^2 \|\zeta_1\|_\sigma\right] \|v_\mu^t - w_\mu^t\| \leq |\mu| d_{\frac{2\sigma}{\sigma-2}} \|\zeta_2\|_\sigma \|v_\mu^t - w_\mu^t\|.$$

Therefore, taking $\mu_2 < \{\overline{\mu}_2, (\lambda_2 - \overline{\lambda})/\lambda_2 (d_{2\sigma'}^2 \|\zeta_1\|_\sigma + d_{\frac{2\sigma}{\sigma-2}} \|\zeta_2\|_\sigma)\}$, we have that $\|v_\mu^t - w_\mu^t\| \leq 0$. Consequently $v_\mu^t = w_\mu^t$, for every $0 < |\mu| < \mu_2$. This concludes the proof of the theorem. □

As a consequence of the argument used in the proof of Theorem 2.13, we verify the continuity of the application $t \to v_\mu^t$.

Corollary 2.14. *Given $(t_n) \subset [t_1, t_2]$ and $t_0 \in [t_1, t_2]$ such that $t_n \to t_0$, there exists $\mu_3 \in (0, \mu_2)$ such that $\|v_\mu^{t_n} - v_\mu^{t_0}\| \to 0$, for every $0 < |\mu| < \mu_3$.*

Proof. As $v_\mu^{t_n}$ and $v_\mu^{t_0}$ are solutions of (2.7), arguing as in Theorem 2.13, we obtain

$$\begin{aligned} \int_\Omega \nabla v_\mu^{t_n} (\nabla v_\mu^{t_n} - \nabla v_\mu^{t_0}) dx &= \lambda \int_\Omega v_\mu^{t_n} (v_\mu^{t_n} - v_\mu^{t_0}) dx \\ &+ \mu \int_\Omega h(x, t_n \varphi_1 + v_\mu^{t_n}, t_n \nabla \varphi_1 + \nabla v_\mu^{t_n})(v_\mu^{t_n} - v_\mu^{t_0}) dx. \end{aligned}$$

and

$$\begin{aligned} \int_\Omega \nabla v_\mu^{t_0} (\nabla v_\mu^{t_n} - \nabla v_\mu^{t_0}) dx &= \lambda \int_\Omega v_\mu^{t_0} (v_\mu^{t_n} - v_\mu^{t_0}) dx \\ &+ \mu \int_\Omega h(x, t_0 \varphi_1 + v_\mu^{t_0}, t_0 \nabla \varphi_1 + \nabla v_\mu^{t_0})(v_\mu^{t_n} - v_\mu^{t_0}) dx. \end{aligned}$$

Thus, from $(h_\nabla)_1$, there exist ζ_1 and $\zeta_2 \in L^\sigma(\Omega)$ such that

$$\begin{aligned}\|v_\mu^{t_n} - v_\mu^{t_0}\|^2 &= \lambda\|v_\mu^{t_n} - v_\mu^{t_0}\|_2^2 \\ &+\mu\int_\Omega [h(x, t_n\varphi_1 + v_\mu^{t_n}, t_n\nabla\varphi_1 + \nabla v_\mu^{t_n}) - h(x, t_n\varphi_1 + v_\mu^{t_n}, t_0\nabla\varphi_1 + \nabla v_\mu^{t_0})](v_\mu^{t_n} - v_\mu^{t_0})dx \\ &+\mu\int_\Omega [h(x, t_n\varphi_1 + v_\mu^{t_n}, t_0\nabla\varphi_1 + \nabla v_\mu^{t_0}) - h(x, t_0\varphi_1 + v_\mu^{t_0}, t_0\nabla\varphi_1 + \nabla v_\mu^{t_0})](v_\mu^{t_n} - v_\mu^{t_0})dx \\ &\leq \frac{\overline{\lambda}}{\lambda_2}\|v_\mu^{t_n} - v_\mu^{t_0}\|^2 + |\mu|\int_\Omega \zeta_1|(t_n - t_0)\nabla\varphi_1 + \nabla v_\mu^{t_n} - \nabla v_\mu^{t_0}||v_\mu^{t_n} - v_\mu^{t_0}|dx \\ &+|\mu|\int_\Omega \zeta_2|(t_n - t_0)\varphi_1 + v_\mu^{t_n} - v_\mu^{t_0}||v_\mu^{t_n} - v_\mu^{t_0}|dx.\end{aligned}$$

Assim, of Hölder Inequality,

$$\begin{aligned}\left(\frac{\lambda_2 - \overline{\lambda}}{\lambda_2}\right)\|v_\mu^{t_n} - v_\mu^{t_0}\|^2 &\leq |\mu|d_{\sigma'}[\|\nabla\varphi_1\|_\infty\|\zeta_1\|_\sigma + \|\varphi_1\|_\infty\|\zeta_2\|_\sigma]\|v_\mu^{t_n} - v_\mu^{t_0}\||t_n - t_0| \\ &+ |\mu|[\|\zeta_1\|_\sigma d_{\frac{2\sigma}{\sigma-2}} + \|\zeta_2\|_\sigma d_{2\sigma'}]\|v_\mu^{t_n} - v_\mu^{t_0}\|^2.\end{aligned}$$

This implies

$$\begin{aligned}&\left[\frac{\lambda_2 - \overline{\lambda}}{\lambda_2} - |\mu|(\|\zeta_1\|_\sigma d_{\frac{2\sigma}{\sigma-2}} + \|\zeta_2\|_\sigma d_{2\sigma'})\right]\|v_\mu^{t_n} - v_\mu^{t_0}\| \\ &\qquad\leq |\mu|d_{\sigma'}[\|\nabla\varphi_1\|_\infty\|\zeta_1\|_\sigma + \|\varphi_1\|_\infty\|\zeta_2\|_\sigma]|t_n - t_0|.\end{aligned}$$

Therefore, to complete the proof of this corollary, is sufficient to choose $0 < \mu_3 < \min\{\mu_2, (\lambda_2 - \overline{\lambda})/\lambda_2[\|\zeta_1\|_\sigma d_{\frac{2\sigma}{\sigma-2}} + \|\zeta_2\|_\sigma d_{2\sigma'}]\}$. □

Lemma 2.15. *Given $\delta > 0$, there exists $\mu_4 \in (0, \mu_3)$ such that, for every $0 < |\mu| < \mu_4$, $\|v_\mu^{t_i}\| < \delta$, for $i = 1, 2$.*

