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W-congruences for minimal surfaces in Nil_3 and Laguerre minimal surfaces in space forms

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Abstract

We obtain a Bäcklund transformation between minimal surfaces in Nil_3 by performing a Calabi correspondence between a CMC-$\frac{1}{2}$ surface in $\mathbb{L}^3$ and its associated minimal surface in Nil_3 and a Ribaucour transform on the original surface in $\mathbb{L}^3$. Next, we relate the geometry of both these surfaces using the Abresch-Rosenberg second form. Furthermore, we extend the definition of a Laguerre minimal surface to space forms whilst relating these to the minimal immersions on $\mathbb{L}^3$ and minimal surfaces on other product spaces $\mathbb{M}^2(k) \times \mathbb{R}$ and $\mathbb{M}^2(k) \times \mathbb{R}_1$, with $k = \pm 1$.

Keywords: Heisenberg space, minimal surfaces, Ribaucour transform, Calabi transform, Abresch-Rosenberg differential form, Hopf differential form, congruence of spheres, Laguerre geometry.

Resumo

Obtemos uma transformação de Bäcklund entre superfícies mínimas em Nil_3 aplicando uma correspondência de Calabi entre uma superfície CMC-$\frac{1}{2}$ em $\mathbb{L}^3$ e sua superfície associada em Nil_3 e fazendo uma transformação de Ribaucour na superfície original em $\mathbb{L}^3$. Em seguida, relacionamos a geometria dessas duas superfícies usando a segunda forma de Abresch-Rosenberg. Adiante, estendemos a definição de superfícies mínimas de Laguerre a formas espaciais enquanto relacionamos estas às superfícies mínimas em $\mathbb{L}^3$ e mínimas em outros espaços produto $\mathbb{M}^2(k) \times \mathbb{R}$ e $\mathbb{M}^2(k) \times \mathbb{R}_1$, com $k = \pm 1$.

Palavras-chave: Espaço de Heisenberg, superfícies mínimas, transformação de Ribaucour, transformação de Calabi, diferencial de Abresch-Rosenberg, diferencial de Hopf, congruência de esferas, geometria de Laguerre.

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Contents

Chapter 1

Introduction

In recent years, the geometry of surfaces immersed in the three dimensional Thurston geometries has generated a great activity among geometers, and many beautiful classical results that were known for surfaces in Euclidean space were extended to this class of spaces [2], [1], [14], [11], [12], [13], [15], [16]. In particular, minimal surfaces in such spaces have received a special attention, due to their obvious geometric appeal and to the strong link with important analytic objects such as holomorphic differentials and harmonic maps.

In this thesis we have studied two topics that are related to minimal surfaces in Thurston geometries. The first one is the creation of a geometric method to produce new examples of minimal surfaces in the Heisenberg space (Nil_3) by starting with a given minimal surface. Our second topic introduces the so called Laguerre minimal surfaces in space forms, and we relate them to minimal surfaces in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

In the first chapters we treat our first topic. The main goal here is to show that the classical theory of transformation of surfaces can be extended to some of the Thurston geometries that are not space forms. More precisely, we will exhibit a geometric transformation, analogous to the classical Bäcklund transformation, for minimal surfaces in the three dimensional Heisenberg space (Nil_3). We also provide a family of new examples of minimal surfaces in Nil_3, by applying our transformation to a concrete example.

The theory of transformations of surfaces is an important chapter in the classical differential geometry of surfaces. It was mainly written in the period between the last decades of the 19th and first decades of the 20th century by mathematicians such as Lie, Bianchi, Bäcklund, Guichard, Darboux, and Eisenhart. In a nutshell, the theory associates to an initial geometric object, such as a surface with constant Gaussian curvature or a surface with constant mean curvature, a new, or transformed surface, with the same geometric property. The elements of this pair of surfaces are related in a nice geometric way. In general, they are either the focal surfaces of a congruence of lines or the envelopes of a congruence of spheres.

The most popular example of this theory is the Bäcklund transformation for surfaces immersed in Euclidean space with constant negative Gaussian curvature. This example is a landmark in the interplay between geometry and partial differential equations and has inspired many works in both geometry and integrable systems. One of the interesting aspects of such transformation is that one may start with with a simple example and produce from it a family of non trivial examples of surfaces with constant negative Gaussian curvature. Examples that would not be easy to produce otherwise.

In the same spirit, since there are not many explicit examples of minimal surfaces in Nil_3, our method can be applied to enlarge this set of examples, which are important to guide theoretical ideas.

Our strategy to obtain the transformation for minimal surfaces in Nil_3 is to compose two geometric transformations. Namely, we will consider the Ribaucour transformation of a spacelike constant mean curvature equal to $\frac{1}{2}$, CMC-$\frac{1}{2}$, surface in the Lorentz-Minkowski space $\mathbb{L}^3$, and the so called generalized Calabi transformation [19] between the mentioned CMC surface and a minimal surface in Nil_3. Our main result is to exhibit an integrable system that depends solely on the geometry of a given simply connected minimal surface immersed in Nil_3, such that its solutions explictly define a new minimal surface immersed in Nil_3. These two surfaces are naturally diffeomorphic and such diffeomorphism preserves (Euclidean) asymptotic lines, and the straight lines joining the corresponding points under this diffeomorphism is tangent to both surfaces at the corresponding points (see Theorem 5 and Proposition 11).

While Ribaucour transformations in Euclidean space are well known and have a simple geometric definition, the generalized Calabi transformation that transforms constant mean curvature from a specific Riemannian three manifold into spacelike surfaces with mean curvature in a specific Lorentzian three manifold is still quite recent and certain of its geometric aspects are yet to be discovered. Our original contribution to the theory of Ribaucour transformation (see Chapter 3) was to verify that some important well known results valid in Euclidean space generalize nicely to $\mathbb{L}^3$. For the generalized Calabi transformation, we have discovered some geometric content in this a priori purely analytical transformation, that might be useful for future research.

The last chapter is devoted to our second topic, namely, Laguerre minimal surfaces in space forms. In the classic book [6], Blaschke and Thomsen study what is known as Lie sphere geometry. This geometry can be divided into two important subgeometries, nowdays called conformal (or Moebius) geometry and Laguerre geometry. There is a notion of minimal surface for both geometries. Moebius minimal surfaces are the critical points of the famous Willmore functional

$$\int_S \left(H^2 - K\right) \, dA,$$

and Laguerre minimal surfaces are the critical points of the Laguerre functional

$$\int_S \frac{H^2 - K}{K} \, dA.$$

It is well known that the Willmore function is invariant under conformal transformations and all three space forms are locally in the same conformal class. Thus, it does not seem interesting to try to define Moebius minimal surfaces in space forms. However, the situation is quite different for the Laguerre functional, and we have identified what looks like a good definition for Laguerre minimal surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3$ as critical points of a certain functional. We have also computed the first variation of this functional: a surface is a critical point of the functional if and only if the average of the radii of curvature is harmonic with respect to a natural metric (see Theorem 7). We have also shown that the well known duality for such surfaces in the Euclidean case extend to our general definition (see Theorem 8). Finally, we relate a particular class of these Laguerre minimal surfaces in space forms to minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}_1$ and $\mathbb{H}^2 \times \mathbb{R}_1$. As an application of this relation, we produce explicit examples of what we call Laguerre minimal surfaces in space forms by starting with well known examples of minimal surfaces in the mentioned product spaces.

This thesis is organized as follows. In Chapter 2, we present some known useful results regarding the classical Ribaucour transformation, the particular Calabi correspondence we will use and the Bäcklund transformation. Also, we present certain aspects of the geometry of Nil_3 and its Gauss map, the Abresch-Rosenberg differential and its relation to the euclidean geometry.

In Chapter 3, we extend the Ribaucour transform to surfaces in $\mathbb{L}^3$ and obtain conditions to guarantee that, if a surface is such that $\alpha + \beta H + \gamma K = 0$, for some real numbers α, β and γ, then its transform retains the same property. In particular, we have means to see that CMC-$\frac{1}{2}$ surfaces are transformed into CMC-$\frac{1}{2}$ surfaces.

In Chapter 4, we obtain a transformation between minimal surfaces in Nil_3 starting with a CMC-$\frac{1}{2}$ in $\mathbb{L}^3$ and a particular Ribaucour transform. Furthermore, we recast this transformation solely in terms of a given minimal surface in Nil_3.

Finally, in Chapter 5, we obtain a generalization to space forms of the Laguerre functional and its Euler-Lagrange equation. In addition, we show a duality between Laguerre minimal surfaces in space forms and conclude by exhibiting a relation between a special type of Laguerre minimal surface and minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$.

Chapter 2

Preliminaries

2.1 Ribaucour Transforms

Ribaucour transformations were studied by Luigi Bianchi in the late 18th century [5] and, with the revival of constant mean curvature surfaces as an active topic of research in the last decades of the the 20th century, many works about such transformations appeared on a steady basis since [9].

Roughly speaking, a congruence of spheres in $\mathbb{R}^3$ is a smooth 2 parameter family of spheres. If a given a congruence of spheres has two distinct surfaces everywhere tangent to the spheres of the congruence, called the envelopes of the congruence, is such that the correspondence defined by the tangency condition preserves lines of curvature, then the envelopes are said to be related by a Ribaucour transformation. More precisely, a modern way to state the definition of Ribaucour transformations in $\mathbb{R}^3$ is the following.

Let Σ and $\tilde{\Sigma}$ be two distinct surfaces in $\mathbb{R}^3$ with orthonormal principal frames given by $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$, and suppose that N and $\tilde{N}$ are unitary normal vector fields defined, at least locally, on each patch of Σ and $\tilde{\Sigma}$, respectively.

Definition 1 (Ribaucour transformation on $\mathbb{R}^3$)**.** *The surfaces Σ and $\tilde{\Sigma}$ are said to be Ribaucour transforms of each other when there exists a diffeomorphism $\phi : \Sigma \to \tilde{\Sigma}$ and a differentiable function $h : \Sigma \to \mathbb{R}$ such that:*

1. *$p + h(p)N(p) = \phi(p) + h(p)\tilde{N}(p)$, for every $p \in \Sigma$;*
2. *$d_p\phi(e_i) = \tilde{e}_i(\phi(p))$, for every $i \in \{1, 2\}$, that is, ϕ preserves lines of curvature.*

Over the years, extensions of this definition were conceived and a linear system of partial differential equations was obtained in order to solve the problem of finding a Ribaucour transform of a given surface. Also, it is known how to preserve certain properties of the original surface such as constant Gaussian or mean curvatures. Thus, Ribaucour transformations can be used to obtain new examples of constant Gaussian or mean curvatures surfaces. In fact, if a surface is linear Weingarten, then these transforms can be adjusted to produce another linear Weingarten surface with the same parameters [10]. In Chapter 3 we will show that with minor modifications these results can be extended to spacelike surfaces in the Lorentz Minkowski space $\mathbb{L}^3$.

2.2 Calabi Duality

We will make use of another transformation that generalizes one introduced by Calabi [7], which is called Calabi transformation for short. These transformations act on constant mean curvature surfaces in a specific Riemannian space and send them into a constant mean curvature surface in a specific Lorentzian space. Classically, a Calabi transform of a minimal surface in $\mathbb{R}^3$ is a maximal surface in Lorentzian space $\mathbb{L}^3$ [19], that is, a surface that is a critical point of the area functional in $\mathbb{L}^3$.

To describe precisely the Calabi transformation, we start by recalling the definition of an important class of Riemannian and Lorentzian three dimensional geometries.

Definition 2 (Bianchi-Cartan-Vranceanu spaces)**.** *Let* $\delta_\kappa(x,y) = 1 + \frac{\kappa}{4}(x^2+y^2)$*. For real numbers* κ *and* τ*, the Bianchi-Cartan-Vranceanu space* $\mathbb{E}(\kappa,\tau)$ *is the Riemmanian manifold* (V,g) *where*

$$V = \left\{(x,y,z) \in \mathbb{R}^3 | \delta_\kappa(x,y) > 0\right\},$$

$$g = \frac{dx^2+dy^2}{\delta_\kappa(x,y)^2} + \left[\tau\left(\frac{ydx - xdy}{\delta_\kappa(x,y)}\right) + dz\right]^2.$$

The Lorentzian counterpart of this space is the pseudo-Riemannian space given by:

$$\mathbb{L}^3(\kappa,\tau) = \left(V, \frac{dx^2+dy^2}{\delta_\kappa(x,y)^2} - \left[\tau\left(\frac{ydx - xdy}{\delta_\kappa(x,y)}\right) + dz\right]^2\right).$$

The special case $\mathbb{E}\left(0,\frac{1}{2}\right)$ is usually called the Heisenberg space and will be denoted by Nil_3. We shall treat the geometry of Nil_3 is detail in the next subsection.

After Calabi's work, his transformation was generalized for minimal surfaces in product spaces by [3] and, more recently, H. Lee has generalized Calabi's transformation to BCV spaces. Roughly speaking, Lee associates to a CMC-H graph in $\mathbb{E}(\kappa,\tau)$ a spacelike CMC-τ graph in $\mathbb{L}^3(\kappa,H)$, see Theorem 1 in [19]. We will use a special case of this correspondence, namely, we will explore the correspondence between a minimal surfaces in Nil_3 and CMC-$\frac{1}{2}$ in $\mathbb{L}^3(0,0) = \mathbb{L}^3$. For the reader's convenience, we state this as follows.

Proposition 1. *(Lee's twin correspondence). Let* $g(x,y)$ *be a function defined over a simply connected domain* U *in the plane* $z=0$*, such that its graph is a spacelike CMC-*$\frac{1}{2}$ *surface in* $\mathbb{L}^3$ *with respect to the downward pointing normal in* $\mathbb{L}^3$*. Then there exists* $f: U \to \mathbb{R}$ *such that its graph is a minimal surface in* Nil_3*. The functions* f *and* g *are related by the following equations.*

$$(f_x, f_y) = \left(\frac{g_y}{\sqrt{1-g_x^2-g_y^2}} - \frac{y}{2}, \frac{-g_y}{\sqrt{1-g_x^2-g_y^2}} + \frac{x}{2}\right),$$

$$(g_x, g_y) = \left(\frac{-f_y + \frac{x}{2}}{\sqrt{1+(f_x+\frac{y}{2})^2+(f_y-\frac{x}{2})^2}}, \frac{f_x + \frac{y}{2}}{\sqrt{1+(f_x+\frac{y}{2})^2+(f_y-\frac{x}{2})^2}}\right).$$

We will say that the surfaces that appear in Proposition 1 are twins. It turns out that the correspondence between twin surfaces given by Proposition 1 is conformal, and the conformal factor between the metrics has a geometric meaning [11]. We state this result as follows.

Proposition 2. *Let S and $S^\star$ be respectively a spacelike CMC-$\frac{1}{2}$ surface in $\mathbb{L}^3$ and a minimal surface in Nil_3 as in Proposition 1. Let g_S and $g_{S^\star}$ be their metrics. Then $g_S = \nu^{-2} g_{S^\star}$, with $\nu = \langle E_3, N \rangle_{Nil_3}$, where $E_3 = (0,0,1)$ and N is a unit normal field $S^\star$.*

2.2.1 Surfaces in Nil_3

The three dimensional nilpotent Lie group Nil_3 can be viewed as the set $\mathbb{R}^3$ with the group structure given by

$$(x_1, y_1, z_1) * \left(x_2, y_2, z_2 \right) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{x_1 y_2 - x_2 y_1}{2} \right),$$

and endowed with the left invariant metric defined by,

$$ds^2 = dx^2 + dy^2 + \left(\frac{ydx - xdy}{2} + dz \right)^2,$$

where (x, y, z) are the canonical coordinates of $\mathbb{R}^3$.

A left invariant orthonormal frame $\{E_1, E_2, E_3\}$ is defined by the following expressions.

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

For more details, we refer to [11] and [17].

We remark that, in coordinates, the inner product of $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ at the point (x, y, z) in is given by

$$\begin{aligned} \langle v, w \rangle_{\mathrm{Nil}_3} = {} & v_1 w_1 \left(1 + \frac{y^2}{4} \right) + v_2 w_2 \left(1 + \frac{x^2}{4} \right) + v_3 w_3 \\ & - \frac{xy}{4}(v_1 w_2 + v_2 w_1) + \frac{y}{2}(v_1 w_3 + v_3 w_1) - \frac{x}{2}(v_3 w_2 + v_2 w_3). \end{aligned} \tag{2.2}$$

The Gauss map in Nil_3

Since Nil_3 is a Lie group, it is natural to relate the tangent space at a point (x, y, z) with the Lie algebra of Nil_3 by a left translation. In particular, the unit normal field to an immersed surface Σ defines, via left translation, a map $\eta : \Sigma \longmapsto \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the standard round sphere viewed as a subset of the Lie algebra of Nil_3. In [11], Daniel noticed an interesting fact: for minimal surfaces such that the image of η is contained in the upper hemisphere of $\mathbb{S}^2$ endowed with a hyperbolic metric, the map η is harmonic.

In Section 4.1 we will show how to recast Lee's twin correspondence in terms of a simple geometric relation between the Gauss map of a minimal surface and the Gauss map of a spacelike CMC-$\frac{1}{2}$ in $\mathbb{L}^3$. For this reason, instead of working directly with η we project the open upper hemisphere of $\mathbb{S}^2$ onto the the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$.

2.3 W-congruences

Following Chern and Terng [8] we will consider a congruence of lines as an immersed surface in the Grassmann manifold of lines in $\mathbb{R}^3$. Locally, we can suppose that this two parameter family of lines are parametrized in the following way,

$$Y(u,v) = X(u,v) + \lambda\xi(u,v),$$

with $\xi(u,v)$ being a unit vector field in the Euclidean metric and λ being a parameter for each line. A regular parametrized curve $(u(t), v(t))$ defines a ruled surface belonging to the congruence. This ruled surface is developable if and only if the determinant $(\xi, dX, d\xi)$ vanishes. In the generic case, for each line L of the congruence there are two such developables that contain L. Now, for each of the developables, there is the point of its striction line that lies in L. These points are called the focal points of the congruence at L. It can be shown, in the generic case, that the set of focal points constitute two surfaces Σ and Σ^*, to be called the focal surfaces of the congruence. Moreover, the lines of the congruence are tangent to both Σ and Σ^* at the focal points and we have the following well known proposition.

Proposition 3. *If there is a diffeomorphism between two immersed surfaces in $\mathbb{R}^3$ such that the straight lines joining corresponding points are tangent to both surfaces at these points, then these two surfaces are the focal surfaces of a congruence of lines.*

A congruence of lines is called a W-congruence if the correspondence between focal points along the lines of the congruence preserves asymptotic lines.

The classical Bäcklund transformation is an example of a W-congruence. The generalization of Backlund transformations for affine minimal surfaces obtained by Chern and Terng in [8] is also an example of a W-congruence.

2.4 The Abresch-Rosenberg quadratic form

Classically, in three-dimensional Euclidean space, the Hopf differential is a quadratic form Qdz^2 defined over an immersed surface Σ with conformal parameter z (cf. [18]). Let N be the Gauss map of Σ and h_{ij} define the coefficients of the second fundamental form with respect to N. The function Q is such that:

$$Q = \frac{h_{11} - h_{22}}{4} + i\frac{h_{12}}{2}.$$

Using the Codazzi equations in $\mathbb{R}^3$ with respect to the second fundamental form II induced by N, it is easy to check, assuming that g_{ij} are the metric coefficients, that

$$\frac{\partial Q}{\partial \overline{z}} = g_{11} H_z.$$

The previous equation implies that the CMC-H surfaces are precisely those over which the Hopf differential is holomorphic. Moreover, using the definition of Q, we may infer whether or not a given conformal parametrization is such that its coordinate curves are also lines of curvature. In this case, Q

is a real valued function. Also, the quadratic form Qdz^2 does not depend on the chosen parameter z and is, in general, known as the Hopf differential of the pair (I, II). In addition, note that (I, II) is umbilical at $p \in \Sigma$ if and only if $Q(p) = 0$. We present the following definitions regarding the pair (I, II).

Definition 3. *[14] A* fundamental pair *on Σ is a pair of real quadratic forms (I, II) on Σ, where I is a Riemannian metric.*

A special case occurs when the shape operator S associated to (I, II) satisfies the Codazzi equations. Thus we may define the following:

Definition 4. *[14] We say that a fundamental pair (I, II), with shape operator S, is a Codazzi pair if*

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = 0, \quad X, Y \in \mathfrak{X}(\Sigma),$$

where ∇ stands for the Levi-Civita connection associated with the Riemannian metric I and $\mathfrak{X}(\Sigma)$ is the set of smooth vector fields on Σ.

In 2004, U. Abresch and H. Rosenberg [1] proposed a Hopf differential for CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ which originated the Abresch-Rosenberg quadratic differential. Following this result, in 2011, M. Batista [4] derived a second form II_S so that the Codazzi pair (I, II_S) defined over a CMC surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ has the Abresch-Rosenberg differential as its Hopf differential. In 2013, Espinar and Trejos [14] extended this result for CMC surfaces in $\mathbb{E}(\kappa, \tau)$, with $\tau \neq 0$. The Abresch-Rosenberg quadratic differential has become an important concept for mathematicians interested in CMC surfaces in BCV spaces.

Definition 5. *[14] Given a local conformal parameter z for I, the Abresch-Rosenberg differential for a minimal surface in Nil_3 with unitary normal vector field N is defined by:*

$$Q_{AR}dz^2 = (iQ + t^2)dz^2,$$

where Q is the usual Hopf differential, $t = \langle \partial_z, T \rangle$ and $T = E_3 - \langle E_3, N \rangle N$.

The differential given by Definition 5 is the Hopf differential relative to the Codazzi pair (I, II_{AR}), where II_{AR} was obtained by Espinar and Trejos [14]. Since we will not use this definition in all its generality, we reduce II_{AR} to our case of interest, which are minimal surfaces ($H = 0$) in Nil_3 ($\mathbb{E}(0, \frac{1}{2})$):

Definition 6. *[14] Given a minimal surface $\Sigma \subset Nil_3$ with unitary vector field N, the Abresch-Rosenberg quadratic form is defined as:*

$$II_{AR}(X, Y) = II(X, Y) + \left\langle \hat{T}, X \right\rangle \left\langle \hat{T}, Y \right\rangle - \frac{|T|^2}{2} \langle X, Y \rangle,$$

where $X, Y \in \mathfrak{X}(\Sigma)$, $T = E_3 - \langle E_3, N \rangle N$, $JT = N \wedge T$ and $\hat{T} = \dfrac{T - JT}{\sqrt{2}}$.

We draw attention to the next propositions regarding some relations between the classical Hopf differential and the Abresch-Rosenberg differential.

Lemma 1. *[20] Let (I, II) be a fundamental pair. Then, any two of the following conditions implies the third:*

1. *(I, II) is a Codazzi pair.*

2. *H is constant.*

3. *The Hopf differential of the pair is holomorphic.*

Proposition 4. *[14] Let (I, II_{AR}) be a fundamental pair. Then $H(I, II_{AR}) = H(I, II)$, where $H(\cdot,\cdot)$ is the mean curvature with respect to the pair $(\cdot,\cdot)$.*

Let Σ be an immersed surface in Nil_3 and η be its Gauss map, obtained by a left translation of the unitary normal vector field N defined over Σ. Consider that g_{ij} are the coefficients of the first fundamental form. On the other hand, consider the same surface immersed in $\mathbb{R}^3$ with Euclidean Gauss map given by $N_{\mathbb{E}}$. Admit that N and $N_{\mathbb{E}}$ define the same orientation on Σ. Furthermore, consider that $\langle\cdot,\cdot\rangle_{\mathbb{E}}$ is the Euclidean inner product and that $\langle\cdot,\cdot\rangle$ is the inner product in Nil_3 defined on (2.2). The following lemma allows us to relate both metrics and the Gauss maps in either space.

Although the quadratic form II_{AR} was introduced in such a way that the Abresch-Rosenberg differential turns out to be the Hopf differential of the pair (I, II_{AR}), it turns out that for the Nil_3 case this quadratic form has a nice relation with Euclidean second fundamental form. Details are available in [21].

Lemma 2. *Let E_3 be the vertical Killing vector field in Nil_3. Then, with the definitions above,*

$$\langle E_3, N\rangle = \frac{\langle E_3, N_{\mathbb{E}}\rangle_{\mathbb{E}}}{\langle N, N_{\mathbb{E}}\rangle_{\mathbb{E}}}.$$

Proposition 5. *Let Σ be a surface in Nil_3. Then the following equation holds.*

$$II_{AR}(X,Y) = \frac{vol_{\mathbb{E}}(\Sigma)}{vol_{Nil_3}(\Sigma)} II(X,Y), \quad X, Y \in \mathfrak{X}(\Sigma),$$

where II is the Euclidean second fundamental form.

Using Proposition 5, we have a useful geometrical result.

Proposition 6. *If Σ is a minimal surface immersed in Nil_3, then there exists a local conformal parametrization of Σ in which coordinate lines are Euclidean asymptotic lines.*

Proof. The Abresch-Rosenberg differential, for a conformal parametrization, is given by

$$Q_{AR} = (h_{22}^{AR} - h_{11}^{AR} + 2ih_{12}^{AR})dz^2.$$

where h_{ij}^{AR} are the coefficients of the second form II_{AR}.

Locally, after a conformal change of parameter, we may assume that Q_{AR} is pure imaginary, i.e., $h_{22}^{AR} = h_{11}^{AR}$. From Proposition 4, we see that $H(I, II) = H(I, II_{AR})$ and therefore $h_{22}^{AR} + h_{11}^{AR} = 0$. Thus, $h_{22}^{AR} = h_{11}^{AR} = 0$. Lastly, from Proposition 5 we know that II_{AR} is a multiple of II, so we conclude that there is a local conformal parameter such that coordinate lines are Euclidean asymptotic lines. $\square$

We quote a result that will be used later.

Proposition 7. *[11] The Abresch-Rosenberg differential Q_{AR} of a nowhere vertical minimal surface in Nil_3 coincides (up to a constant) with the Hopf differential Q of its Gauss map. Namely, $Q_{AR} = 4i \cdot Q$.*

Another useful result is the following.

Proposition 8. *Let $\Sigma \subset Nil_3$ be a surface with a given conformal parametrization Z by Euclidean asymptotic lines. Then Σ is minimal.*

Proof. Using Proposition 5, we have $h_{ij}^{AR} = \lambda h_{ij}^{\mathbb{E}}$, for $i \neq j$ and a nonvanishing function λ, and $h_{ij}^{\mathbb{E}} = 0$, for $i = j$, we have $h_{11}^{\mathbb{AR}} = h_{22}^{\mathbb{AR}} = 0$. Since the parametrization is conformal, this implies that $H(I, II_{AR}) = 0$. Finally, from Proposition 4 we conclude that Σ is minimal in Nil_3. $\square$

Chapter 3

Ribaucour transformations that preserve linear Weingarten surfaces in $\mathbb{L}^3$

Ribaucour transformations for surfaces immersed in Euclidean three space is a classical subject that has been revisited in recent decades and is still a topic of interest. Our goal here is to extend to surfaces in $\mathbb{L}^3$ the results in [26]. We will show how to non-trivially transform CMC-$\frac{1}{2}$ spacelike surfaces into CMC-$\frac{1}{2}$. As mentioned in the introduction, this transformation is a key step to transform minimal surfaces in Nil_3. We start our discussion with the following definition. For a different and broader approach on Ribaucour transformations in Lorentzian spaces we refer to [23]. Throughout this chapter, the inner product used in $\mathbb{L}^3$ is given by $\langle (v_1, v_2, v_3), (w_1, w_2, w_3)\rangle = v_1w_1 + v_2w_2 - v_3w_3$.

Definition 7. *Let M and $\tilde{M}$ be umbilic-free spacelike surfaces in $\mathbb{L}^3$ with orthonormal principal vector fields $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ and normal vector fields N and $\tilde{N}$, respectively. We say that M and $\tilde{M}$ are associated by a Ribaucour transformation when there exists a diffeomorphism $\phi : M \to \tilde{M}$ and a differentiable function $h : M \to \mathbb{R}$ such that:*

1. $p + h(p)\cdot N(p) = \phi(p) + h(p)\cdot \tilde{N}(\phi(p))$;
2. $d_p\phi(e_i) = \tilde{e}_i(\phi(p))$;
3. *ϕ preserves lines of curvature.*

If this is the case, we will say that p and $\tilde{p} := \phi(p)$ are corresponding points of this transformation.

Let X be a local parametrization of M with unitary normal vector field N in which coordinate lines are lines of curvature. Similarly, let $\tilde{X}$ be a local parametrization of $\tilde{M}$ with unitary vector field $\tilde{N}$. Thus, we may write:

$$\tilde{X} = X + h(N - \tilde{N}). \tag{3.1}$$

The geometric content of this definition is that M and $\tilde{M}$ are the envelopes of a congruence (i.e., a 2-parameter family) of Lorentzian spacelike spheres. In the classical Euclidean case, the surface comprised of the centers of such spheres was given and the difficult task was to find a convenient radius function h such that the envelopes shared some common geometric property such as having constant mean curvature. Darboux was the first one to note that this problem could be recast in terms of a linear system of first order PDEs, and this point of view will be adopted here.

Let M and $\tilde{M}$ be related by a Ribaucour transformation. Then, since M is spacelike, we set $e_1 = \dfrac{X_u}{|X_u|}$ and $e_2 = \dfrac{X_v}{|X_v|}$ so that $\{e_1, e_2, N\}$ form an orthonormal frame for $\mathbb{L}^3$ along M. Surely, we must have $\langle N, N\rangle = -1$. Next, we write $\tilde{N}$ in terms of this frame:

$$\tilde{N} = \sum_{i=1}^{2} b_i e_i + b_3 N. \tag{3.2}$$

Since $\tilde{M}$ is also spacelike, we obtain $b_1^2 + b_2^2 - b_3^2 = -1$. Moreover, expressing $dN(e_i)$ in terms of this frame, we set:

$$d\tilde{N}(e_i) = \sum_{k=1}^{2} L_i^k e_k + L_i^3 N. \tag{3.3}$$

Since X is a parametrization by lines of curvature, then its transform $\tilde{X}$ also shares this property and so, both X and $\tilde{X}$ must be umbilic-free. More precisely, we may write $dN(e_i) = \lambda_i e_i$ and we obtain $\left\langle d\tilde{N}(e_i), d\tilde{X}(e_j)\right\rangle = 0$, for $i \neq j$. In this setting, we will prove two lemmas that refer to the expression of the coefficients b_i and L_i^k. They will enable us to rewrite $\tilde{N}$ in a more convenient way.

Lemma 3. *If $\tilde{N} = \sum_{i=1}^{2} b_i e_i + b_3 N$, then*

$$\begin{aligned} b_i &= \frac{dh(e_i)(b_3 - 1)}{1 + h\lambda_i}, \text{ for } i \in \{1, 2\}, \\ b_3 &= \frac{\Delta + 1}{\Delta - 1}, \\ \tilde{N} &= \frac{1}{\Delta - 1}\left[(\Delta + 1)N + \sum_{i=1}^{2} 2Z_i e_i\right]. \end{aligned}$$

where $Z_i = \dfrac{dh(e_i)}{1 + h\lambda_i}$ and $\Delta = Z_1^2 + Z_2^2$, for $i \in \{1, 2\}$.