Proof. Using (2.6), the Hölder Inequality and (2.9), we obtain

$$\left|\int_\Omega h(x, t_i\varphi_1 + v_\mu^{t_i}, t_i\nabla\varphi_1 + \nabla v_\mu^{t_i})v_\mu^{t_i}dx\right| \leq \int_\Omega f|v_\mu^{t_i}|dx \leq \|f\|_\sigma\|v_\mu^{t_i}\|_{\sigma'} \leq d_{\sigma'}\|f\|_\sigma\|v_\mu^{t_i}\|.$$

Thus, from (2.7) and the above inequality it follows that

$$\|v_\mu^{t_i}\|^2 = \lambda\|v_\mu^{t_i}\|_2^2 + \mu\int_\Omega h(x, t_i\varphi_1 + v_\mu^{t_i}, t_i\nabla\varphi_1 + \nabla v_\mu^{t_i})v_\mu^{t_i}dx \leq \frac{\overline{\lambda}}{\lambda_2}\|v_\mu^{t_i}\|^2 + |\mu|d_{\sigma'}\|f\|_\sigma\|v_\mu^{t_i}\|.$$

Thus

$$\|v_\mu^{t_i}\| \leq \frac{\lambda_2}{\lambda_2 - \overline{\lambda}}|\mu|d_{\sigma'}\|f\|_\sigma.$$

Taking $\mu_4 < \min\{\mu_3, \delta(\lambda_2 - \overline{\lambda})/\lambda_2 d_{\sigma'}\|f\|_\sigma\}$, we complete the proof of the lemma. □

2.2.1 Proof of Theorem 2.8

First we prove the Theorem 2.8 assuming that $\mu > 0$ and h satisfies $(h_\nabla)_1$, (2.6) and $(h_\nabla^+)_2$ there exist t_1 and $t_2 \in \mathbb{R}$, $t_1 < t_2$, such that

$$\int_\Omega h(x, t_1\varphi_1, t_1\nabla\varphi_1)\varphi_1 dx > 0 > \int_\Omega h(x, t_2\varphi_1, t_2\nabla\varphi_1)\varphi_1 dx.$$

To achieve this goal it is suffices to verify that there exits $t_0 \in (t_1, t_2)$ such that

$$\int_\Omega \nabla u \nabla\varphi_1 dx = \lambda \int_\Omega u\varphi_1 dx + \mu \int_\Omega h(x, u, \nabla u)\varphi_1 dx, \text{ for every }, \tag{2.30}$$

where $u = t_0\varphi_1 + v_\mu^{t_0}$.

From (2.6) and of the fact that $H_0^1(\Omega)$ is continuously embedded in $L^1(\Omega)$, we may apply the Lebesgue Dominated Convergence Theorem to ensure that the functional $T_i : \langle\varphi_1\rangle^\perp \longrightarrow \mathbb{R}$, $i = 1, 2$, given by

$$T_i(z) = \int_\Omega h(x, t_i\varphi_1 + z, t_i\nabla\varphi_1 + \nabla z)\varphi_1 dx,$$

is continuous. Therefore, from $(h_\nabla^+)_2$, there exists $\delta > 0$ such that, for every $z \in \langle\varphi_1\rangle^\perp$, with $\|z\| < \delta$,

$$\int_\Omega h(x, t_1\varphi_1 + z, t_1\nabla\varphi_1 + \nabla z)\varphi_1 dx > \frac{T_1(0)}{2}$$
$$0 > \frac{T_2(0)}{2} > \int_\Omega h(x, t_2\varphi_1 + z, t_2\nabla\varphi_1 + \nabla z)\varphi_1 dx. \tag{2.31}$$

Then applying Lemma 2.15, there exists $\mu^* \in (0, \mu_3)$ such that $\|v_\mu^{t_1}\| < \delta$ and $\|v_\mu^{t_2}\| < \delta$, for every $\mu \in (0, \mu^*)$. Thus, as a consequence from (2.31) we have, for every $\mu \in (0, \mu^*)$,

$$\int_\Omega h(x, t_1\varphi_1 + v_\mu^{t_1}, t_1\nabla\varphi_1 + \nabla v_\mu^{t_1})\varphi_1 dx > \frac{T_1(0)}{2}$$
$$0 > \frac{T_2(0)}{2} > \int_\Omega h(x, t_2\varphi_1 + v_\mu^{t_2}, t_2\nabla\varphi_1 + \nabla v_\mu^{t_2})\varphi_1 dx.$$

Taking $\nu^*<(\lambda_1/2\|\varphi_1\|^2)\min\{T_1(0)/(|t_1|+1),\ -T_2(0)/(|t_2|+1)\}$, by the above inequality, for every $\mu\in(0,\mu^*)$ and $|\lambda-\lambda_1|<\mu\nu^*$, we obtain

$$\frac{\lambda-\lambda_1}{\lambda_1\mu}\|\varphi_1\|^2 t_1+\int_\Omega h(x,t_1\varphi_1+v_\mu^{t_1},t_1\nabla\varphi_1+\nabla v_\mu^{t_1})\varphi_1dx>-\frac{|\lambda-\lambda_1|}{\lambda_1\mu}\|\varphi_1\|^2|t_1|+\frac{T_1(0)}{2}>0$$

and

$$\frac{\lambda-\lambda_1}{\lambda_1\mu}\|\varphi_1\|^2 t_2+\int_\Omega h(x,t_2\varphi_1+v_\mu^{t_2},t_2\nabla\varphi_1+\nabla v_\mu^{t_2})\varphi_1dx<\frac{|\lambda-\lambda_1|}{\lambda_1\mu}\|\varphi_1\|^2|t_2|+\frac{T_2(0)}{2}<0.$$

Next, We define $g:[t_1,t_2]\longrightarrow\mathbb{R}$ by

$$g(t)=\frac{\lambda-\lambda_1}{\lambda_1\mu}t+\int_\Omega h(x,t\varphi_1+v_\mu^t,t\nabla\varphi_1+\nabla v_\mu^t)\varphi_1dx,\ \text{for every } t\in[t_1,t_2].$$

In view of (2.6), Corollary 2.14 and the Lebesgue Dominated Convergence Theorem we may assert that g is a continuous function. Since $g(t_1)>0>g(t_2)$, there exists $t_0\in(t_1,t_2)$ such that $g(t_0)=0$. This implies that $u=t_0\varphi_1+v_\mu^{t_0}$ satisfies (2.30). This concludes the proof of Theorem 2.8 when h satisfies $(h_\nabla^+)_2$. Arguing in an analogous way we may prove Theorem 2.8 with $\mu>0$ and h satisfying $(h_\nabla)_1$, (2.6) and

$(h_\nabla^-)_2$ there exist t_1 and $t_2\in\mathbb{R}$, $t_1<t_2$, such that

$$\int_\Omega h(x,t_1\varphi_1,t_1\nabla\varphi_1)\varphi_1dx<0<\int_\Omega h(x,t_2\varphi_1,t_2\nabla\varphi_1)\varphi_1dx.$$

If $\mu<0$, setting $\tilde{\mu}=-\mu>0$ and $\tilde{h}(x,u,\nabla u)=-h(x,u,\nabla u)$ we have that $\tilde{h}$ satisfies (2.6), $(h_\nabla)_1$ and $(h_\nabla)_2$. Therefore, the result follows directly from the case $\mu>0$. This concludes the proof of the theorem. □

2.3 Proofs of the main results

In order to apply Theorem 2.8 we consider the truncated function h_R defined by

$$h_R(x,s,\xi)=\chi(s)h(x,s,\xi\Psi(\xi)),\ \text{for every } x\in\overline{\Omega},\ s\in\mathbb{R},\ \xi\in\mathbb{R}^N, \tag{2.32}$$

where $R > \max\{|t_1|,|t_2|\}\max\{\|\varphi_1\|_\infty, \|\nabla\varphi_1\|_\infty\}$ and $\chi \in C^\infty(\mathbb{R},[0,1])$ and $\Psi \in C^1(\mathbb{R}^N,[0,1])$ are functions satisfiying

$$\chi(t) = \begin{cases} 1 \text{ if } |s| \le R+1 \\ 0 \text{ if } |s| \ge R+2 \end{cases} \quad \text{and} \quad \Psi(\xi) = \begin{cases} 1 \text{ if } |\xi| \le R+1, \\ 0 \text{ if } |\xi| \ge R+2. \end{cases}$$

Associated with h_R, we consider the problem

$$\begin{cases} -\Delta u = & \lambda u + \mu h_R(x,u,\nabla u) \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{2.33}$$

As $\|t_i\varphi_1\|_\infty$, $\|t_i\nabla\varphi_1\|_\infty < R$, $i = 1,2$, it follows from (2.32) and (h_2) that h_R satisfies the hypotesis (h_2). Moreover, from (h_0), there exists $f_{R+2} \in L^\sigma(\Omega)$ such that $|h(x,s,\xi)| \le f_{R+2}(x)$, for every $|s| \le R+2$, $|\xi| \le R+2$ and for almost every $x \in \overline{\Omega}$. Consequently, from (2.32),

$$|h_R(x,s,\xi)| \le f_{R+2}(x), \text{ for every } s \in \mathbb{R},\ \xi \in \mathbb{R}^N, \text{ for almost every } x \in \overline{\Omega}. \tag{2.34}$$

On the other hand, from $(h_\nabla)_0$ and $(h_\nabla)_1$, given A_1 and A_2 such that $|s_1|$, $|s_2| \le A_1$ and $|\xi| \le A_2$ there exist ζ_1 and $f_A \in L^\sigma(\Omega)$, with $A = \max\{A_1, A_2\}$, such that

$$\begin{aligned} &|h_R(z,s_1,\xi) - h_R(z,s_2,\xi)| \\ &\qquad \le |\chi(s_1)||h(z,s_1,\xi\Psi(\xi)) - h(z,s_2,\xi\Psi(\xi))| + |h(z,s_2,\xi\Psi(\xi))||\chi(s_1) - \chi(s_2)| \\ &\qquad \le \zeta_1(z)|s_1 - s_2| + f_A(z)|\chi'(\theta)||s_1 - s_2|, \end{aligned}$$

where $\theta \in (s_1,s_2)$. Thus, since χ' is bounded, there exists $\hat{\zeta}_1 \in L^\sigma(\Omega)$ such that, for every $z \in \overline{\Omega}$, $|s_1|$, $|s_1| \le A_1$ and $|\xi| \le A_2$

$$|h_R(z,s_1,\xi) - h_R(z,s_2,\xi)| \le \hat{\zeta}_1(z)|s_1 - s_2|. \tag{2.35}$$

Arguing in an analogous way we find $\hat{\zeta}_2 \in L^\sigma(\Omega)$ such that, for every $z \in \overline{\Omega}$, $|s_1| \le A_1$ and $|\xi_1|$, $|\xi_1| \le A_2$,

$$|h_R(z,s,\xi_1) - h_R(z,s,\xi_1)| \le \hat{\zeta}_2(z)|s_1 - s_2|. \tag{2.36}$$

Before proving Theorem 2.1, we state a lemma which proof is analogous to the proof Lemma 1.19 with hypothesis (2.34) replacing the hipothesis (1.25).

Lemma 2.16. *Suppose h_R satisfies* (2.34) *and that there exist positive constants μ^* and*

ν^* such that, for every $\mu \in (0,\mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, $u_\mu = t\varphi_1 + v$ is a solution of Problem (2.33), with $t \in (t_1,t_2)$ and $v \in \langle\varphi_1\rangle^\perp$. Then there exist $q > N$ and $b_q > 0$ such that $\|v\|_{2,q} \le b_q\mu$.

Proof of Theorem 2.1. Since h_R satisfies $(h_\nabla)_2$, from (2.34), (2.35), (2.36) and Theorem 2.8, there exist positive constants $\hat{\mu}$ and ν^* such that, for every $\mu \in (0,\hat{\mu})$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (2.33) has a solution $u_\mu = t\varphi_1 + v$ of class $C^{1,\gamma}(\overline{\Omega})$, with $t \in (t_1,t_2)$ and $v \in \langle\varphi_1\rangle^\perp$. Since $q > N$ we may apply the Sobolev Embedding Theorem to conclude that there exists K such that $\|v\|_{1,\gamma} \le K\|v\|_{2,q} \le Kb_q\mu$. Consequently, $\|v\|_\infty$, $\|\nabla v\|_\infty \to 0$, as $\mu \to 0$. This implies that there exists $\mu^* \in (0,\hat{\mu})$ such that $\|v\|_\infty$, $\|\nabla v\|_\infty < 1$, for every $\mu \in (0,\mu^*)$. Thus, for every $\mu \in (0,\mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$,

$$\|u_\mu\|_\infty = \|t\varphi_1 + v\|_\infty \le \|t\varphi_1\|_\infty + \|v\|_\infty < R + 1$$

and

$$\|\nabla u_\mu\|_\infty = \|t\nabla\varphi_1 + \nabla v\|_\infty \le \|t\nabla\varphi_1\|_\infty + \|\nabla v\|_\infty < R + 1$$

Hence $\chi(u_\mu) = 1$, $\Psi(\nabla u_\mu) = 1$ and $h_R(x,u_\mu) = h(x,u_\mu)$, for every $\mu \in (0,\mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. Consequently u_μ is a solution of (2.1). This concludes the proof of Theorem 2.1. □