Proof. Since $\tilde{X}$ is a Ribaucour transform, we have $d\tilde{X} = dX + dh(N - \tilde{N}) + h(dN - d\tilde{N})$ so that

$$\begin{aligned} d\tilde{X}(e_i) &= dX(e_i) + (N - \tilde{N})dh(e_i) + h(dN(e_i) - d\tilde{N}(e_i)) \\ &= e_i + dh(e_i)(N - \tilde{N}) - h(\lambda_i e_i d\tilde{N}(e_i)) \\ &= (1 + h\lambda_i)e_i + dh(e_i)(N - \tilde{N}) - hd\tilde{N}(e_i). \end{aligned} \tag{3.4}$$

We relate the coefficients b_i by the following:

$$
\begin{aligned}
\left\langle d\tilde{X}(e_i), \tilde{N} \right\rangle &= 0 \\
\left\langle (1+h\lambda_i)e_i + dh(e_i)(N-\tilde{N}) - hd\tilde{N}(e_i), \tilde{N} \right\rangle &= 0 \\
(1+h\lambda_i)b_i - dh(e_i)b_3 + dh(e_i) &= 0 \\
(1+h\lambda_i)b_i &= dh(e_i)b_3 - dh(e_i) \\
b_i &= \frac{dh(e_i)(b_3-1)}{1+h\lambda_i}.
\end{aligned}
$$

Recall that $\left\langle \tilde{N}, \tilde{N} \right\rangle = -1$. Hence, by setting $Z_i = \dfrac{dh(e_i)}{1+h\lambda_i}$ and $\Delta = Z_1^2 + Z_2^2$, we have

$$
\begin{aligned}
b_1^2 + b_2^2 - b_3^2 &= -1 \\
(b_3-1)^2\Delta - (b_3^2 - 1) &= 0 \\
(b_3-1)((b_3-1)\Delta - (b_3+1)) &= 0 \\
\Delta b_3 - b_3 &= \Delta + 1 \\
b_3 &= \frac{\Delta+1}{\Delta-1}.
\end{aligned}
$$

Also, $b_i = Z_i(b_3 - 1) = \dfrac{2Z_i}{\Delta-1}$. From these relations, we conclude that

$$
\tilde{N} = \frac{1}{\Delta-1}\left[(\Delta+1)N + \sum_{i=1}^{2} 2Z_i e_i\right].
$$

□

Lemma 4. *If $d\tilde{N}(e_i) = \sum_{k=1}^{2} L_i^k e_k + L_i^3 N$, then:*

$$
\begin{aligned}
L_i^j &= \frac{2dZ_j(e_i)}{\Delta-1} - \frac{2Z_j d\Delta(e_i)}{(\Delta-1)^2} + \frac{2}{\Delta-1}\left[\frac{\Delta+1}{2}\lambda_i\delta_{ij} + \sum_k Z_k\omega_{kj}(e_i)\right], \\
L_i^3 &= -\frac{2d\Delta(e_i)}{(\Delta-1)^2} + \frac{2Z_i\lambda_i}{\Delta-1}.
\end{aligned}
$$

Proof. Using equation (3.2), we deduce that $\tilde{N} = \sum_{k=1}^{2} b_k e_k + b_3 N$. Therefore, we obtain

$$
d\tilde{N}(e_i) = \sum_{k=1}^{2} db_k(e_i)e_k + b_k de_k(e_i) + db_3(e_i)N + b_3 dN(e_i).
$$

Noticing that $de_i(e_j) = \sum \omega_{ik}(e_j)e_k$, we get, for $1 \le j \le 2$,

$$
\begin{aligned}
L_i^j = \left\langle d\tilde{N}(e_i), e_j \right\rangle &= \left\langle \sum_{k=1}^{2} db_k(e_i)e_k + b_k de_k(e_i) + db_3(e_i)N + b_3 dN(e_i), e_j \right\rangle \\
&= db_j(e_i) + \left\langle \sum_{k=1}^{2} b_k \sum_{l=1}^{2} \omega_{kl}(e_i)e_l, e_j \right\rangle + b_3 \left\langle dN(e_i), e_j \right\rangle,
\end{aligned}
$$

so that $L_i^j = db_j(e_i) + \sum_{k=1}^2 b_k\omega_{kj}(e_i) + b_3\lambda_i\delta_{ij}$.

We now compute L_i^3. Indeed, notice that

$$\begin{aligned} -L_i^3 = \left\langle d\tilde{N}(e_i), N\right\rangle &= -db_3(e_i) + \left\langle \sum_{k=1}^2 b_k de_k(e_i), N\right\rangle \\ &= -db_3(e_i) + b_i \left\langle de_i(e_i), N\right\rangle . \end{aligned}$$

Next, since X is a parametrization by lines of curvature, $dN(e_i) = \lambda_i e_i$. The previous expression becomes

$$L_i^3 = db_3(ei) + b_i\lambda_i.$$

The expression for L_i^j may be rewritten using the equations for b_i as:

$$\begin{aligned} L_i^3 &= db_3(e_i) + b_i\lambda_i \\ &= d\left(\frac{\Delta+1}{\Delta-1}\right)(e_i) + Z_i(b_3-1)\lambda_i \\ &= d\left(1+\frac{2}{\Delta-1}\right)(e_i) + Z_i\frac{2}{\Delta-1}\lambda_i \\ &= -\frac{2d\Delta(e_i)}{(\Delta-1)^2} + \frac{2Z_i\lambda_i}{\Delta-1}. \end{aligned} \tag{3.5}$$

From this, it follows that

$$\begin{aligned} L_i^j &= db_j(ei) + \sum_k b_k\omega_{kj}(e_i) + b_3\lambda_i\delta_{ij} \\ &= d\left(Z_j\frac{2}{\Delta-1}\right)(ei) + \sum_k \frac{2Z_k}{\Delta-1}\omega_{kj}(e_i) + \frac{\Delta+1}{\Delta-1}\lambda_i\delta_{ij} \\ &= \frac{2dZ_j(e_i)}{\Delta-1} - \frac{2Z_jd\Delta(e_i)}{(\Delta-1)^2} + \frac{2}{\Delta-1}\left[\frac{\Delta+1}{2}\lambda_i\delta_{ij} + \sum_k Z_k\omega_{kj}(e_i)\right]. \end{aligned} \tag{3.6}$$

□

Proceeding, we obtain some differential equations that h must satisfy. This is the core of it, so it is worth mentioning as a theorem.

Theorem 1. *If there exists some function $h : M \to \mathbb{R}$ such that X and $\tilde{X}$ are associated by a Ribaucour transformation, then the following equations hold:*

$$dZ_j(e_i) + \sum_k Z_k\omega_{kj}(e_i) - Z_iZ_j\lambda_i = 0, \text{ for } i \neq j. \tag{3.7}$$

Proof. Since ϕ preserves lines of curvature, we have $\left\langle d\tilde{N}(e_i), d\tilde{X}(e_j)\right\rangle = 0$, for $i \neq j$. Hence

$$\left\langle \sum_{k=1}^{2} L_i^k e_k + L_i^3 N, (1+h\lambda_j)e_j + dh(e_j)(N-\tilde{N}) - hd\tilde{N}(e_j) \right\rangle = 0$$
$$L_i^j(1+h\lambda_j) - L_i^3 dh(e_j) = 0$$
$$L_i^j - L_i^3 Z_j = 0, \text{ for } i \neq j.$$

Using our recently derived expressions for L_i^j, we have:

$$\frac{2dZ_j(e_i)}{\Delta-1} - \frac{2Z_j d\Delta(e_i)}{(\Delta-1)^2} + \frac{2}{\Delta-1}\left[\delta_{ij}\frac{\Delta+1}{2}\lambda_i + \sum Z_k\omega_{kj}(e_i)\right] + \frac{2Z_j d\Delta(e_i)}{(\Delta-1)^2} - \frac{2Z_iZ_j\lambda_i}{\Delta-1} = 0$$
$$dZ_j(e_i)(\Delta-1) - Z_j d\Delta(e_i) + (\Delta-1)\sum_k Z_k\omega_{kj}(e_i) + d\Delta(e_i)Z_j - (\Delta-1)Z_iZ_j\lambda_i = 0$$
$$dZ_j(e_i) + \sum_k Z_k\omega_{kj}(e_i) - Z_iZ_j\lambda_i = 0.$$

□

These are the PDE satisfied by h so that X has a Ribaucour transform. Since these PDE coincide with those obtained by [26], the next proposition is valid. See [26] for more details.

Proposition 9. *Let h be a non-vanishing function satisfying* (3.7)*. Then*

$$\phi = \frac{1}{h}\sum_k Z_k\omega_k.$$

is a closed 1-form and there exists a non-vanishing function Ω defined on a simply connected domain such that $d\Omega(e_i) = \frac{\Omega}{h}Z_i$, for $i \in \{1,2\}$.

Essentially, this means that the system (3.7) is always integrable. By defining $\Omega_i = d\Omega(e_i)$ and $W = \frac{\Omega}{h}$ it is possible to recast the PDE given by (3.7) as an integrable linear system. This is the content of the following theorem.

Theorem 2. *Let X be an umbilic-free spacelike immersion in $\mathbb{L}^3$ defined on a simply-connected domain and consider the positive orthonormal frame given by $\{e_1, e_2, N\}$, where $e_i = \frac{X_{u_i}}{|X_{u_i}|}$, $i = 1, 2$, and N is a timelike vector field normal to X satisfying $dN(e_i) = \lambda_i X_{u_i}$. If $\Omega_1, \Omega_2, \Omega$ and W are such that*

$$
\begin{aligned}
d\Omega_i(e_j) &= \sum_{k=1}^{2} \Omega_k \omega_{ik}(e_j),\ \textit{for } i \neq j, \\
d\Omega &= \sum_{i=1}^{2} \Omega_i \omega_i, \\
dW &= -\sum_{i=1}^{2} \Omega_i \lambda_i \omega_i \\
&\textit{and} \\
h &= \frac{\Omega}{W},
\end{aligned}
\tag{3.8}
$$

then $\tilde{X} = X - \frac{2\Omega}{S}\left(WN + \sum_{i=1}^{2} \Omega_i e_i\right)$ *is a parametrization of a Ribaucour transform of* X *with normal vector field given by* $\tilde{N} = N + \frac{2W}{S}\left(WN + \sum_{i=1}^{2} \Omega_i e_i\right)$*, in which* $S = \Omega_1^2 + \Omega_2^2 - W^2$.

Proof. The equations in (3.8) are easily verified using (3.7) and the definitions of Ω and W.

Next, using (3.2) and setting $S = \sum_{i=1}^{2} \Omega_i - W^2$:

$$
\begin{aligned}
\tilde{N} &= \frac{1}{\Delta - 1}\left[(\Delta + 1)N + \sum_{i=1}^{2} 2Z_i e_i\right] \\
&= \frac{1}{\left(\frac{\Omega_1}{W}\right)^2 + \left(\frac{\Omega_2}{W}\right)^2 - 1}\left[\left(\left(\frac{\Omega_1}{W}\right)^2 + \left(\frac{\Omega_2}{W}\right)^2 + 1\right) N + \sum_{i=1}^{2} 2\frac{\Omega_i}{W} e_i\right] \\
&= \frac{W^2}{\Omega_1 + \Omega_2 - W^2}\left[\frac{\Omega_1 + \Omega_2 + W^2}{W^2} N + \sum_{i=1}^{2} 2\frac{\Omega_i}{W} e_i\right] \\
&= \frac{2W}{S}\sum_{i=1}^{2} \Omega_i e_i + \frac{\Omega_1 + \Omega_2 + W^2}{S} N \\
&= \frac{2W}{S}\sum_{i=1}^{2} \Omega_i e_i + \frac{\Omega_1 + \Omega_2 + W^2}{S} N - \frac{2W^2}{S} N + \frac{2W^2}{S} N \\
&= N + \frac{2W}{S}\left(WN + \sum_{i=1}^{2} \Omega_i e_i\right).
\end{aligned}
\tag{3.9}
$$

Hence, the parametrization of the transform $\tilde{X}$, using (3.1) is given by

$$
\begin{aligned}
\tilde{X} &= X + h(N - \tilde{N}) \\
&= X + \frac{\Omega}{W}\left[N - \left(N + \frac{2W}{S}\left(WN + \sum_{i=1}^{2} \Omega_i e_i\right)\right)\right] \\
&= X - \frac{2\Omega}{S}\left(WN + \sum_{i=1}^{2} \Omega_i e_i\right).
\end{aligned}
\tag{3.10}
$$

$\square$

In local coordinates, we may rewrite Theorem 2:

Corollary 1. *Under the conditions of Theorem 2, let $a_i = |X_{u_i}|$ and* (3.8) *reduces to:*

$$
\begin{aligned}
&\frac{\partial \Omega_i}{\partial u_j} = \Omega_j \frac{1}{a_i}\frac{\partial a_j}{\partial u_i}, \text{ for } i \neq j,\\
&\frac{\partial \Omega}{\partial u_i} = \Omega_i a_i\\
&\qquad \text{and}\\
&\frac{\partial W}{\partial u_i} = -\Omega_i \lambda_i a_i.
\end{aligned}
\tag{3.11}
$$

Further we will discuss Ribaucour transformations that preserves linear Weingarten surfaces so, it will be useful to compute the principal curvatures of $\tilde{X}$.

Proposition 10. *Let X and $\tilde{X}$ be related by a Ribaucour transformation satisfying* (3.8). *Denote the principal curvatures of X and $\tilde{X}$ by λ_i and $\tilde{\lambda}_i$, $i = 1, 2$, respectively. Then the following equations hold:*

$$\tilde{\lambda}_i = \frac{dS(e_i)W + \Omega_i \lambda_i S}{\Omega_i S - \Omega dS(e_i)}, \quad i = 1, 2. \tag{3.12}$$

Proof. Since $\tilde{X}$ is also a parametrization by lines of curvature, we have the following

$$\tilde{\lambda}_i = \frac{\left\langle d\tilde{N}(e_i), d\tilde{X}(e_i)\right\rangle}{\left\langle d\tilde{X}(e_i), d\tilde{X}(e_i)\right\rangle}.$$

We will compute the previous numerator and denominator separately. From equation (3.4), we obtain

$$(1 + h\tilde{\lambda}_i)d\tilde{X}(e_i) = (1 + h\lambda_i)e_i + dh(e_i)(N - \tilde{N}).$$

By applying the inner product on each side with itself, we have

$$
\begin{aligned}
(1 + h\tilde{\lambda}_i)^2 \left\langle d\tilde{X}(e_i), d\tilde{X}(e_i)\right\rangle &= (1 + h\lambda_i)^2 - 2dh(e_i)(1 + h\lambda_i)b_i - 2dh(e_i)^2 + 2dh(e_i)^2 b_3\\
&= (1 + h\lambda_i)^2 - 2dh(e_i)^2(b_3 - 1) - 2dh(e_i)^2 + 2dh(e_i)^2 b_3\\
&= (1 + h\lambda_i)^2,
\end{aligned}
$$

which can be written as $\left\langle d\tilde{X}(e_i), d\tilde{X}(e_i)\right\rangle = \dfrac{(1 + h\lambda_i)^2}{(1 + h\tilde{\lambda}_i)^2}$.