Proof of Proposition 2.2. From Theorem 2.1, for each $i = 1,\ldots,k$, there exist positive constants μ_i and ν_i such that, for every $0 < |\mu| < \mu_i$ and $|\lambda - \lambda_1| < |\mu|\nu_i$, Problem (2.1) has a solution $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{1,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i,t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$. Therefore, taking $0 < \mu^* < \min\{\mu_i : i = 1,\ldots,k\}$ and $0 < \nu^* < \min\{\nu_i : i = 1,\ldots,k\}$, we conclude that Problem (2.1) has k solutions, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$. □

In order to verify Remark 2.3 we consider the function $\Phi_\nabla : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi_\nabla(t) = \int_\Omega h(x,t\varphi_1,t\nabla\varphi_1)\varphi_1 dx, \quad \text{for every } t \in \mathbb{R}. \tag{2.37}$$

Theorem 2.17. *Suppose h satisfies $(h_\nabla)_0$, $(h_\nabla)_1$ and $(h_\nabla)_2$. Consider the positive constants μ^* and ν^* given by Theorem 2.1. Then there exists $\mu^{**} \in (0,\mu^*)$ such that, if $\mu \in (0,\mu^{**})$ and $|\lambda - \lambda_1| < \mu\nu^*$, the solution of Problem (2.1), $u_\mu = t\varphi_1 + v$, with $t \in (t_1,t_2)$ and $v \in \langle\varphi_1\rangle^\perp$, is positive or negative provided $t_1 \ge 0$ or $t_2 \le 0$, respectively.*

Proof. without loss generality we suppose h satisfies $(h_\nabla^+)_2$, com $t_1 \ge 0$. Note that, we may assume $t_1 > 0$. Indeed, if $t_1 = 0$, from $(h_\nabla^+)_2$ and the continuity of Φ in $[t_1,t_2]$, there exists $\tilde{t}_1 > t_1$ satisfying $(h_\nabla^+)_2$.

By Lemma 2.16, there exist positive constants $q > N$ and b_q such that, for every $\mu \in (0,\mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, the solution of Problem (2.1), $u_\mu = t\varphi_1 + v$, with

$t = t(\mu) \in (t_1, t_2)$, $v = v(\mu) \in \langle \varphi_1 \rangle^\perp$, satisfies $\|v\|_{2,q} \le b_q \mu$. Thus, of Sobolev Embedding Theorem, $v \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$.

We claim that

$$\lim_{\mu \to 0} \frac{|v(x)|}{d(x, \partial\Omega)} = 0, \quad \text{for every } x \in \Omega,$$

where $d(x, \partial\Omega)$ denote the distance from x to $\partial\Omega$. Suppose the claim is true. Since $t \ge t_1 > 0$, from (1.26), we have that

$$\frac{u_\mu(x)}{d(x, \partial\Omega)} = \frac{t\varphi_1(x) + v(x)}{d(x, \partial\Omega)} \ge \frac{t_1 K d(x, \partial\Omega) + v(x)}{d(x, \partial\Omega)}, \text{ for every } x \in \Omega.$$

Hence, from the above claim, there exists $\mu^{**} \in (0, \mu^*)$ such that, for $\mu \in (0, \mu^{**})$,

$$u_\mu(x) \ge \frac{t_1 K}{2} d(x, \partial\Omega), \text{ for every } x \in \Omega.$$

Therefore, u_μ is positive in Ω.

Finally, to complete the proof of the theorem it remains to prove the claim: suppose there exist $\varepsilon > 0$, $(x_n) \subset \Omega$ and $(\mu_n) \subset \mathbb{R}^+ \setminus \{0\}$, with $\mu_n \to 0$, such that $(|v_{\mu_n}(x_n)|/d(x_n, \partial\Omega)) \ge \varepsilon$, for every $n \in \mathbb{N}$. As $(x_n) \subset \overline{\Omega}$ there exists $x_0 \in \overline{\Omega}$ such that $x_n \to x_0$, up to a subsequence.

If $x_0 \in \Omega$, as $d(x_n, \partial\Omega) \to d(x_0, \partial\Omega)$ and $\|v\|_\infty \to 0$, as $\mu \to 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\varepsilon}{2} d(x_0, \partial\Omega) \le \varepsilon d(x_{n_0}, \partial\Omega) \le |v_{\mu_{n_0}}(x_{n_0})| \le \|v_{\mu_{n_0}}\|_\infty \le \frac{\varepsilon}{4} d(x_0, \partial\Omega),$$

which is a contradiction.

If $x_0 \in \partial\Omega$, since $x_n \to x_0$ and $\|\nabla v\|_\infty \to 0$, as $\mu \to 0$, for n_0 sufficiently large there exists $y_{n_0} \in \partial\Omega$ such that $d(x_{n_0}, \partial\Omega) = |x_{n_0} - y_{n_0}|$ and $\|\nabla v_{n_0}\| \le \varepsilon/2$. Consequently, from $v_{n_0}(y_{n_0}) = 0$ and the Mean Value Theorem, there exists $\theta \in (0, 1)$ such

$$\begin{aligned} \varepsilon \le \frac{|v_{n_0}(x_{n_0})|}{d(x_{n_0}, \partial\Omega)} &= \frac{|v_{n_0}(x_{n_0}) - v_{n_0}(y_{n_0})|}{|x_{n_0} - y_{n_0}|} \\ &= |\langle \nabla v_{\mu_{n_0}}(x_{n_0} + \theta(y_{n_0} - x_{n_0})), \frac{x_{n_0} - y_{n_0}}{|x_{n_0} - y_{n_0}|} \rangle| \le \|v_{\mu_{n_0}}\|_\infty \le \frac{\varepsilon}{2}. \end{aligned}$$

In both cases we obtain a contradiction. This completes the proof of theorem. □

Arguing as in the proof of Theorem 2.17, we have:

Proposition 2.18. *Suppose h satisfies $(h_\nabla)_0$, $(h_\nabla)_1$ and $(h_\nabla)_2$. Consider the positive constants μ^* and ν^* given by Proposition 2.2. Then there exists $\mu^{**} \in (0, \mu^*)$ such that,* *if* $0 < |\mu| < \mu^{**}$ *and* $|\lambda - \lambda_1| < |\mu|\nu^*$*, the solutions of Problem* (2.1)*,* $u_i = \hat{t}_i\varphi_1 + v_i$*, with* $\hat{t}_i \in (t_i, t_{i+1})$ *and* $v_i \in \langle\varphi_1\rangle^\perp$*, are ordered, i.e.,* $u_i < u_{i+1}$ *in* Ω*,* $i = 1, \cdots, k$*.*

Proof. From $(h_\nabla)_2$ and the continuity of Φ in $[t_i, t_{i+1}]$, with Φ_∇ given in (2.37), there exist t_i^i, $t_{i+1}^i \in (t_i, t_{i+1})$, with $t_i^i < t_{i+1}^i$, satisfying $(h_\nabla)_2$. By Lemma 2.16, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, the solutions of Problem (2.1), $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i = \hat{t}_i(\mu) \in (t_i^i, t_{i+1}^i)$ and $v_i = v_i(\mu) \in \langle\varphi_1\rangle^\perp$, satisfy $v_i \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$. Consequently, $u_{i+1} - u_i = (\hat{t}_{i+1} - \hat{t}_i)\varphi_1 + (v_{i+1} - v_i)$, with $\hat{t}_{i+1} - \hat{t}_i > t_{i+1}^{i+1} - t_{i+1}^i > 0$ and $(v_{i+1} - v_i) \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$, Therefore, arguing as in Theorem 2.17, there exists $\mu^{**} \in (0, \mu^*)$ such that $u_{i+1} - u_i > 0$ in Ω, for every $0 < |\mu| < \mu^{**}$. □

2.4 No solubility of Problem (2.1)

The goal of this section is to establish the non-existence of solutions for Problem (2.1) when the condition $(h_\nabla)_2$ is not satisfied. More specifically we proof Theorem 2.4.

Proof of Theorem 2.4. To prove that Problem (2.1) has no solution, we assume by contradiction that there exist $(\mu_k) \subset \mathbb{R} \setminus \{0\}$, with $|\mu_k| \to 0$, and $(\hat{\lambda}_k) \subset \mathbb{R}$ such that $|\hat{\lambda}_k - \lambda_1| < |\mu_k|/k$ and $u_k = \tau_k\varphi_1 + v_k$, with $\tau_k \in [t_1, t_2]$ and $v_k \in \langle\varphi_1\rangle^\perp$, such that u_μ is solution of (2.1). Then, $\hat{\lambda}_k < |\mu_k|/k + \lambda_1 \leq |\mu_k| + \lambda_1$.

From (1.1), $(h_\nabla)_3$, Hölder Inequality and (2.9), we have that

$$\begin{aligned}
\frac{\lambda_2 - |\mu_k| - \lambda_1}{\lambda_2}\|v_k\|^2 \leq& \ \|v_k\|^2 - \hat{\lambda}_k\|v_k\|_2^2 = \mu_k\int_\Omega h(x, \tau_k\varphi_1 + v_k, \tau_k\nabla\varphi_1 + \nabla v_k)v_k dx \\
\leq& |\mu_k|K\int_\Omega f(1 + |v_k| + |\nabla v_k|)|v_k|dx \\
\leq& |\mu_k|K\|f\|_\sigma(d_{\sigma'}\|v_k\| + d_{2\sigma'}^2\|v_k\|^2 + \|\nabla v_k\|_2 \ d_{\frac{2\sigma}{\sigma-2}}\|v_k\|),
\end{aligned}$$

where $K = K(\max\{|t_1|, |t_2|\}, \|\varphi_1\|_\infty, \|\nabla\varphi_1\|_\infty)$. Therefore

$$\frac{1}{\lambda_2}\left[(\lambda_2 - \lambda_1) - |\mu_k|[1 + \lambda_2K\|f\|_\sigma(d_{2\sigma'}^2 + d_{\frac{2\sigma}{\sigma-2}})]\right]\|v_k\| \leq |\mu_k|K\|f\|_\sigma d_{\sigma'}.$$

As $|\mu_k| \to 0$, it follows that $\|v_k\| \to 0$. Then, up to a subsequence,

$$\begin{cases} v_k \to 0 \text{ and } \nabla v_k \to 0 \text{ a.e. in } \Omega, \\ |v_k| \leq \eta_1 \in L^\sigma(\Omega) \text{ a.e. in } \Omega, \\ |\nabla v_k| \leq \eta_2 \in L^2(\Omega) \text{ a.e. in } \Omega. \end{cases}$$

Moreover, from the compactness of $[t_1, t_2]$, we may suppose that there exists $\tau_0 \in [t_1, t_2]$

such that $\tau_k\varphi_1 + v_k \to \tau_0\varphi_1$ a.e. in Ω. Thus, invoking $(h_\nabla)_3$, we may apply the Lebesgue Dominated Convergence Theorem to obtain

$$\int_\Omega h(x, \tau_k\varphi_1 + v_k, \tau_k\nabla\varphi_1 + \nabla v_k)\varphi_1 dx \to \int_\Omega h(x, \tau_0\varphi_1, \tau_0\nabla\varphi_1)\varphi_1 dx.$$

Consequently, since $|\hat{\lambda}_k - \lambda_1||\tau_k|/|\mu_k| \leq \max\{|t_1|, |t_2|\}/k$, we have that

$$\lim_{k\to\infty}\left[\frac{\hat{\lambda}_k - \lambda_1}{\mu_k\lambda_1}\|\varphi_1\|^2\tau_k + \int_\Omega h(x, \tau_k\varphi_1 + v_k, \tau_k\nabla\varphi_1 + \nabla v_k)\varphi_1 dx\right] = \int_\Omega h(x, \tau_0\varphi_1, \tau_0\nabla\varphi_1)\varphi_1 dx.$$

Hence, from $(h_\nabla)_4$, there exists $k \in \mathbb{N}$ sufficiently large such that

$$\begin{aligned} \int_\Omega \nabla u_k \nabla\varphi_1 dx - \hat{\lambda}_k \int_\Omega u_k\varphi_1 dx - \mu_k \int_\Omega h(x, \tau_k\varphi_1 + v_k, \tau_k\nabla\varphi_1 + \nabla v_k)\varphi_1 dx \\ = -\mu_k\left[\frac{\hat{\lambda}_k - \lambda_1}{\mu_k\lambda_1}\|\varphi_1\|^2\tau_k + \int_\Omega h(x, \tau_k\varphi_1 + v_k, \tau_k\nabla\varphi_1 + \nabla v_k)\varphi_1 dx\right] \neq 0. \end{aligned}$$

This contradicts the fact that u_k is solution of Problem (2.1). This completes the proof of the theorem. □

2.5 Applications

Proof of Proposition 2.5. Considering $u = \beta^{\frac{1}{p-q}}w$, with $p = p_1 + p_2$ and $q = q_1 + q_2$, in (2.2) we obtain

$$-\beta^{\frac{1}{p-q}}\Delta w = \lambda\beta^{\frac{1}{p-q}}w + \beta b_1\beta^{\frac{q}{p-q}}w^{q_1}|\nabla w|^{q_2} + b_2\beta^{\frac{p}{p-q}}w^{p_1}|\nabla w|^{p_2}.$$