On the other hand, by using equations (3.4) and (3.3), we deduce

$$
\begin{aligned}
\left\langle d\tilde{X}(e_i), d\tilde{N}(e_i)\right\rangle &= \left\langle \frac{(1+h\lambda_i)e_i + dh(e_i)N}{1+h\tilde{\lambda}_i}, \sum_{k=1}^{2} L_i^k e_i + L_i^3 N \right\rangle \\
&= \frac{1+h\lambda_i}{1+h\tilde{\lambda}_i} L_i^i - \frac{dh(e_i)}{1+h\tilde{\lambda}_i} L_i^3 \\
&= \frac{1+h\lambda_i}{1+h\tilde{\lambda}_i}\left(L_i^i - \frac{dh(e_i)}{1+h\lambda_i} L_i^3\right) \\
&= \frac{1+h\lambda_i}{1+h\tilde{\lambda}_i}\left(L_i^i - Z_i L_i^3\right).
\end{aligned}
$$

From which it follows that $\tilde{\lambda}_i = \dfrac{1+h\tilde{\lambda}_i}{1+h\lambda_i}(L_i^i - Z_i L_i^3)$. The expressions for L_i^j and L_i^3 were obtained in (3.6) and (3.5). So, we will compute $Z_i L_i^i - Z_i^2 L_i^3$. Indeed, notice that

$$
\begin{aligned}
Z_i L_i^i &= Z_i \left[\frac{2dZ_i(e_i)}{\Delta - 1} - \frac{2Z_i d\Delta(e_i)}{(\Delta-1)^2} + \frac{2}{\Delta-1}\left(\sum_{k=1}^{2} Z_k \omega_{ki}(e_i) + \frac{\Delta+1}{2}\lambda_i\right)\right] \\
&= \frac{2}{\Delta-1}\left[Z_i dZ_i(e_i) - \frac{Z_i^2 d\Delta(e_i)}{\Delta-1} + \sum_{k=1}^{2} Z_i Z_k \omega_{ki}(e_i) + \frac{\Delta+1}{2}\lambda_i Z_i\right] \\
&= \frac{2}{\Delta-1}\left[Z_i dZ_i(e_i) + \sum_{k=1}^{2} Z_i Z_k \omega_{ki}(e_i)\right] - \frac{2Z_i^2 d\Delta(e_i)}{(\Delta-1)^2} + \frac{\Delta+1}{\Delta-1}\lambda_i Z_i.
\end{aligned}
\tag{3.13}
$$

At the same time, using equation (3.7), we achieve

$$
\begin{aligned}
dZ_j(e_i) + \sum_{k=1}^{2} Z_k \omega_{kj}(e_i) - Z_i Z_j \lambda_i &= 0, \text{ for } i \neq j, \\
dZ_j(e_i) + \sum_{k\neq i} Z_k \omega_{kj}(e_i) - Z_i Z_j \lambda_i &= -Z_i \omega_{ij}(e_i), \\
dZ_j(e_i) + \sum_{k\neq i} Z_k \omega_{kj}(e_i) - Z_i Z_j \lambda_i &= Z_i \omega_{ji}(e_i),
\end{aligned}
$$

and hence, we get

$$
\sum_{j\neq i}\left(Z_j dZ_j(e_i) + \sum_{k\neq i} Z_j Z_k \omega_{kj}(e_i) - Z_i Z_j^2 \lambda_i\right) = \sum_{j\neq i} Z_j Z_i \omega_{ji}(e_i).
$$

Substituting the previous relation in (3.13) and adjusting indexes, we arrive at

$$
\begin{aligned}
Z_i L_i^i &= \frac{2}{\Delta - 1}\left[Z_i dZ_i(e_i) + \sum_{k\neq i}(Z_k dZ_k(e_i) - Z_i Z_k^2 \lambda_i)\right] - \frac{2Z_i^2 d\Delta(e_i)}{(\Delta-1)^2} + \frac{\Delta+1}{\Delta-1}\lambda_i Z_i \\
&= \frac{2}{\Delta - 1}\left[\frac{d\Delta(e_i)}{2} - Z_i\lambda_i \sum_{k\neq i} Z_k^2\right] - \frac{2Z_i^2 d\Delta(e_i)}{(\Delta-1)^2} + \frac{\Delta+1}{\Delta-1}\lambda_i Z_i \\
&= \frac{2}{\Delta - 1}\left[\frac{d\Delta(e_i)}{2} - Z_i\lambda_i(\Delta - Z_i^2)\right] - \frac{2Z_i^2 d\Delta(e_i)}{(\Delta-1)^2} + \frac{\Delta+1}{\Delta-1}\lambda_i Z_i \qquad (3.14)\\
&= \frac{d\Delta(e_i)}{\Delta-1} - Z_i\lambda_i + Z_i^2\left[-\frac{2d\Delta(e_i)}{(\Delta-1)^2} + \frac{2Z_i\lambda_i}{\Delta-1}\right] \\
&= -\frac{\Delta-1}{2}\left[-\frac{2d\Delta(e_i)}{(\Delta-1)^2} + \frac{2Z_i\lambda_i}{\Delta-1}\right] + Z_i^2 L_i^3 \\
&= -\frac{\Delta-1}{2}L_i^3 + Z_i^2 L_i^3.
\end{aligned}
$$

Therefore, we conclude that

$$
L_i^i - Z_i L_i^3 = \frac{1-\Delta}{2Z_i}L_i^3. \qquad (3.15)
$$

This provides the following expression for $\tilde{\lambda_i}$:

$$
\tilde{\lambda_i} = \frac{1 + h\tilde{\lambda_i}}{1 + h\lambda_i}\cdot\frac{1-\Delta}{2Z_i}L_i^3,
$$

that is,

$$
\tilde{\lambda_i} = -\frac{(\Delta-1)L_i^3}{2dh(e_i) + (\Delta-1)L_i^3 h}.
$$

Notice that $\Delta - 1 = \dfrac{S}{W^2}$ so that

$$
\begin{aligned}
(\Delta-1)L_i^3 &= (\Delta-1)\cdot\left[-\frac{2d\Delta(e_i)}{(\Delta-1)^2} + \frac{2Z_i\lambda_i}{\Delta-1}\right] \\
&= -\frac{2d\Delta(e_i)}{\Delta-1} + 2Z_i\lambda_i \\
&= -2d(log(\Delta-1))(e_i) + 2Z_i\lambda_i \\
&= -2d\left(log\left(\frac{S}{W^2}\right)\right)(e_i) + 2\frac{\Omega_i}{W}\lambda_i \\
&= -2d\left(logS - logW^2\right)(e_i) + 2\frac{\Omega_i}{W}\lambda_i \\
&= -2\left[\frac{dS(e_i)}{S} - \frac{2dW(e_i)}{W}\right] + 2\frac{\Omega_i}{W}\lambda_i \\
&= -2\frac{dS(e_i)}{S} - 2\frac{\lambda_i\Omega_i}{W}.
\end{aligned}
$$

We may also compute $2dh(e_i) + (\Delta-1)L_i^3 h$ as follows

$$
\begin{aligned}
2dh(e_i) + (\Delta - 1)L_i^3 h &= 2\frac{\Omega_i}{W}\left(1 + \frac{\Omega}{W}\lambda_i\right) - 2\frac{\Omega}{W}\left(\frac{dS(e_i)}{S} + \frac{\lambda_i\Omega_i}{W}\right) \\
&= 2\frac{\Omega_i}{W} - 2\frac{\Omega dS(e_i)}{WS}.
\end{aligned}
$$

Thus concluding that $\tilde{\lambda}_i = \dfrac{dS(e_i)W + \Omega_i\lambda_i S}{\Omega_i S - \Omega dS(e_i)}$. This finishes the proof of the proposition.

□

Let us define $T_i = 2\left(d\Omega_i(e_i) + \sum_{k=1}^{2}\Omega_k\omega_{ki}(e_i) + W\lambda_i\right)$. It is useful to express $dS(e_i)$ in terms of T_i, and using (3.8) it is easy to check through direct computation that $dS(e_i) = \Omega_i T_i$. This yields $\tilde{\lambda}_i = \dfrac{\Omega_i T_i W + \Omega_i\lambda_i S}{\Omega_i S - \Omega\Omega_i T_i} = \dfrac{T_i W + \lambda_i S}{S - \Omega T_i}$.

Theorem 3. *Under the same conditions as Theorem 2, suppose S satisfy, for some constants c, α, β and γ, the additional relation $S = 2c\left(\alpha\Omega^2 + \beta\Omega W + \gamma W^2\right)$. In this case, $\alpha + \beta H + \gamma K = 0$ if, and only if, $\alpha + \beta\tilde{H} + \gamma\tilde{K} = 0$, where $H, \tilde{H}, K$ and $\tilde{K}$ are the mean and Gaussian curvatures of X and $\tilde{X}$, respectively. Furthermore, let p and $\tilde{p}$ be corresponding points. Then p is umbilic if, and only if, $\tilde{p}$ is umbilic.*

Proof. Given a linear Weingarten surface M, we aim to solve (3.8) so that $\tilde{M}$ is also linear Weingarten. Suppose that the additional condition on S is given:

$$
S = 2c(\alpha\Omega^2 + \beta\Omega W + \gamma W^2).
$$

Using 2, we have

$$
dS = 2c\sum_{k=1}^{2}\left[(2\alpha\Omega + \beta W) - (\beta\Omega + 2\gamma W)\lambda_i\right]\Omega_i\omega_i.
$$

From this, let $P = \alpha\Omega^2 + \beta\Omega W + \gamma W^2$ and we compute both the numerator and denominator of $\tilde{\lambda}_i$ as follows

$$
\begin{aligned}
WdS(e_i) + S\Omega_i\lambda_i &= 2c\Omega_i\left[((2\alpha\Omega + \beta W) - (\beta\Omega + 2\gamma W)\lambda_i)W + \lambda_i P\right], \\
S\Omega_i - \Omega dS(e_i) &= 2c\Omega_i\left[P - \Omega((2\alpha\Omega + \beta W) - (\beta\Omega + 2\gamma W)\lambda_i)\right].
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\tilde{\lambda}_i &= \frac{2\alpha\Omega W + \beta W^2 - \beta\Omega W\lambda_i - 2\gamma W^2\lambda_i + \lambda_i\alpha\Omega^2 + \lambda_i\beta\Omega W + \lambda_i\gamma W^2}{\alpha\Omega^2 + \beta\Omega W\gamma W^2 - 2\alpha\Omega^2 - \beta\Omega W + \beta\Omega^2\lambda_i + 2\gamma W\lambda_i\Omega} \\
&= \frac{(2\alpha\Omega W + \beta W^2) + \lambda_i(\alpha\Omega^2 - \gamma W^2)}{(2\gamma\Omega W + \beta\Omega^2)\lambda_i - (\alpha\Omega^2 - \gamma W^2)}.
\end{aligned}
$$

By setting $T = \alpha\Omega^2 - \gamma W^2$, $L = 2\alpha\Omega W + \beta W^2$ and $Q = 2\gamma\Omega W + \beta\Omega^2$, we have

$$\tilde{\lambda}_i = \frac{L + \lambda_i T}{Q\lambda_i - T}.$$

Let $A = (Q\lambda_1 - T)(Q\lambda_2 - T) = Q^2K + T^2 + 2QTH$, suppose $\alpha + \beta H + \gamma K = 0$ and let us compute $\alpha + \beta\tilde{H} + \gamma\tilde{K}$.

$$\begin{aligned}\alpha + \beta\tilde{H} + \gamma\tilde{K} &= \frac{1}{A}\left[(\alpha T^2 + \beta LT + \gamma L^2) + H(\beta LQ + 2\alpha QT - 2\gamma LT - \beta T^2)\right.\\ &\quad \left.+ K\left(\alpha Q^2 - \beta QT + \gamma T^2\right)\right]\\ &= \frac{1}{A}\left(\alpha Q - \beta T - \gamma L\right)\left(QK + 2TH - L\right).\end{aligned}$$

The converse statement is analogous.

Since $\alpha Q - \beta T - \gamma L = 0$, we conclude that, with the additional equation involving S, two surfaces associated by Ribaucour are either both linear Weingarten or neither one is. It is worth noticing that

$$\begin{aligned}\tilde{\lambda}_2 - \tilde{\lambda}_1 &= \frac{L + \lambda_1 T}{Q\lambda_2 - T} - \frac{L + \lambda_2 T}{Q\lambda_2 - T}\\ &= \frac{LQ\lambda_2 + \lambda_1\lambda_2 TQ - LT - \lambda_1 T^2 - LQ\lambda_1 + LT - \lambda_1\lambda_2 TQ + T^2\lambda_2}{A}\\ &= \frac{\lambda_2(LQ + T^2) - \lambda_1(LQ + T^2)}{A}\\ &= \frac{LQ + T^2}{A}(\lambda_2 - \lambda_1).\end{aligned}$$

That is, corresponding points are both umbilic or neither one is. □

The particular case of Ribaucour transformations that preserve the constant mean curvature $\frac{1}{2}$ property is essential for our arguments in the next chapter, and we will state it as follows.

Theorem 4. *Let $X(u_1, u_2)$ be an umbilic free spacelike immersion in $\mathbb{L}^3$ defined on a simply connected domain such that the coordinate curves are lines of curvature. Consider the positive orthonormal frame given by $\{e_1, e_2, N\}$, where $e_i = \dfrac{X_{u_i}}{|X_{u_i}|}$, $i = 1, 2$, and N is a unit timelike vector field normal to X. Let $a_i = |X_{u_i}|$ and λ_i be such that $dN(e_i) = \lambda_i X_{u_i}$. Suppose that the set of functions $\{\Omega_1, \Omega_2, \Omega, W\}$ is a solution of the completely integrable system*

$$\begin{aligned}\frac{\partial\Omega_i}{\partial u_j} &= \Omega_j \frac{1}{a_i}\frac{\partial a_j}{\partial u_i}, \text{ for } i \neq j,\\ \frac{\partial\Omega}{\partial u_i} &= \Omega_i a_i,\\ \frac{\partial W}{\partial u_i} &= -\Omega_i\lambda_i a_i.\end{aligned} \tag{3.16}$$

Then

$$\tilde{X} = X - \frac{2\Omega}{S}\left(WN + \sum_{i=1}^{2}\Omega_i e_i\right) \tag{3.17}$$

is a parametrization of a Ribaucour transform of X with normal vector field given by

$$\tilde{N} = N + \frac{2W}{S}\left(WN + \sum_{i=1}^{2}\Omega_i e_i\right), \tag{3.18}$$

where $S = \Omega_1^2 + \Omega_2^2 - W^2$.

In addition, if X is a CMC-$\frac{1}{2}$ immersion and $S = 2c(2\Omega W - \Omega^2)$, where $c \in \mathbb{R}$, $c \neq 0$, then $\tilde{X}$ is also a CMC-$\frac{1}{2}$ immersion.

Chapter 4

Bäcklund transformations for minimal surfaces in Nil_3

In this chapter we present our main result, namely, we will show that minimal surfaces in Nil_3 admit Bäcklund transformations. More precisely, given a minimal surface in Nil_3 we can construct a W-congruence such that our minimal surface is a focal surface and the other focal surface is also minimal. Moreover, this Bäcklund transformation turns out to be the composition of the transformations we have already discussed in this work: Calabi and Ribaucour transformations. As an application we will produce new examples of minimal surfaces in Nil_3.

4.1 The generalized Calabi correspondence in geometric form

The generalized Calabi correspondence given by Proposition 1 can be recast in a geometric way that is suitable for applications. Instead of dealing with graphs and the corresponding constant mean curvature PDEs, we express the correspondence in terms of the Gauss maps of the CMC-$\frac{1}{2}$ surface in $\mathbb{L}^3$ and the minimal surface in Nil_3. The advantage of this point of view is that it does not assume that the surfaces involved are given as graphs.

Let Y be a spacelike CMC-$\frac{1}{2}$ immersion into $\mathbb{L}^3$ defined on a simply connected domain $U \in \mathbb{R}^2$ with unit normal N pointing downwards. We will write $Y = (F, h)$, where F is the horizontal part and h the vertical part of Y. We argue that we can associate to Y a minimal immersion into Nil_3 defined on U and given $Z = (F, g)$. Note that the horizontal part of Z coincides with the horizontal part of Y, so we just have to show how the function g is defined. This is done in the following way. The normal field $N = (N_1, N_2, N_3)$ is $\mathbb{H}^2$ valued if we identify the lower sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$ with $\mathbb{H}^2$. Now consider the vector field $\eta = (\eta_1, \eta_2, \eta_3)$ with values in the upper hemisphere of the unit sphere $\mathbb{S}^2$ defined by

$$(\eta_1, \eta_2, \eta_3) = \left(-\frac{N_2}{N_3}, \frac{N_1}{N_3}, -\frac{1}{N_3}\right). \tag{4.1}$$

The expression above corresponds to the following geometric procedure to obtain η from N: first flip N to the upper sheet of hyperboloid, then perform an anti-clockwise rotation with respect to the positive vertical axis and finally project from the upper sheet of the hyperboloid onto the upper hemisphere of $\mathbb{S}^2$ with a radial projection with respect to the point $(0,0,-1)$. We will call this projection Π.

The unknown function g can be determined up to an additive constant using equation (4.1). More precisely, we will show that g is a solution of a first order integrable system of PDEs. If we denote the horizontal part of both immersions as $F=(x_1,x_2)$, then η is given by

$$\eta = \frac{1}{\Delta}\cdot\left(\beta x_{2,x}-\alpha x_{2,y}, -\beta x_{1,x}+\alpha x_{1,y}, x_{1,x}x_{2,y}-x_{2,x}x_{1,y}\right),$$

where Δ normalizes η and α and β are given by

$$\begin{aligned} \alpha &= g_x + \frac{1}{2}(x_2 x_{1,x} - x_1 x_{2,x}), \\ \beta &= g_y + \frac{1}{2}(x_2 x_{1,y} - x_1 x_{2,y}). \end{aligned} \tag{4.2}$$

, see [17].

To determine the system of PDEs for g we need an expression for α and β. From equation (4.1), by looking at the third coordinate, we obtain

$$N_3 = -\frac{\Delta}{x_{1,x}x_{2,y}-x_{1,y}x_{2,x}}.$$

Substitution of this expression into the other two coordinate in (4.1) yields a linear system for α and β, which has the following solution.

$$\begin{aligned} \alpha &= N_1 x_{2,x} - N_2 x_{1,x}, \\ \beta &= N_1 x_{2,y} - N_2 x_{1,y}. \end{aligned}$$

Using equation (4.2), we arrive at the following PDEs for g

$$\begin{aligned} g_x &= N_1 x_{2,x} - N_2 x_{1,x} + \frac{1}{2}(x_1 x_{2,x} - x_2 x_{1,x}), \\ g_y &= N_1 x_{2,y} - N_2 x_{1,y} + \frac{1}{2}(x_1 x_{2,y} - x_2 x_{1,y}). \end{aligned} \tag{4.3}$$

Now, it can be checked that the integrability condition for (4.3) is equivalent to the CMC-$\frac{1}{2}$ condition on Y. Alternatively, it is not difficult to verify that the system (4.3), for the special case of graphs, is precisely the one that appears in Proposition 1.

An interesting geometric aspect of the twin relation defined by the Calabi correspondence is given by the following proposition.

Proposition 11. *Let S and $S^\star$ be a twin pair, with S being a umbilic-free CMC-$\frac{1}{2}$ spacelike surface in $\mathbb{L}^3$ and $S^\star$ being a minimal surface in Nil$_3$. Then the Calabi correspondence sends lines of curvature of S into Euclidean asymptotic lines of $S^\star$.*

Proof. It is well known that we can find a local conformal coordinate system $z=(x,y)$ for S such that the coordinate curves are lines of curvature. This in turn implies that the Hopf differential associated to the harmonic Gauss map N is expressed as $Q = kdz^2$, where k is a real constant. We claim that the coordinate curves of $S^\star$ associated to (x, y) are Euclidean asymptotic lines. To see this, note first that the Gauss map η is harmonic if we consider the hyperbolic metric on the upper hemisphere of $\mathbb{S}^2$ induced by the projection Π. Alternatively, we can use Π to define a harmonic map $\hat{\eta} = \Pi \circ \eta$ with values in $\mathbb{H}^2$. The harmonic maps N and $\hat{\eta}$ differ by a $\frac{\pi}{2}$ rotation with respect to the vertical axis. It is then easy to see that their Hopf quadratic differentials coincide.

Now, By Proposition 7, the Abresch-Rosenberg quadratic differential Q_{AR} associated to $S^\star$ is $4i$ times the Hopf differential of $\hat{\eta}$. Thus, Q_{AR} is a pure imaginary constant, and therefore $h_{11}^{AR} - h_{22}^{AR} = 0$. Since $S^\star$ is minimal and $H(I, II) = H(I, II_{AR})$ it follows that $h_{11}^{AR} = h_{22}^{AR} = 0$. Finally, from Proposition 5 we conclude that the coordinate curves are asymptotic lines (in the Euclidean sense). $\square$

4.2 Bäcklund transformation from Calabi and Ribaucour transformations

We are now in position to establish a Bäcklund type transformation for minimal surfaces in Nil_3. The process works as follows. Given a non vertical minimal immersion into Nil_3, we will consider its twin CMC-$\frac{1}{2}$ surface in $\mathbb{L}^3$, and apply to this surface a Ribaucour transformation to obtain another CMC-$\frac{1}{2}$ in $\mathbb{L}^3$. Finally, we consider the associated twin minimal surface in Nil_3. We will show that there is a choice of the arbitrary additive constant for the twin correspondence such that the two minimal surfaces in Nil_3 are the focal surfaces of a W-congruence.

At a first look, this process to obtain new examples of minimal surfaces in Nil_3 might seem quite complicated, because it appears to involve three stages of integration (i.e. solving PDEs). Namely, one for the first twin correspondence, another for the Ribaucour transformation and one more for the last twin correspondence. Fortunately, it turns out that we can formulate the Bäcklund type transformation solely in terms of the geometric data of the initial minimal surface. So, there is in fact only one integration, namely, the one associated to the Ribaucour system.

Theorem 5. *Let $Z : U \subset \mathbb{R}^2 \to Nil_3$, be a non vertical minimal umbilic-free immersion defined on a simply connected domain and let Y be a spacelike CMC-$\frac{1}{2}$ twin immersion in $\mathbb{L}^3$. Let $\tilde{Y}$ be a CMC-$\frac{1}{2}$ Ribaucour transform of Y and $\tilde{Z}$ be a non vertical twin immersion. Then, the additive constant in the twin correspondence can be chosen in such a way that $\tilde{Z}$ and Z are the focal surfaces of a W-congruence.*

Proof. To perform the necessary computations, we will need a bit of notation. Let η and $\tilde{\eta}$ be the Gauss maps of $Z = (F, g)$ and $\tilde{Z} = (\tilde{F}, \tilde{g})$, respectively, where $\tilde{g}$ is defined up to an additive constant. Let $(\varphi_1, \varphi_2) = (\tilde{F}_1 - F_1, \tilde{F}_2 - F_2)$, where F_i and $\tilde{F}_i$ are the i-th coordinates of F and $\tilde{F}$. Finally, the immersions Y and $\tilde{Y}$ are written as $Y = (x_1, x_2, x_3)$ and $\tilde{Y} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and the respective unit normal fields as $N = (N_1, N_2, N_3)$ and $\tilde{N} = (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$. Without loss of generality, we will consider that the coordinate curves of Z are asymptotic lines, and from Proposition 11 it follows that the coordinate curves of Y and $\tilde{Y}$ are lines of curvature and that the coordinate curves of $\tilde{Z}$ are also asymptotic lines.

From Remark 3, to show that $Z(U)$ and $\tilde{Z}(U)$ are the focal surfaces of the congruence of lines defined by the straight lines passing through Z and $\tilde{Z}$, it suffices to show that these lines are tangent to both immersions.

Now, a tangent vector to one such line at Z is given by

$$Z - \tilde{Z} = \varphi_1 E_1 + \varphi_2 E_2 + (g - \tilde{g} + \frac{1}{2}(\varphi_1 x_2 - \varphi_2 x_1))E_3. \tag{4.4}$$

In the same way, a tangent vector to this straight line at $\tilde{Z}$ is given by

$$\tilde{Z} - Z = \varphi_1 E_1 + \varphi_2 E_2 + (\tilde{g} - g + \frac{1}{2}(\varphi_1 \tilde{x}_2 - \varphi_2 \tilde{x}_1))E_3, \tag{4.5}$$

where the frame $\{E_1, E_2, E_3\}$ is evaluated at Z in (4.4) and at $\tilde{Z}$ in (4.5).

After left translating these vectors to the Lie algebra and taking the inner product with η and $\tilde{\eta}$, we can say that the straight line passing through Z and $\tilde{Z}$ is tangent to both Z and $\tilde{Z}$ if and only if

$$\begin{aligned} \varphi_1 \eta_1 + \varphi_2 \eta_2 + \left(\tilde{g} - g + \frac{x_2 \varphi_1}{2} - \frac{x_1 \varphi_2}{2}\right) \eta_3 = 0, \\ \varphi_1 \tilde{\eta}_1 + \varphi_2 \tilde{\eta}_2 + \left(\tilde{g} - g + \frac{\tilde{x}_2 \varphi_1}{2} - \frac{\tilde{x}_1 \varphi_2}{2}\right) \tilde{\eta}_3 = 0. \end{aligned} \tag{4.6}$$

Solving both equations for $\tilde{g} - g$, we get two expressions for $\tilde{g} - g$ that must coincide if our claim is true. This necessary condition is written as

$$\frac{\varphi_1 \eta_1 + \varphi_2 \eta_2}{\eta_3} - \frac{\varphi_1 \tilde{\eta}_1 + \varphi_2 \tilde{\eta}_2}{\tilde{\eta}_3} + \frac{\varphi_1}{2}(x_2 - \tilde{x}_2) + \frac{\varphi_2}{2}(\tilde{x}_1 - x_1) = 0. \tag{4.7}$$

Using the above notation, Theorem 2 and defining $\psi = \Omega_1 e_1 + \Omega_2 e_2 + WN$, we have

$$\tilde{x}_i - x_i = -\frac{2\Omega}{S}\psi_i, \tag{4.8}$$

$$\tilde{N}_i - N_i = \frac{2W}{S}\psi_i, \tag{4.9}$$

where ψ_i is the i-th coordinate of ψ, for $i = 1, 2$.

Furthermore, using (4.8), (4.9) and (4.1) to simplify (4.7), we get

$$\begin{aligned} 0 &= \left[\frac{\varphi_1 \eta_1 + \varphi_2 \eta_2}{\eta_3} - \frac{\varphi_1 \tilde{\eta}_1 + \varphi_2 \tilde{\eta}_2}{\tilde{\eta}_3}\right] + \frac{\varphi_1}{2}(\tilde{x}_2 - x_2) + \frac{\varphi_2}{2}(\tilde{x}_1 - x_1) \\ &= \left[\varphi_1 N_2 - \varphi_2 N_1 - \varphi_1 \tilde{N}_2 + \varphi_2 \tilde{N}_1\right] - \frac{\Omega}{S}(\varphi_1 \psi_2 - \varphi_2 \psi_1) \\ &= \left[\frac{2W}{S}\left(\varphi_1 \psi_2 - \varphi_2 \psi_1\right)\right] - \frac{\Omega}{S}(\varphi_1 \psi_2 - \varphi_2 \psi_1) \\ &= \frac{2W - \Omega}{S}\left(\varphi_1 \psi_2 - \varphi_2 \psi_1\right). \end{aligned}$$

From the definition of φ and ψ, we have $\varphi_1 \psi_2 - \varphi_2 \psi_1 = 0$. Thus, indeed (4.7) holds.

In the sequel, we will choose the expression of $\tilde{g}-g$ obtained from the top equation of (4.6) and show that it coincides with the expression obtained for $\tilde{g}-g$ when we apply the Ribaucour and Calabi transformations.

Indeed, using (4.1) and taking the derivative with respect to x in (4.6), we obtain

$$\tilde{g}_x - g_x = -\varphi_{1,x}N_2 - \varphi_1 N_{2,x} + \varphi_{2,x}N_1 + \varphi_2 N_{1,x} - \left(\frac{x_{2,x}\varphi_1 + x_2\varphi_{2,x}}{2} - \frac{x_{1,x}\varphi_2 + x_1\varphi_{2,x}}{2}\right). \tag{4.10}$$

On the other hand, using the first equation of (4.3) for g and $\tilde{g}$ we have

$$\begin{aligned}\tilde{g}_x - g_x = \tilde{N}_1\tilde{x}_{2,x} - \tilde{N}_2\tilde{x}_{1,x} + \frac{\tilde{x}_1\tilde{x}_{2,x}}{2} - \frac{\tilde{x}_2\tilde{x}_{1,x}}{2}\\ - N_1 x_{2,x} + N_2 x_{1,x} - \frac{x_1 x_{2,x}}{2} + \frac{x_2 x_{1,x}}{2}.\end{aligned} \tag{4.11}$$

Next, we use (4.8) and (4.9) to rewrite (4.11) as

$$\begin{aligned}\tilde{g}_x - g_x =& \left(N_1 + \frac{2W}{S}\psi_1\right)\left(x_{2,x} - \left(\frac{2\Omega}{S}\psi_2\right)_x\right) - \left(N_2 + \frac{2W}{S}\psi_2\right)\left(x_{1,x} - \left(\frac{2\Omega}{S}\psi_1\right)_x\right)\\ &+ \frac{1}{2}\left(x_1 - \frac{2\Omega}{S}\psi_1\right)\left(x_{2,x} - \left(\frac{2\Omega}{S}\psi_2\right)_x\right) - \frac{1}{2}\left(x_2 - \frac{2\Omega}{S}\psi_2\right)\left(x_{1,x} - \left(\frac{2\Omega}{S}\psi_1\right)_x\right)\\ &- N_1 x_{2,x} + N_2 x_{1,x} - \frac{x_1 x_{2,x}}{2} + \frac{x_2 x_{1,x}}{2}.\end{aligned} \tag{4.12}$$

The condition for the expressions of $\tilde{g}_x - g_x$ given by (4.10) and (4.12) to coincide is, after some simplifications, given by

$$\left[\frac{2W-\Omega}{S}\frac{2\Omega}{S}\right][\psi_1\psi_{2,x} - \psi_2\psi_{1,x}] + [\psi_2 x_{1,x} - \psi_1 x_{2,x}]\left[\frac{2W-2\Omega}{S}\right] + \frac{2\Omega}{S}[\psi_1 N_{2,x} - \psi_2 N_{1,x}] = 0.$$

Since Y is parametrized by lines of curvature, we have $N_{i,x} = \lambda_i x_{i,x}$, for $i = 1, 2$. Also, since $\tilde{Y}$ is obtained by a Ribaucour transformation which preserves CMC-$\frac{1}{2}$ property, it is true that $S = 2c(2\Omega W - \Omega^2)$, for some real constant $c \neq 0$. This implies $\dfrac{1}{2c\Omega} = \dfrac{2W-\Omega}{S}$. The previous equation then simplifies to

$$(\psi_1\psi_{2,x} - \psi_2\psi_{1,x}) + 2c\,(\psi_2 x_{1,x} - \psi_1 x_{2,x})\,(W - \Omega(\lambda_1 + 1)) = 0. \tag{4.13}$$

To work with (4.13) we need an expression for ψ_x. Now, by definition of ψ, to obtain such expression we need an expression for $\Omega_{1,x}$. Also, the latter can be determined if we differentiate S with respect to x and recall that $S = \Omega_1^2 + \Omega_2^2 - W^2 = 2c(2W\Omega - \Omega^2)$. Using (3.11) and solving for $\Omega_{1,x}$ we arrive at

$$\Omega_{1,x} = 2ca_1\,(W - \Omega(1+\lambda_1)) - W\lambda_1 a_1 - \frac{\Omega_2}{a_2}\frac{\partial a_1}{\partial y}.$$

Now, if we compute ψ_x and express it with respect to the frame $\{Y_x, Y_y, N\}$ we get the following expression.

$$\begin{aligned}\psi_x = Y_x &\left[2c\left(W-\Omega(1+\lambda_1)\right) - \frac{\Omega_2}{a_1a_2}\frac{\partial a_1}{\partial y} + \frac{\Gamma_{11}^1\Omega_1}{a_1} - \frac{\Omega_1}{a_1^2}\frac{\partial a_1}{\partial y} + \frac{\Omega_2}{a_2}\Gamma_{12}^1\right] \\ &+ Y_y\left[\frac{\Omega_1}{a_1}\Gamma_{11}^2 + \frac{\Omega_1}{a_2^2}\frac{\partial a_1}{\partial y} + \frac{\Gamma_{12}^2\Omega_2}{a_2} - \frac{\Omega_2}{a_2^2}\frac{\partial a_2}{\partial x}\right] \\ &+ N\left[h_{11}\frac{\Omega_1}{a_1} - \Omega_1 a_1\lambda_1\right],\end{aligned}$$

where $a_1 = \sqrt{g_{11}}$, $a_2 = \sqrt{g_{22}}$ and h_{ij} are the second fundamental form coefficients. The left-hand side of (4.13) is a polynomial in $x_{i,x}$ and $x_{i,y}$, for $i = 1, 2$. A tedious computation gives us the coefficient of, for instance, $x_{1,x}x_{2,y}$. This coefficient is given by the following expression

$$\begin{aligned}&\frac{\Omega_1}{a_1}\left[\frac{\Gamma_{11}^2}{a_2}\Omega_1 + \frac{\Omega_1}{a_2^2}\frac{\partial a_1}{\partial y} + \frac{\Gamma_{12}^2}{a_2}\Omega_2 - \frac{\Omega_2}{a_2^2}\frac{\partial a_2}{\partial x}\right] \\ -&\frac{\Omega_2}{a_2}\left[-\frac{\Omega_2}{a_1a_2} + \frac{\Gamma_{11}^1}{a_1}\Omega_1 - \frac{\Omega_1}{a_1^2}\frac{\partial a_1}{\partial x} + \frac{\Gamma_{12}^1}{a_2}\Omega_2\right].\end{aligned}$$

Notice that the expression above no longer depends on the constant c. The Christoffel symbols above are given by

$$\begin{aligned}\Gamma_{11}^1 &= \frac{1}{a_1}\frac{\partial a_1}{\partial x}, & \Gamma_{11}^2 &= -\frac{a_1}{a_2^2}\frac{\partial a_1}{\partial y}, \\ \Gamma_{12}^1 &= \frac{1}{a_1}\frac{\partial a_1}{\partial y}, & \Gamma_{12}^2 &= -\frac{1}{a_2}\frac{\partial a_2}{\partial x}.\end{aligned}$$

Using these equations, we see that the coefficient of $x_{1,x}x_{2,y}$ in the left hand side of (4.13) is zero. By using an analogous procedure to evaluate the other coefficients, we conclude that the left hand side of (4.13) is indeed null. Therefore, we conclude that Z and $\tilde{Z}$ are related by a W-congruence. So, this finishes the proof of the theorem. □

In addition, it is possible to obtain an algebraic relation between the principal curvatures of the surface Y and the geometry of Z related by a Calabi correspondence.