This implies that

$$\begin{cases} -\Delta w = & \lambda w + \mu(b_1 w^{q_1}|\nabla w|^{q_2} + b_2 w^{p_1}|\nabla w|^{p_2}) \text{ in } \Omega, \\ w = & 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{2.38}$$

where $\mu = \beta^{\frac{p-1}{p-q}}$. Defining $h : \overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^N \to \mathbb{R}$ by $h(x, s, \xi) = b_1(x)s^{q_1}|\xi|^{q_2} + b_2(x)s^{p_1}|\xi|^{p_2}$ we have that h is a Carathéodory function and it satisfies (h_0) and (h_1). Considering the function Φ_∇ given in (2.37), we have that

$$\begin{aligned}\Phi_\nabla(t) &= \int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \\ &= t^q \int_\Omega b_1 \varphi_1^{q_1+1}|\nabla\varphi_1|^{q_2}dx + t^p \int_\Omega b_2\varphi_1^{p_1+1}|\nabla\varphi_1|^{p_2}dx = r_1t^q + r_2t^p,\end{aligned}$$

for every $t \in \mathbb{R}^+$.

(i) As $p > q \geq 1$ and $r_1r_2 < 0$ we have the following cases:

(i_1) If $r_1 > 0 > r_2$, taking $\delta = \sqrt[p-q]{-r_1/r_2} > 0$ we have that $\Phi(t) > 0$, for every $t \in (0, \delta)$, and $\Phi(t) < 0$, for every $t > \delta$. Therefore, there exist $t_1 \in (0, \delta)$ and $t_2 \in (\delta, \infty)$ such that (h_2^+) is satisfied. The existence of a solution it follows from Theorem 2.1, with $\beta_1^* = (\mu^*)^{\frac{p-q}{p-1}}$ and μ^* given by Theorem 2.1.

(i_2) If $r_2 > 0 > r_1$, the result follows arguing as in item (i_1) and from Theorem 2.1, with $(h_\nabla^-)_2$ instead of $(h_\nabla^+)_2$.

Remark 2.19. *Arguing as in the proof of Theorem* 2.17*, we may conclude that the solution given by Proposition* 2.5 *is positive in* Ω*.*

Now we present the proof of Proposition 2.7.

Proof of Proposition 2.7. By Theorem 2.1, it suffices verify that there exist real numbers t_1 and t_2, with $t_1 < t_2$, satisfying $(h_\nabla)_2$: remembering that $g_i^- := \liminf_{s\to-\infty} g(s)$, $g(s) + M \geq 0$ and $\Gamma(x, s\varphi_1, s\nabla\varphi_1) + \alpha \geq 0$, for every $s \leq 0$, we may use Fatou's Lemma and (LL_∇) to get

$$\begin{aligned}\liminf_{t\to-\infty} \int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx &= \liminf_{t\to-\infty} \int_\Omega (f + g(t\varphi_1) + \Gamma(x, t\varphi_1, t\nabla\varphi_1))\varphi_1 dx \\ &= \int_\Omega f\varphi_1 dx + \liminf_{t\to-\infty} \int_\Omega g(t\varphi_1)\varphi_1 dx + \liminf_{t\to-\infty} \int_\Omega \Gamma(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \\ &\geq \int_\Omega (f + g_i^- - \alpha)\varphi_1 dx > 0.\end{aligned}$$

Analogously, as $g_s^+ := \limsup_{s\to\infty} g(s)$, $M - g(s) \geq 0$ and $\alpha - \Gamma(x, s\varphi_1, s\nabla\varphi_1) \geq 0$, for every $s \geq 0$, Fatou's Lemma and (LL^+) assure us that

$$\begin{aligned}\limsup_{t\to\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx &= \limsup_{t\to\infty} \int_\Omega (f + g(t\varphi_1) + \Gamma(x, t\varphi_1, t\nabla\varphi_1))\varphi_1 dx \\ &= \int_\Omega f\varphi_1 dx + \limsup_{t\to\infty} \int_\Omega g(t\varphi_1)\varphi_1 dx + \limsup_{t\to\infty} \int_\Omega \Gamma(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \\ &\leq \int_\Omega (f + g_s^+ + \alpha)\varphi_1 dx < 0.\end{aligned}$$

Consequently, there exist real numbers t_1 and t_2, with $t_1 < 0 < t_2$, such that the condition (h_2^+) is valid for t_1 and t_2. This concludes the proof of the proposition. □

Before ending this section, we give an application of Proposition 2.2: we consider $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$h(x,t,\xi) = \sum_{i,j=0}^{m} \alpha_i(x) t^i |\xi|^j, \text{ where } \alpha_{ij} \in L^\sigma(\Omega), \text{ with } \sigma > N \text{ if } N \geq 3 \text{ and } \sigma > 2 \text{ if } N = 1, 2. \tag{2.39}$$

In this case, the function Φ_∇, defined in (2.37), is given by

$$\Phi_\nabla(t) = \sum_{i,j=0}^{m} d_{ij} t^{i+j}, \text{ with } d_{ij} = \int_\Omega \alpha_{ij}(x) \varphi_1^{i+1} |\nabla \varphi_1|^j dx.$$

The existence of solutions provided by Proposition 2.2 depends on the multiplicity of the roots of Φ_∇:

Proposition 2.20. *Suppose h is given by* (2.39). *If the function Φ_∇ has $\tau_1, \ldots, \tau_k$ roots of multiplicity odd, then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (2.1) *has k solutions $u_i = \hat{t}_i \varphi_1 + v_i$ of class $C^{1,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in \mathbb{R}$ and $v_i \in \langle \varphi_1 \rangle^\perp$, $i = 1, \cdots, k$. Furthermore, if $(\lambda - \lambda_1)/\mu \to 0$, as $\mu \to 0$, the solutions converge to $\tau_i \varphi_1$, as $\mu \to 0$, for $i = 1, \ldots, k$.*

Proof. We consider only the case where Φ_∇ has k roots of odd multiplicity, with k even, the case k even may be argued in similar way.