Proposition 12. *Let $Y \subset \mathbb{L}^3$ be a nowhere vertical CMC-$\frac{1}{2}$ surface parametrized by lines of curvature and with given principal curvatures k_1 and k_2 and $Z \subset Nil_3$ be a minimal surface related to Y by a Calabi correspondence with conformal parametrization by asymptotic lines. Then the following relations hold:*

$$k_1 = \frac{1}{2} - \frac{h_{12}^{AR}}{g_{11}^Z\theta^2},$$

$$k_2 = \frac{1}{2} + \frac{h_{12}^{AR}}{g_{11}^Z\theta^2},$$

where h_{12}^{AR} is the non-vanishing Abresch-Rosenberg second form coefficient and θ is the angle function given by proposition 2.

Proof. Since Y and Z share the same horizontal part, we may assume that $Y = (u(x,y), v(x,y), \phi(x,y))$ and $Z = (u(x,y), v(x,y), \psi(x,y))$. By imposing that Y and Z are related by a Calabi correspondence, we have that the Gauss maps of both are related by the construction given in the beginning of Section 4.1. More precisely, if $\eta = (\eta_1, \eta_2, \eta_3)$ is the Gauss map of Z and $N = (N_1, N_2, N_3)$ is the Gauss map of Y, then we must have

$$\begin{aligned} -\frac{\eta_2}{\eta_3} &= N_1, \\ \frac{\eta_1}{\eta_3} &= N_2, \\ \eta_3 \cdot N_3 &= -1. \end{aligned}$$

We solve this system of equations for ϕ_x and ϕ_y in order to express the immersion Y using the derivatives of ψ.

Next, since Y is a parametrization by lines of curvature, then, by Proposition 11, Z is a parametrization by Euclidean asymptotic lines. This enables us to express the second order derivatives of ψ in terms of its first order derivatives as follows:

$$\begin{aligned} \psi_{xx} &= \frac{(v_x\psi_y - v_y\psi_x)u_{xx} + (v_y\psi_x - v_x\psi_y)v_{xx}}{u_xv_y - u_yv_x}, \\ \psi_{yy} &= \frac{(v_x\psi_y - v_y\psi_x)u_{yy} + (v_y\psi_x - v_x\psi_y)v_{yy}}{u_xv_y - u_yv_x}. \end{aligned}$$

Thus, if we wish to compute the second fundamental form coefficients h_{ij} of Y in terms of ψ, these previous equations may be used to substitute ψ_{xx} and ψ_{yy} in h_{ij}. So, since Y is a parametrization by lines of curvature, $k_1 = \dfrac{h_{11}}{g_{11}}$ and $k_2 = \dfrac{h_{22}}{g_{22}}$, where g_{ij} are the first fundamental form coefficients of Y. By computing both k_1 and k_2 in terms of ψ and its derivatives we may check that

$$\begin{aligned} k_1 - \frac{1}{2} + Q &= 0, \\ k_2 - \frac{1}{2} - Q &= 0, \end{aligned}$$

where $Q = \dfrac{\langle Z_{xy}, Z_x \wedge^{\mathbb{E}} Z_y \rangle_{\mathbb{E}}}{\langle Z_x \wedge^{\mathbb{E}} Z_y, E_3 \rangle_{\mathbb{E}}^2}$ and E_3 is the vertical left-invariant vector field on Nil_3. Using propositions 5 and 2 we check that $Q = \dfrac{h_{12}^{AR}}{g_{11}^{Z}\theta^2}$ and the result follows. $\square$

Together with the fact that Y and Z have conformal metrics, Proposition 12 allows us to rewrite the Ribaucour system for Y given in (3.11), which depends on the eigenvalues λ_i of the second fundamental form in terms of the geometry of Z, since $k_i = -\lambda_i$.

4.3 Obtaining an explicit parametrization of the W-congruence transform

Now, we aim to obtain an explicit parametrization of $\tilde{Z}$ by using the Ribaucour transform parametrization given by Theorem 2 and, by using the relations between Y, $\tilde{Y}$, Z and $\tilde{Z}$, we will rewrite that equation in terms of objects related to Z.

First, consider the left-invariant frame $\{E_1, E_2, E_3\}$ given by equation (2.1). Suppose Y is a CMC-$\frac{1}{2}$ spacelike surface in $\mathbb{L}^3$ and Z is minimal in Nil_3 obtained from Y by a Calabi transform. It is a fact that both have identical horizontal components, i.e.,

$$Y = (x_1, x_2, h)\,,$$
$$Z = (x_1, x_2, g)\,.$$

in which x_1, x_2, h and g are functions defined on some open subset of $\mathbb{R}^2$.

We attempt to rewrite the tangent frame $\{Z_x, Z_y\}$ using $\{E_1, E_2, E_3\}$. For instance, $Z_x = (x_{1,x}, x_{2,x}, g_x) = \alpha E_1 + \beta E_2 + \gamma E_3$, for some $\alpha, \beta, \gamma \in \mathbb{R}$. So, we may write the following:

$$Z_x = (x_{1,x}, x_{2,x}, g_x) = \alpha\left(1, 0, -\frac{x_2}{2}\right) + \beta\left(0, 1, \frac{x_2}{2}\right) + \gamma\,(0, 0, 1)\,,$$

so that

$$\begin{cases} \alpha = x_{1,x} \\ \beta = x_{2,x} \\ \gamma = g_x + \dfrac{x_2 x_{1,x} - x_1 x_{2,x}}{2}. \end{cases}$$

An analogous computation is carried to write Z_y in terms of the left-invariant frame $\{E_i\}$.

Now we may compare the coordinates of Z_x in the Lie Algebra of Nil_3, denoted by Z_x^I with those of Y_x. Then the following equation holds:

$$Y_x - Z_x = \left(h_x - g_x - \frac{x_2 x_{1,x} - x_1 x_{2,x}}{2}\right) E_3.$$

We recall equation (4.3) and use it to obtain

$$Y_x - Z_x = \left(h_x - N_1 x_{2,x} + N_2 x_{1,x}\right) E_3, \tag{4.14}$$

where N_i is the i-th coordinate of Y lorentzian unitary normal vector field N. Let η be the Gauss map for Z, then it is true that $(\eta_1, \eta_2, \eta_3) = \left(-\frac{N_2}{N_3}, \frac{N_1}{N_3}, -\frac{1}{N_3}\right)$. Along with the definition of N, we obtain the following PDEs:

$$h_x = \eta_2 x_{1,x} - \eta_1 x_{2,x},$$
$$h_y = \eta_1 x_{1,y} - \eta_2 x_{2,y}.$$

The third coordinate in equation (4.14) yields the following

$$\begin{aligned} h_x - N_1 x_{2,x} + N_2 x_{1,x} &= \eta_2 x_{1,x} - \eta_1 x_{2,x} - \frac{1}{\eta_3}\left(-\eta_2 x_{2,x} - \eta_1 x_{1,x}\right) \\ &= \left[\frac{\eta_1}{\eta_3} + \eta_2\right] x_{1,x} + \left[\frac{\eta_2}{\eta_3} - \eta_1\right] x_{2,x}. \end{aligned}$$

Let us define $\theta := \left[\frac{\eta_1}{\eta_3} + \eta_2\right] x_{1,x} + \left[\frac{\eta_2}{\eta_3} - \eta_1\right] x_{2,x}$. Thus, we rewrite equation (4.14) as

$$Y_x - Z_x = \theta. \tag{4.15}$$

Let $\{e_1, e_2, N\}$ be an orthonormal frame in $\mathbb{L}^3$, in which $e_1 := \frac{Y_x}{\|Y_x\|}$, $e_2 := \frac{Y_y}{\|Y_y\|}$ and $\|(u_1, u_2, u_3)\| = u_1^2 + u_2^2 - u_3^2$. Dividing equation (4.15) by $\|Y_x\|$, which is not null, we obtain

$$\frac{Y_x}{\|Y_x\|} - \frac{1}{\|Y_x\|}\frac{\|Z_x^I\|}{\|Z_x^I\|} Z_x^I = \frac{1}{\|Y_x\|}\left(0, 0, \theta\right).$$

It is important to notice that $\|\cdot\|$ is being referred to two different norms. The context will make it clear which norm is being used, e.g., since Z_x is a vector in Nil$_3$, $\|\cdot\|$ is defined with respect to Nil$_3$ metric.

Recall that Calabi transform is conformal, thus so are the metrics on Y and Z. Let $k := \frac{\|Z_x\|}{\|Y_x\|}$ be the conformal factor, $\varepsilon_1 := \frac{Z_x}{\|Z_x\|}$ and $\varepsilon_2 := \frac{Z_y}{\|Z_y\|}$. This yields:

$$e_1 - k\varepsilon_1 = \frac{\theta}{\|Y_x\|} E_3.$$

Theorem 6. *Let Σ be a nowhere vertical minimal surface in Nil$_3$ given by the immersion $Z = (Z_1, Z_2, g)$ and Gauss' map $\eta = (\eta_1, \eta_2, \eta_3)$. If $(\Omega, \Omega_1, \Omega_2, W)$ solves the system given in Theorem 4, where*

- $k = |\langle E_3, \eta\rangle|$,
- $a_i = \frac{\|Z_x\|}{k}$,
- $\lambda_i = -k_i$ *given by Proposition 12,*

for some $c \in \mathbb{R}\backslash\{0\}$, then $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2, \tilde{g})$ is a minimal immersion of $\tilde{\Sigma}$, given in coordinates by the following

$$\begin{aligned}
\tilde{Z}_1 - Z_1 &= \frac{2\Omega}{S}\left(k\Omega_1(\varepsilon_1)_1 + k\Omega_2(\varepsilon_2)_1 - W\frac{\eta_2}{\eta_3}\right),\\
\tilde{Z}_2 - Z_2 &= \frac{2\Omega}{S}\left(k\Omega_1(\varepsilon_1)_2 + k\Omega_2(\varepsilon_2)_2 + W\frac{\eta_1}{\eta_3}\right),\\
\tilde{g} &= g - \frac{\varphi_1\eta_1 + \varphi_2\eta_2}{\eta_3} + \frac{x_1\varphi_2 - x_2\varphi_1}{2},
\end{aligned}$$

where $\varphi_i := \tilde{Z}_i - Z_i$.

Proof. It is known that a CMC-$\frac{1}{2}$ immersion Y in $\mathbb{L}^3$ that spans, via Calabi correspondence, the minimal immersion Z in Nil_3 shares the same horizontal part as Z, and $\tilde{Y}$, a Ribaucour transform that preserves the CMC-$\frac{1}{2}$ property by Theorem 4, does the same with $\tilde{Z}$. As a consequence, we have $\tilde{Y}_i - Y_i = \tilde{Z}_i - Z_i$, for $i = 1, 2$, where Y_i denotes the i-th coordinate of Y in $\mathbb{L}^3$, which is analogous to $\tilde{Y}$, Z and $\tilde{Z}$.

Since $\tilde{Y}$ is a Ribaucour transform of Y, then the following equation is satisfied, provided $\Omega, \Omega_1, \Omega_2$ and W satisfy the relations in (3.11)

$$\tilde{Y}_i - Y_i = \tilde{Z}_i - Z_i = \frac{2\Omega}{S}\left(\Omega_1 e_1 + \Omega_2 e_2 + WN\right)_i.$$

In turn, we have the following equations

$$\begin{aligned}
\tilde{Z}_1 - Z_1 &= \frac{2\Omega}{S}\left(k\Omega_1(\varepsilon_1)_1 + k\Omega_2(\varepsilon_2)_1 - W\frac{\eta_2}{\eta_3}\right),\\
\tilde{Z}_2 - Z_2 &= \frac{2\Omega}{S}\left(k\Omega_1(\varepsilon_1)_2 + k\Omega_2(\varepsilon_2)_2 + W\frac{\eta_1}{\eta_3}\right).
\end{aligned}$$

For the third coordinate of $\tilde{Z}$, since, by Theorem 5, $\tilde{Z}$ and Z are related by a W-congruence, we use the height function $\tilde{g}$ obtained by such transform. Thus $\tilde{g}$ must satisfy the following equation

$$\left\langle \eta, \tilde{Z} - Z \right\rangle_Z = \varphi_1\eta_1 + \varphi_2\eta_2 + \left(\tilde{g} - g + \frac{x_2\varphi_1}{2} - \frac{x_1\varphi_2}{2}\right)\eta_3 = 0,$$

where $\varphi_i := \tilde{x}_i - x_i$. Solving the above equation for $\tilde{g}$, we obtain:

$$\tilde{g} = g - \frac{\varphi_1\eta_1 + \varphi_2\eta_2}{\eta_3} + \frac{x_1\varphi_2 - x_2\varphi_1}{2}.$$

The resulting parametrization $\tilde{Z}$ is given using only objects related directly to Z. Thus, given $(\Omega, \Omega_1, \Omega_2, W)$ a solution to equations (3.8), one can generate transforms of a surface in Nil_3. If, in addition, the given solution to the Ribaucour system is such that $\frac{1}{2c\Omega} = \frac{2W - \Omega}{S}$, for some $c \in \mathbb{R}\backslash\{0\}$, then $\tilde{Z}$ is a minimal surface in W-congruence to Z. Indeed, since $\tilde{Y}$ is also a CMC-$\frac{1}{2}$ surface in $\mathbb{L}^3$ then the $\tilde{Z}$ obtained via Calabi correspondence is also minimal, by construction. $\square$

4.4 Examples

The following spacelike surface Σ in $\mathbb{L}^3$ is CMC-$\frac{1}{2}$. Let Y be a local coordinate chart for Σ and set

$$Y(x,y) = \left(-2\cosh(y) + \sqrt{5}\cdot x, \sinh(y), \frac{-2\sqrt{5}\cdot x + 5\cosh(y)}{\sqrt{5}}\right).$$

Computing the first fundamental form coefficients and denoting $\langle\cdot,\cdot\rangle$ as the inner product on $\mathbb{L}^3$ with signature $(++-)$, we have $a_1 := \sqrt{g_{11}} = \langle Y_x, Y_x\rangle = 1$, $g_{12} = \langle Y_x, Y_y\rangle = 0$ and $a_2 := \sqrt{g_{22}} = \langle Y_y, Y_y\rangle = 1$. Furthermore, let N be the unitary normal vector field to Σ and we obtain the second fundamental form coefficients $h_{11} = \langle Y_{xx}, N\rangle = 0$, $h_{12} = \langle Y_{xy}, N\rangle = 0$ and $h_{22} = \langle Y_{yy}, N\rangle = 1$. According to the latter, if $-\lambda_1$ and $-\lambda_2$ are the principal curvatures given by the direction Y_x and Y_y, respectively, then $\lambda_1 = 0$ and $\lambda_2 = -1$. This means Y is conformal and orthogonal and sets the scenario for a Ribaucour transformation.

Substituting these values in equations (3.11) yields

$$\begin{aligned}
\frac{\partial\Omega_1}{\partial y} &= 0, & \frac{\partial\Omega_2}{\partial x} &= 0,\\
\frac{\partial\Omega}{\partial x} &= \Omega_1, & \frac{\partial\Omega}{\partial y} &= \Omega_2,\\
\frac{\partial W}{\partial x} &= 0, & \frac{\partial W}{\partial y} &= \Omega_2.
\end{aligned}$$

Since Y is linear Weingarten, our aim is to use Theorem 2 and solve the complete Ribaucour system in order to obtain the solution $(\Omega, \Omega_1, \Omega_2, W)$. Let us use $\alpha = -1$, $\beta = 2$ e $\gamma = 0$ in Theorem 3 and keep the constant c unassigned. The additional algebraic condition is now $\Omega_1^2 + \Omega_2^2 - W^2 + c\Omega^2 - 2c\Omega W = 0$ and differentiating it w.r.t x and y gives us the following pair of equations:

$$\begin{aligned}
\frac{\partial\Omega_2}{\partial y} &= W(c+1),\\
\frac{\partial\Omega_1}{\partial x} &= c(W-\Omega).
\end{aligned} \tag{4.16}$$

The first equation above allows us to rewrite the equation for $\dfrac{\partial W}{\partial y}$ using only W as follows

$$\begin{aligned}
\frac{\partial W}{\partial y} &= \Omega_2, \text{ differentiating w.r.t } y \quad &\Rightarrow\\
\frac{\partial^2 W}{\partial y^2} &= \frac{\partial\Omega_2}{\partial y} &\Rightarrow\\
\frac{\partial^2 W}{\partial y^2} &= W(c+1).
\end{aligned}$$

The nature of the solution will depend on the chosen constant c. Essentially, we are able to make three choices: either $c < -1$ or $c = -1$ or $c > -1$. In each case, the solutions for various values of c are alike.

4.4.1 Example 1 ($c = 3 > -1$)

Firstly, we consider $c = 3$. So, according to the previous equations, we have $W = c_1 e^{2y} + c_2 e^{-2y}$, for some constants c_1 and c_2. Since $\dfrac{\partial W}{\partial y} = \Omega_2$, we solve this last equation for Ω_2

$$\Omega_2 = 2c_1 e^{2y} - 2c_2 e^{-2y}.$$

We aim to obtain a solution for Ω_2. To do that, we will use the second equation on (4.16) and differentiate it w.r.t x. This yields

$$\begin{aligned}
\frac{\partial \Omega_1}{\partial x} &= c(W - \Omega), \text{ differentiating w.r.t } x && \Rightarrow \\
\frac{\partial^2 \Omega_1}{\partial x^2} &= c\left(\frac{\partial W}{\partial x} - \frac{\partial \Omega}{\partial x}\right) && \Rightarrow \\
\frac{\partial^2 \Omega_1}{\partial x^2} &= -c\Omega_1.
\end{aligned}$$

Thus, for $c = 3$, we have $\Omega_1 = d_1 \sin\left(\sqrt{3}x\right) + d_2 \cos\left(\sqrt{3}x\right)$, for some constants d_1 and d_2. Recall the following equations

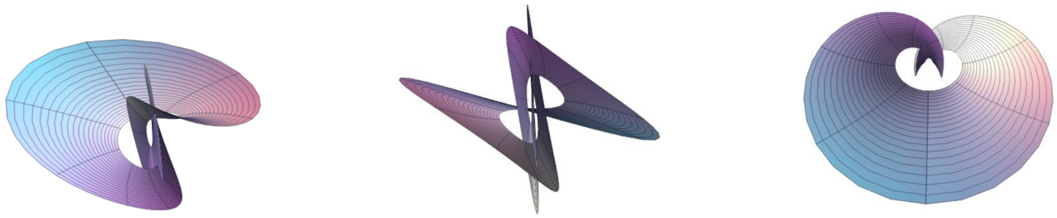
$$\begin{aligned}
\frac{\partial W}{\partial x} &= 0, & \frac{\partial \Omega}{\partial y} &= \frac{\partial W}{\partial y}, \\
\frac{\partial \Omega_1}{\partial y} &= 0, & \frac{\partial \Omega}{\partial x} &= \Omega_1.
\end{aligned}$$

Upon inspecting the equations we conclude that $\Omega = \int \Omega_1 dx + W$. Lastly, we need to check the algebraic condition imposed so we may choose c_1, c_2, d_1 and d_2 accordingly. Indeed,

$$\begin{aligned}
c\Omega^2 - 2c\Omega W + \Omega_1^2 + \Omega_2^2 - W^2 = 0 \quad &\Leftrightarrow \\
d_1^2 + d_2^2 - 16c_1c_2 = 0.&
\end{aligned}$$

Now we choose the constants $c_1 = c_2 = \dfrac{1}{4}$, $d_1 = 0$ and $d_2 = 1$. Summarizing, we obtained the following solution to (3.11)

$$\begin{aligned}
\Omega &= \frac{\cosh(2y)}{2} + \frac{\sin(\sqrt{3}x)}{\sqrt{3}}, & \Omega_1 &= \cos(\sqrt{3}x), \\
W &= \frac{\cosh(2y)}{2}, & \Omega_2 &= \sinh(2y).
\end{aligned}$$



(a) (b) (c)

Figure 4.1: Views on example 1 using polar coordinates centered at $(x, y) = (0.9, 0)$

Let Z be a Calabi transform of Y, a minimal surface in Nil_3, that is. If we denote ε_1 and ε_2 the orthonormal frame associated to Z_x and Z_y, respectively. Furthermore, let η be the Gauss' map of Z, g the height function of Z and k be the conformal factor between Y and Z. Then, by Theorem 6, and denoting by $\tilde{Z}$ the Bäcklund transform of Z, we have

$$\begin{aligned}
\tilde{Z}_1 &= Z_1 - \varphi_1, \\
\tilde{Z}_2 &= Z_2 - \varphi_2, \\
\tilde{Z}_3 &= g - \frac{\varphi_1\eta_1 + \varphi_2\eta_2}{\eta_3} + \frac{x_1\varphi_2 - x_2\varphi_1}{2},
\end{aligned}$$

where the quantities φ_i are given by

$$\begin{aligned}
\varphi_1 &= -\frac{2\Omega}{\Omega_1^2 + \Omega_2^2 - W^2}\left(\frac{\Omega_1\varepsilon_{1,1}}{k} + \frac{\Omega_2\varepsilon_{2,1}}{k} - \frac{\eta_2 W}{\eta_3}\right), \\
\varphi_2 &= -\frac{2\Omega}{\Omega_1^2 + \Omega_2^2 - W^2}\left(\frac{\Omega_1\varepsilon_{1,2}}{k} + \frac{\Omega_2\varepsilon_{2,2}}{k} + \frac{\eta_1 W}{\eta_3}\right).
\end{aligned}$$

Given the size of the expression above, although the functions $(\Omega, \Omega_1, \Omega_2, W)$ are relatively simple, the resulting parametrization of $\tilde{Z}$ is big enough to not be worth exhibiting. It is true, however, that $\tilde{g}_{11} = \tilde{g}_{22}$ and $\tilde{g}_{12} = 0$. The first fundamental form factors nicely to

$$\tilde{g}_{11} = \frac{\left[\cosh(2y)\left(10\cosh^3 y + 15\cosh y + 8\sqrt{5}\cos(\sqrt{3}x)\right) - 20\cosh y\sin^2(\sqrt{3}x)\right]^2}{5\left(\sqrt{3}\cosh(2y) - 2\sin(\sqrt{3}x)\right)^4}.$$

Notice that the denominator on $\tilde{g}_{11}$ vanishes within a curve whilst the numerator does not, this implies the metric blows up near this curve. Upon further inspection, we notice that the coordinate functions of $\tilde{Z}$ also blows up near the same curve. We would then ask whether or not $\tilde{Z}$ is complete, and what type of ends it has. In fact, the metric expression is sufficiently complicated to make the computation of divergent curves rather complex.

Recall that in classical Ribaucour transformations over $\mathbb{R}^3$, if the obtained surface $\tilde{\Sigma}$ had a metric with any singularities, it would have a countable set of points over which the metric $\tilde{g}_{ij}$ is not well defined. On the other hand, in lorentzian Ribaucour transformations we end up with a curve of singularities. This is due to the fact that the expression for the metric $\tilde{g}$ of a transform has an expression given by the metric g_{ij} of the original surface Σ and the solution to the Ribaucour PDEs $(\Omega, \Omega_1, \Omega_2, W)$. More specifically, there is a division by S on the expression of $\tilde{g}$. In real Euclidean space, $S := \Omega_1^2 + \Omega_2^2 + W^2$ as is lorentzian space $S := \Omega_1^2 + \Omega_2^2 - W^2$. It then makes sense that the nature of new singularities behaves differently like they do in real Euclidean space and the Lorentzian space.

4.4.2 Example 2 ($c = -1$)

Let $c = -1$. By using the general solution, we have $W = c_1 y + c_2$, for some constants c_1 and c_2. Since $\dfrac{\partial W}{\partial y} = \Omega_2$, we have Ω_2

$$\Omega_2 = c_1.$$

We aim to obtain a solution for Ω_1. For that, we'll use the second equation on (4.16) and differentiate it w.r.t x. Proceeding as in the first example and supposing $c = -1$, we obtain $\Omega_1 = d_1 e^x + d_2 e^{-x}$, for some constants d_1 and d_2. Recall the following equations:

$$\begin{aligned} \frac{\partial W}{\partial x} &= 0, & \frac{\partial \Omega}{\partial y} &= \frac{\partial W}{\partial y}, \\ \frac{\partial \Omega_1}{\partial y} &= 0, & \frac{\partial \Omega}{\partial x} &= \Omega_1. \end{aligned}$$

Upon inspecting the equations we conclude that $\Omega = \int \Omega_1 dx + W$. Lastly, we need to check the algebraic condition imposed so we may choose c_1, c_2, d_1 and d_2 accordingly. Indeed, for $c = -1$, we achieve

$$c\Omega^2 - 2c\Omega W + \Omega_1^2 + \Omega_2^2 - W^2 = 0$$

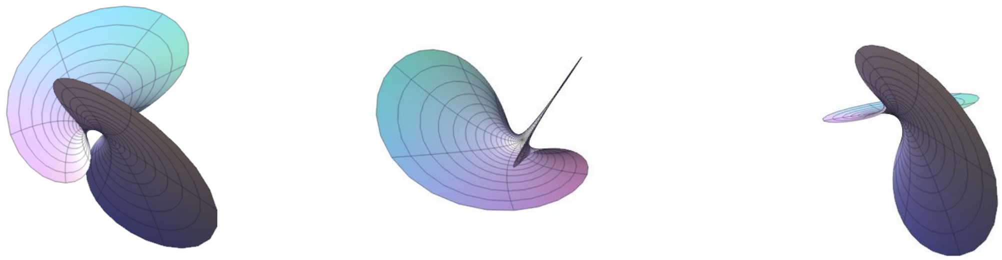
and therefore we get

$$c_1^2 + 4d_1 d_2 = 0.$$

Now we choose the constants $c_1 = 0$, $c_2 = 1$, $d_1 = 1$ and $d_2 = 0$. Summarizing, we obtained the following solution to (3.11)

$$\begin{aligned} \Omega &= 1 - e^{-x}, & \Omega_1 &= e^{-x}, \\ W &= 1, & \Omega_2 &= 0. \end{aligned}$$

To define the coordinate functions of $\tilde{Z}$, we proceed in similar fashion as in example 1 and obtain



(a) (b) (c)

Figure 4.2: Views on example 2 using polar coordinates centered at $(x, y) = (0.9, 0)$

$$\tilde{Z} = \frac{1}{e^{-x}+1}\Big(\sqrt{5}xe^{-x} - 2e^{-x}\cosh y + 2\sqrt{5}e^{-x} + \sqrt{5}x + 2\cosh y,\\ (e^{-x}-1)\sinh y,\\ \frac{\sqrt{5}xe^{-x}\sinh y + 2\sqrt{5}e^{-x}\sinh y - \sqrt{5}x\sinh y + 2ye^{-x} + 2y}{2}\Big).$$

We compute the expression of the metric $\tilde{g}_{11}$:

$$\tilde{g}_{11} = \frac{\left[5\cosh^2 y\left(e^{-2x}+1\right)^2 + 16e^{-2x}\left(\sqrt{5}\cosh x\cosh y + 1\right)\right]}{\left(e^{-x}+1\right)^4}.$$

Notice that $\tilde{g}_{11} \geq \frac{5(e^{-2x}+1)^2}{(e^{-x}+1)^4} \geq \frac{5}{4}$, so this example is geodesically complete.

Remark 1. It is reasonable to ask if it is possible that $\tilde{Z}$ is just another parametrization of the original surface. This is not the case. Indeed, notice that if the Ribaucour transformation between Y and $\tilde{Y}$ is not trivial, then the associated W-correspondence between Z and $\tilde{Z}$ in Nil_3 cannot be trivial. This is due to the fact that the Calabi transformation is invertible. If it were to be that Z and $\tilde{Z}$ are a change of parameter apart even though Y and $\tilde{Y}$ are not, this would imply the same minimal surface in Nil_3, parametrized by Z and $\tilde{Z}$ would yield geometrically different surfaces under the inverse Calabi transformation, parametrized by Y and $\tilde{Y}$, which is a contradiction, since the Gaussian curvature of Y is null and the Gaussian curvature of $\tilde{Y}$ is non-vanishing.

Chapter 5

Laguerre minimal surfaces in space forms

5.1 Introduction

In this chapter we will extend the notion of Laguerre minimal surfaces in euclidean space to surfaces in three dimensional space forms by considering an appropriate functional in the space of oriented spheres. We will show that the Euler Lagrange equation for this functional is equivalent to the harmonicity of the average of the radii of curvature with respect to a natural metric. We will show that, under generic assumptions, we have a well defined notion of duality for such surfaces, analogous to the well known euclidean case. Finally, we will relate a particular class of Laguerre minimal surfaces, called Bonnet surfaces, with minimal surfaces in the Riemannian products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. This relation will be used to produce some examples of Laguerre minimal surfaces.

5.2 The space of oriented spheres $\mathbb{H}^3$ and $\mathbb{S}^3$ and the Laguerre functional

The space of oriented spheres in $\mathbb{R}^3$ can be identified with $\mathbb{L}^4$ and plays a key role in understanding Laguerre differential geometry, a role analogous to the identification of oriented spheres in $\mathbb{S}^3$ with points of the De Sitter space that is appropriate to study conformal (or Moebius) geometry.

To guide us in our study of Laguerre minimal surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3$ we will identify the space of oriented spheres in these spaces with appropriate four dimensional Lorentzian manifolds. The basic idea is that an oriented immersion X of a surface Σ into $\mathbb{H}^3$ or $\mathbb{S}^3$ will induce a spacelike immersion Y into the space of oriented spheres via the so called middle sphere congruence. We will then say that the immersion X is Laguerre minimal if Y itself is a minimal surface in the usual sense.

The Lorentzian manifolds that will be used are actually very simple, namely, we will consider the Lorentzian products $\mathbb{M}^3(k) \times \mathbb{R}_1$, $k = \pm 1$, where $\mathbb{M}^3(1) = \mathbb{S}^3$ and $\mathbb{M}^3(-1) = \mathbb{H}^3$, with product metrics with a minus sign in the second factor to make the metric Lorentzian. To each oriented sphere in $\mathbb{M}(k)$ with center C and signed radius R (to keep track of orientation) we can associate the point (C, R) in $\mathbb{M}^3(k) \times \mathbb{R}$.

Now let $X : \Sigma \to \mathbb{M}^3(k)$ be an umbilic free immersion of a surface Σ and with unit normal field N. We will consider the map $Y : \Sigma \to \mathbb{M}^3(k) \times \mathbb{R}$, called the middle sphere congruence, defined by

$$Y = (\cos rX + \sin rN, r), \text{ if } k = 1,$$

$$Y = (\cosh rX + \sinh rN, r), \text{ if } k = -1,$$

where $r = \frac{r_1+r_2}{2}$ is the average of the curvature radii, r_i of X. Since we are interested in the case where Y is minimal, we will compute the area of Y in terms of the geometry of X.