As h is given by (2.39), it is sufficient to prove that there exist $t_i \in \mathbb{R}$, with $t_i < t_{i+1}$, $i = 1, \cdots, k$, satisfying $(\hat{h}_\nabla)_2$. Indeed, as observed above, $\Phi_\nabla(t)$ is a polynomial function. Therefore, we can express Φ_∇ in the form

$$\Phi(t) = (t - \tau_1)^{2n_1+1} \cdots (t - \tau_k)^{2n_k+1} (t - c_1)^{2z_1} \cdots (t - c_l)^{2z_l} p(t),$$

where c_j are the roots of multiplicity even from $\Phi_\nabla(t)$, with $j = 1, \cdots, l$ and the τ_i are ordered in an increasing manner, n_i, $z_i \in \mathbb{N}$ and $p(t)$ is the product of quadratic polynomials irreducible. Without loss of generality we may assume that $p(t) > 0$. Thus, as k is even, for $t \notin \{c_1, \ldots, c_l\}$, we have that

$$\begin{cases} \Phi(t) > 0 & \text{for every } t < \tau_1, \\ \Phi(t) < 0 & \text{for every } \tau_1 < t < \tau_2, \\ & \vdots \\ \Phi(t) < 0 & \text{for every } \tau_{k-1} < t < \tau_k, \\ \Phi(t) > 0 & \text{for every } t > \tau_k. \end{cases}$$

This implies that there exist t_i, $i = 1, \cdots, k+1$, with $t_1 \in (-\infty, \tau_1)$, $t_{k+1} \in (\tau_k, \infty)$ and $t_i \in (\tau_{i-1}, \tau_i)$, $i = 2, \cdots, k$, satisfying $(\hat{h}_\nabla)_2$ such that $c_j \notin (t_i, t_{i+1})$, for every $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Consequently, by Proposition 2.2, there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (2.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{1,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.

Finally, we prove that the solutions u_i, $i = 1, \ldots, k$, converge to $\tau_i\varphi_1$, $i = 1, \ldots, k$. Fixing i, by Lemma 2.16, $v_i = v_i(\mu) \to 0$ in $C^1(\overline{\Omega})$, as $\mu \to 0$, and there exists $\hat{t}_0 \in [t_i, t_{i+1}]$ such that, up to a subsequence, $\hat{t}_i = \hat{t}_i(\mu) \to \hat{t}_0$, $u_i = u_i(\mu) = \hat{t}_i(\mu)\varphi_1 + v_i(\mu) \to \hat{t}_0\varphi_1$ and $\nabla u_i = \nabla u_i(\mu) = \hat{t}_i(\mu)\nabla\varphi_1 + \nabla v_i(\mu) \to \hat{t}_0\nabla\varphi_1$, for every $x \in \Omega$, as $\mu \to 0$. Moreover, as u_i is solution from (2.1),

$$0 = \frac{\lambda - \lambda_1}{\mu}\|\varphi_1\|_2^2\hat{t}_i + \int_\Omega h(x, \hat{t}_i\varphi_1 + v_i, \hat{t}_i\nabla\varphi_1 + \nabla v_i)\varphi_1 dx.$$

Therefore, since $(\lambda-\lambda_1)/\mu \to 0$, as $\mu \to 0$, the Lebesgue Dominated Convergence Theorem assures us that

$$\begin{aligned} \Phi(\hat{t}_0) &= \int_\Omega h(x, \hat{t}_0\varphi_1, \hat{t}_0\nabla\varphi_1)\varphi_1 dx \\ &= \lim_{\mu\to 0}\left[\frac{\lambda - \lambda_1}{\mu}\|\varphi_1\|_2^2\hat{t}_i + \int_\Omega h(x, \hat{t}_i\varphi_1 + v_i, \hat{t}_i\nabla\varphi_1 + \nabla v_i)\varphi_1 dx\right] = 0. \end{aligned}$$

Consequently, from de $(\hat{h}_2)$, $\hat{t}_0 \in (t_i, t_{i+1})$. Thus, as τ_i is the only root of Φ in (t_i, t_{i+1}) we conclude that $u_i = \hat{t}_i\varphi_1 + v_i \to \hat{t}_0\varphi_1 = \tau_i\varphi_1$. □

Remark 2.21. (*i*) *It follows from Propositon* 2.18 *that the solutions* u_i, *provided by Proposition* 2.20, *are ordered.*

(*ii*) *Given* $p(t) = \sum_{i,j=0}^m d_{ij}t^{i+j}$, *with* d_{ij}, $i, j = 1, \ldots, m$, *constants, we may find* $h(x, t, \xi) = \sum_{i,j=0}^m \alpha_{ij}t^i|\xi|^j$ *such that* $p(t) = \Phi_\nabla(t)$. *Indeed, for this it suffices to take* $d_{ij} = \alpha_{ij}\int_\Omega \varphi_1^{i+1}|\nabla\varphi_1|^i dx$.

Bibliography

[1] Agmon S.; Douglis A.; Nirenberg L. *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I.* Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] Alama, S.; Tarantello, G. *On semilinear elliptic equations with indefinite nonlinearities.* Calc. Var. Partial Differential Equations 1 (1993), 439-475.

[3] Alama, S.; Del Pino, M. *Solutions of elliptic equations with indefinite nonlinearities via morse theory and linkings.* Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), 95-115.

[4] Alama, S.; Tarantello, G. *Elliptic problems with nonlinearities indefinite in sign.* J. Funct. Anal. 141 (1996), 159-215.

[5] Alves, C. O.; De Freitas L. R.; Soares, S. H. M. *Indefinite quasilinear elliptic equations in exterior domains with exponential critical growth.* Differential Integral Equations 24 (2011), 1047-1062.

[6] Amann H.; Crandall M. G. *On some existence theorems for semi-linear elliptic equations.* Indiana Univ. Math. J., 27 (1978), 779-790.

[7] Ambrosetti, A.; Brezis, H.; Cerami, G. *Combined effects of concave and convex nonlinearities in some elliptic problems.* J. Funct. Anal. 122 (1994), 519-543.

[8] Ambrosetti, A.; Arcoya, D. *On a quasilinear problem at strong resonance.* Topol. Methods Nonlinear Anal., 6 (1995), 255-264.

[9] Arcoya, D.; Martinez-Aparicio P. J. *Quasilinear equations with natural growth.* Rev. Mat. Iberoam., 24 (2008), 597-616.

[10] Bartsch, T.; Willem, M. *On an elliptic equation with concave and convex nonlinearities.* Proc. Amer. Math. Soc.123 (1995), 3555-3561.

[11] Berestycki, H.; Capuzzo-Dolcetta, I.; Nirenberg, L. *Superlinear indefinite elliptic problems and nonlinear Liouville theorems.* Topol. Methods Nonlinear Anal. 4 (1994), 59-78.

[12] Boccardo, L.; Gallouët, T. *Strongly nonlinear elliptic equations having natural growth terms and L^1 data.* Nonlinear Anal., 19 (1992), 573-579.

[13] Boccardo, L. *Dirichlet problems with singular and gradient quadratic lower order terms.* ESAIM Control Optim. Calc. Var., 14 (2008), 411-426.

[14] Brézis, H.; Turner, R. E. L. *On a class of superlinear elliptic problems.* Comm. Partial Differential Equations 2 (1977), 601-614.

[15] Brézis H.; Nirenberg L. *Positive Solutions of Nonlinear Elliptic Equations involving Critical Exponents.* Communications on Pure and Applied Mathematics, 34 (1983), 437

[16] Brown, K. J.; Wu, T. F. *A fibering map approach to a semilinear elliptic boundary value problem.* Electron. J. Differential Equations 2007, 9 pp.

[17] Bueno, H.; Ercole, G.; Ferreira, W.M.; Zumpano, A.*Existence of positive solutions for the p-Laplacian with dependence on the gradient.* Nonlinearity 25 (2012), 1211-1234.

[18] Cañada A. *Nonselfadjoint semilinear elliptic boundary value problems.* Ann. Mat. Pura Appl., 148 (1987), pp. 237-250.

[19] Castro, A.; Lazer, A. C. *Critical point theory and the number of solutions of a nonlinear Dirichlet problem.* Ann. Mat. Pura Appl. (4) 120 (1979), 113-137.

[20] Castro A. *Reduction methods via minimax.* First Latin American School of Differential Equations, Held at São Paulo, Brazil, June 29-July 17, 1981.

[21] Chang, K. C.; Jiang, M. Y. *Dirichlet problem with indefinite nonlinearities.* Calc. Var. Partial Differential Equations 20 (2004), 257-282.

[22] Cossio, J. *Contribution to the study of elliptic partial differential equations.* (*Spanish*) *Mathematics and statistics* (*Spanish*). Rev. Acad. Colombiana Cienc. Exact. Fís. Natur. 28 (2004), 135-145.

[23] Costa D. G.; Tehrani, H. *Existence of positive solutions for a class of indefinite elliptic problems in* $\mathbb{R}^N$. Calc. Var. Partial Differential Equations 13 (2001), 159-189.

[24] Courant R.; Hilbert D. *Methods of Mathematical Physics. Vol II: Partial differential equations.* Interscience Publishers, New York-Lon don 1962.

[25] De Figueiredo D.; Girardi M.; Matzeu M. *Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques.* Differential Integral Equations, 17 (2004),119-126.

[26] De Figueiredo, D. G.; Gossez, J. P.; Ubilla, P. *Local superlinearity and sublinearity for indefinite semilinear elliptic problems.* J. Funct. Anal. 199 (2003), 452-467.

[27] De Figueiredo, D. G.; Gossez, J. P.; Ubilla, P. *Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity.* J. Eur. Math. Soc. (JEMS) 8 (2006), 269-286.

[28] De Figueiredo, G. M. *Quasilinear equations with dependence on the gradient via Mountain Pass techniques in* $\mathbb{R}^N$. Applied Mathematics and Computation 203 (2008), 14-18.

[29] Faraci, F.; Motreanu, D.; Puglisi, D. *Positive solutions of quasi-linear elliptic equations with dependence on the gradient.* Calc. Var. Partial Differential Equations 54 (2015), 525-538.

[30] Gilbarg, D.; Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order.* Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[31] Girardi M.; Matzeu M. *Positive and negative solutions of a quasi-linear elliptic equation by a Mountain Pass method and truncature techniques.* Nonlinear Anal., 59 (2004), 199-210.

[32] Girardi M.; Matzeu M. *A compactness result for quasilinear elliptic equations by Mountain Pass techniques*, Rend. Mat. Appl. (7), 29 (2009), 83-95.

[33] Iturriaga, L.; Lorca S.; Sánchez, J. *Existence and multiplicity results for the p-Laplacian with a p-gradient term.* Nonlin. Diff. Eqns Applic. 15 (2008), 729-743.

[34] Ladyzhenskaya, O. A.; Uraltseva, N. N. *Linear and quasilinear elliptic equations.* Izdat. "Nauka", Mosacow (1964) (in Russian. English translation: Academic Press, New York (1968). 2nd Russian edition (1973).

[35] Landesman, E. M.; Lazer, A. C. *Nonlinear perturbations of linear elliptic boundary value problems at resonance.* J. Math. Mech. 19 (1970), 609-623.

[36] Landesman, E. M.; Lazer, A. C.; Meyers, David R. *On Saddle Point Problems in the calculus of Variations, the Ritz Algorithm, and Monotone Convergence.* J. Math. Anal. Appl. 52 (1975), 594-614.

[37] Matzeu M.; Servadei R. *A variational approach to a class of quasilinear elliptic equations not in divergence form.* Discrete Cont. Din. Syst. 5(2012), 819-830.

[38] Medeiros, E. S.; Severo, U. B.; Silva, E. A. B. *On a class of elliptic problems with indefinite nonlinearities.* Calc. Var. Partial Differential Equation 50 (2014), 751-777.

[39] Mironescu, P.; Rădulescu, V. D. *The study of a bifurcation problem associated to an asymptotically linear function.* Nonlinear Anal. 26 (1996), 857-875.

[40] Nagle K.; Pothoven K.; Singkofer k. *Nonlinear elliptic equations at resonance where the nonlinearity depends essentially on the derivatives.* J. Differential Equations, 38 (1980), pp. 210-225

[41] Ouyang, T. *On the positive solutions of semilinear equations* $\Delta u + \lambda u + hu^p = 0$ *on compact manifolds II.* Indiana Univ. Math. J. 40 (1991), 1083-1141.

[42] Pohožaev S. *Equations of the type* $\Delta u = f(x, u, Du)$. Mat. Sb. (N.S.), 113(155) (1980), 324-338, 351.

[43] Rabinowitz, P. H. *Minimax methods in critical point theory with applications to diferential equations.* CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[44] Rezende, M. C.; Santos, C. A. *Quasilinear problems with two parameters including superlinear and gradient terms.* Electron. J. Differential Equations, 2014, 2014(220): 1-22.

[45] Ruiz, D. *A priori estimates and existence of positive solutions for strongly nonlinear problems.* J. Differ. Equ. 199, 96-114 (2004).

[46] Shaw, H. *Nonlinear elliptic boundary value problems at resonance.* J. Differential Equations 26 (1977), 335-346.

[47] Silva, E. A. B.; Silva, M. L. *Continuous dependence of solutions for indefinite semilinear elliptic problems*. Electron. J. Differential Equations 2013, 17 pp.

[48] Struwe M. *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer-Verlag, Zürich, 2007.

[49] Xavier J. B. M. *Some existence theorems for equations of the form* $-\Delta u = f(x, u, Du)$. Nonlinear Anal., 15 (1990), 59-67.

[50] Yan Z. *A note on the solvability in* $W^{2,p}(\Omega)$ *for the equation* $-\Delta u = f(x, u, Du)$. Nonlinear Anal., 24 (1995), 1413-1416.