Basically, the arguments and computations are almost identical for the cases $k = 1$ and $k = -1$, so we have decided to treat in detail the case $k = 1$ and only point out the minor changes that occur in the case $k = -1$.

Assuming for the moment that Y is an immersion, a simple computation shows that the first fundamental form of Y is given by

$$I_Y = \cos^2(r) I_X - 2\cos(r)\sin(r) II_X + \sin^2(r) III_X \,, \tag{5.1}$$

where I_X, II_X and III_X denote respectively the first, second and third fundamental forms of X.

For the computation that follows, it is convenient to use local coordinates such that the coordinate curves are lines of curvature. Let g_{ij}, h_{ij} and l_{ij} denote the coefficients of the first, second and third fundamental forms of X, respectively for the above mentioned local coordinates. We may then write

$$\begin{cases} h_{ij} = k_i g_{ij} \delta_{ij} \\ l_{ij} = k_i^2 g_{ij} \delta_{ij}, \end{cases} \tag{5.2}$$

where k_i, $i = 1, 2$, denotes the principal curvatures of X. Let $\hat{g}_{ij}$ denote the coefficients of the first fundamental of Y. The following equation holds

$$\det(\hat{g}) = \det(g) \cdot \left(\cos^2 r + k_1^2 \sin^2 r - 2k_1 \cos r \sin r\right)\left(\cos^2 r + k_2^2 \sin^2 r - 2k_2 \cos r \sin r\right),$$

where g and $\hat{g}$ are the matrices with entries $\hat{g}_{ij}$ and g_{ij} respectively.

Using the fact that $r_i = \operatorname{arccot} k_i$ if $k = 1$ ($r_i = \operatorname{arccoth} k_i$ if $k = -1$), a long but straightforward computation yields the following expression for the area of $Y(\Sigma)$

$$\int_{\tilde{\Sigma}} dA_Y = \int_{\Sigma} K + 1 - \sqrt{(K-1)^2 + (2H)^2} dA_X,$$

where K and H are the extrinsic and mean curvatures of X, respectively, and dA_Y and dA_X area elements of Y and X. Notice that, as a consequence of the Gauss-Bonnet theorem, the term $\int_{\Sigma} (K+1) dA$ can be considered irrelevant if we are interested in variations with compact support of the area functional. Therefore, we will deal with the functionals

$$\mathscr{L}_S(X) = \int_\Sigma \sqrt{(K-1)^2 + (2H)^2} dA, \tag{5.3}$$

if $k = 1$ and

$$\mathscr{L}_H(X) = \int_\Sigma \sqrt{(K+1)^2 + (2H)^2} dA, \tag{5.4}$$

for $k = -1$.

Remark 2. For surfaces in $\mathbb{H}^3$, we will assume that the principal curvatures are greater than 1, this assumption guarantees that the radii of curvature are well defined.

Definition 8. *An oriented immersion $X : \Sigma \to \mathbb{M}(k)$ is called Laguerre minimal if it is a critical point of (5.3) if $k = 1$ or (5.4) if $k = -1$ for normal variations with compact support.*

It is interesting to note that for immersions into $\mathbb{S}^3$ the Laguerre functional was already considered by [22] by looking at the image of the Gauss map with values in the Grassmannian $G(2, 4)$ which can be naturally identified with the product of spheres $\mathbb{S}^2 \times \mathbb{S}^2$. Lagrangian surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ (with the canonical product metric) that are critical points of the area functional under Hamiltonian variations are called Hamiltonian minimal surfaces. Thus, the image under the Gauss maps of Laguerre minimal surfaces in $\mathbb{S}^3$ are Hamiltonian minimal surfaces. We think that although we have reached a class of surfaces already considered in previous works, it is interesting to explore this alternative point of view, that can hopefully lead to contributions to the study of Hamiltonian minimal surfaces. For example, we will show that up to some degenerate cases a given Laguerre minimal surfaces admits a dual Laguerre minimal surface, so this duality also exists for minimal Hamiltonian surfaces, which we believe is not yet well known.

5.3 The Euler-Lagrange equation for $\mathscr{L}_S$ and $\mathscr{L}_H$

In this section we show that the Euler-Lagrange equation for $\mathscr{L}_S$ and $\mathscr{L}_H$ is equivalent to the harmonicity of $r = \frac{1}{2}(r_1 + r_2)$ with respect to the metrics $I_X + III_X$ if $k = 1$ or $I_X - III_X$ if $k = -1$. We recall that for Euclidean Laguerre minimal surfaces, we have a similar result, but the harmonicity is with respect to the third fundamental form of the surface.

To simplify our computations we shall use local coordinates such that the coordinate curves are lines of curvature and separate the steps using lemmas.

Lemma 5. *Let $X : \Sigma \to \mathbb{S}^3$ be an oriented immersion with unit normal field N, and consider the normal variation given by $X + t\varphi N$, where $\varphi : \Sigma \to \mathbb{R}$ has compact support. Then the first variation of $\mathscr{L}_S$ is given by the following expression*

$$\delta\mathscr{L}_S = \int_\Sigma \frac{(K-1) \cdot \operatorname{div}(L\nabla\varphi) + (2H) \cdot \Delta\varphi}{\sqrt{(K-1)^2 + (2H)^2}} dA, \tag{5.5}$$

where the div, ∇ and Δ denote the divergence, gradient and Laplace-Beltrami operators with respect to I_X, $\delta := \left.\dfrac{d}{dt}\right|_{t=0}$ and

$$L = \begin{bmatrix} k_2 & 0 \\ 0 & k_1 \end{bmatrix}.$$

Proof. The core of this proof is to pass the differentiation operator under the integral sign and using formulae for the derivatives of the mean and extrinsic curvatures. Since our integrand in $\mathscr{L}_S$ is a smooth function, we may use theorem A in [25] to compute these derivatives. In order to simplify our computations, let $F = \sqrt{(K-1)^2 + (2H)^2}$. Then,

$$\delta \mathscr{L}_S = \int\limits_{\Sigma} \left(\frac{(K-1)\delta K + 2H\delta(2H)}{F} - 2HF\varphi \right) dA.$$

Since, using [25], we have $\delta K = 2HK\varphi + 2H\varphi + \operatorname{div}(L\nabla\varphi)$ and $\delta(2H) = \varphi(4H^2 - 2K) + \Delta\varphi + 2\varphi$. Upon substitution on the previous equation, it simplifies to

$$\int\limits_{\Sigma} \left(\frac{(K-1)\operatorname{div}(L\nabla\varphi) + 2H[\varphi(4H^2 - 2K) + \Delta\varphi + 2\varphi] - 2HF^2\varphi}{F} \right) dA.$$

From this, it follows that

$$\delta \mathscr{L}_S = \int\limits_{\Sigma} \frac{(K-1)\cdot \operatorname{div}(L\nabla\varphi) + (2H)\cdot \Delta\varphi}{\sqrt{(K-1)^2 + (2H)^2}} \, dA.$$

So, the proof is completed. $\square$

Lemma 6. *Let $\alpha = r_1 + r_2$, then (5.5) can be rewritten as*

$$\delta \mathscr{L}_S = \int \cos\alpha \operatorname{div}(L\nabla\varphi) + \sin\alpha \Delta\varphi dA \tag{5.6}$$

Proof. Using the fact that $r_i = \operatorname{arccot} k_i$, it is easy to check that

$$\cos\alpha = \frac{K-1}{\sqrt{(K-1)^2 + (2H)^2}} \text{ and } \sin\alpha = \frac{2H}{\sqrt{(K-1)^2 + (2H)^2}}, \tag{5.7}$$

and substitution into (5.5) yields (5.6). $\square$

We are now ready to prove the following theorem.

Theorem 7. *An oriented immersion $X : \Sigma \to \mathbb{M}(k)$ is Laguerre minimal if and only if $\Delta_\omega r = 0$, where $\omega = I + III$ if $k = 1$ and $\omega = I - III$ if $k = -1$.*

Proof. We will limit ourselves to prove the assertion for $k = 1$. First note that using the divergence theorem and the fact our variations have compact support, we may write

$$\int\limits_{\Sigma} \sin\alpha \cdot \Delta\varphi \, dA = -\int\limits_{\Sigma} \varphi\Delta(\sin\alpha) + 2\left\langle \nabla \sin\alpha, \nabla\varphi \right\rangle \, dA, \tag{5.8}$$

Using the fact that the operator L is self-adjoint, we get

$$\operatorname{div}(L\nabla(\varphi\cdot\cos\alpha)) = \varphi\operatorname{div}(L\nabla\cos\alpha) + \cos\alpha\operatorname{div}(L\nabla\varphi) + 2\left\langle L\nabla\varphi, \nabla\cos\alpha\right\rangle.$$

The divergence theorem then yields

$$\int\limits_{\Sigma}\cos\alpha\operatorname{div}(L\nabla\varphi)\,dA = -\int\limits_{\Sigma}\varphi\operatorname{div}(L\nabla\cos\alpha) + 2\left\langle L\nabla\varphi, \nabla\cos\alpha\right\rangle\,dA. \tag{5.9}$$

Using (5.8) and (5.9) we rewrite (5.6) as follows

$$\delta\mathscr{L}_S = -\int_{\Sigma}\varphi\underbrace{(\Delta(\sin\alpha) + \operatorname{div}(L\nabla\cos\alpha))}_{P} + 2\underbrace{(\left\langle L\nabla\varphi, \nabla\cos\alpha\right\rangle + \left\langle\nabla\sin\alpha, \nabla\varphi\right\rangle)}_{Q}\,dA. \tag{5.10}$$

A simple computation shows that the terms denoted by P and Q in (5.10) can be written as follows $P = \operatorname{div}(\tilde{L}\nabla\alpha)$ and $Q = \left\langle\tilde{L}\nabla\varphi, \nabla\alpha\right\rangle$, where $\tilde{L} = \cos\alpha I - \sin\alpha L$, I being the identity matrix.

Note also that $\operatorname{div}(\varphi\tilde{L}\nabla\alpha) = Q + \varphi P$. This last observation, using the divergence theorem, implies that

$$\delta\mathscr{L}_S = -\int\limits_{\Sigma}\varphi P + 2Q\,dA = -\int\limits_{\Sigma}Q\,dA$$

.

Before we finish the proof we need to show that Q can be expressed as

$$Q = -\sqrt{1+k_1^2}\sqrt{1+k_2^2}\left\langle\nabla_\omega\varphi, \nabla_\omega\alpha\right\rangle_\omega, \tag{5.11}$$

where ∇_ω and $\langle\,,\,\rangle_\omega$ denote respectively the gradient operator and the metric with respect to $\omega = I + III$.

To check that (5.11) in fact holds, we consider local coordinates (x_1, x_2) such that the coordinate curves are lines of curvature. We then have the following expressions:

- $\nabla\varphi = \dfrac{1}{g_{11}}\dfrac{\partial\varphi}{\partial x_1}\dfrac{\partial}{\partial x_1} + \dfrac{1}{g_{22}}\dfrac{\partial\varphi}{\partial x_2}\dfrac{\partial}{\partial x_2}$
- $\nabla\alpha = \dfrac{1}{g_{11}}\dfrac{\partial\alpha}{\partial x_1}\dfrac{\partial}{\partial x_1} + \dfrac{1}{g_{22}}\dfrac{\partial\alpha}{\partial x_2}\dfrac{\partial}{\partial x_2}$
- $\tilde{L}\nabla\varphi = \dfrac{\cos\alpha - k_2\sin\alpha}{g_{11}}\varphi_1\dfrac{\partial}{\partial x_1} + \dfrac{\cos\alpha - k_1\sin\alpha}{g_{22}}\varphi_1\dfrac{\partial}{\partial x_2}$

For such local coordinates we have

$$Q = \left\langle\tilde{L}\nabla\varphi, \nabla\alpha\right\rangle = \frac{\cos\alpha - k_2\sin\alpha}{g_{11}}\varphi_1\alpha_1 + \frac{\cos\alpha - k_1\sin\alpha}{g_{22}}\varphi_2\alpha_2$$

The coefficients of the metric ω in our local coordinates are $g_{11}^\omega = (1+k_1^2)g_{11}$, $g_{12}^\omega = 0$ and $g_{22}^\omega = (1+k_2^2)g_{11}$. Using (5.7) it is then easy to check that (5.11) holds.

Now, using (5.11), we may write

$$\delta \mathscr{L}_S = -\int_\Sigma Q\, dA = \int_\Sigma \sqrt{1+k_1^2}\sqrt{1+k_2^2}\, \langle \nabla_\omega \varphi, \nabla_\omega \alpha \rangle_\omega \, dA$$

.

Note that the area element with respect to the metric ω is $dA_\omega = \sqrt{1+k_1^2}\sqrt{1+k_2^2} dA$. Thus, using again the divergence theorem, we have

$$\delta \mathscr{L}_S = \int_\Sigma \langle \nabla_\omega \varphi, \nabla_\omega \alpha \rangle_\omega \, dA_\omega = -\int_\Sigma \varphi \Delta_\omega \alpha \, dA_\omega,$$

and so $\delta \mathscr{L}_S = 0$ if and only if $\Delta_\omega \alpha = 0$. This ends the proof for $k = 1$. □

For $k = -1$ the proof is analogous with minor and obvious modifications.

5.4 Duality for Laguerre minimal surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$

It is well known that for Laguerre minimal surfaces in Euclidean space we have a notion of duality. Namely, up to some degenerate cases, given a Laguerre minimal surface M there exists a dual Laguerre minimal surface M^*, and both surfaces are envelopes of the middle sphere congruence associated to M, which is also the middle sphere congruence associated to M^*.

In this section we will show that this duality property extends naturally to Laguerre minimal surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$.

There are three cases where the dual surface is in a sense trivial or degenerate. The first case is when the Laguerre minimal surface M is a minimal surface ($H = 0$), then the middle sphere congruence is degenerate in the sense that $M = M^*$. The second case is what we will call Laguerre surface of spherical type, using the established terminology for the Euclidean case. These are surfaces such that their middle sphere congruence has a great sphere (if $k = 1$) or a hyperbolic plane (if $k = -1$) as the other envelope. The third degenerate case is when the centers of the spheres of the middle sphere congruence lies in a great sphere (if $k = 1$) or in a hyperbolic plane (if $k = -1$), these surfaces are classically called Bonnet surfaces in the Euclidean case, and we will maintain the terminology for the space forms.

The duality follows from the following proposition.

Proposition 13. *Let Y be a spacelike immersion into $\mathbb{S}^3 \times \mathbb{R}_1$ associated to a congruence of spheres that has one of its envelopes given by the immersion X with unit normal field N and has a radius function given by ρ. Then the mean curvature vector field $\mathbf{H}$ of Y is parallel to the lightlike field*

$$\eta = (-\sin \rho X + \cos \rho N, 1),$$

if and only if $\rho = \frac{r_1+r_2}{2}$.

Proof. By hypothesis, the immersion Y can be written as

$$Y = (\cos\rho X + \sin\rho N, \rho).$$

Let Δ_{I_Y} denote the Laplace-Beltrami operator with respect to the first fundamental form of Y. Using the classical formula

$$\Delta_{I_Y} Y = 2\mathbf{H},$$

we can assert that $\mathbf{H}$ is parallel to η if and only if $\Delta_{I_Y} Y$ is parallel to η.

Note that if the immersion $\tilde{X}$ with unit normal field $\tilde{N}$ is the other envelope of Y, then the field

$$\tilde{\eta} = (-\sin\rho\tilde{X} + \cos\rho\tilde{N}, 1),$$

together with η span the normal bundle of Y. Thus, $\mathbf{H}$ must be written as

$$\mathbf{H} = \mathbf{c_1}\eta + \mathbf{c_2}\tilde{\eta},$$

and, if the envelopes do not coincide, then $\mathbf{H}$ is parallel to η if and only if $\langle \mathbf{H}, \eta \rangle = 0$.

For our local coordinates (x_1, x_2), it is convenient to assume that the coordinate curves are lines of curvature. The coefficients of the metric I_Y for such coordinates are given by

$$g^Y_{11} = g_{11}(\cos\rho - k_1\sin\rho)^2, \;\; g^Y_{12} = 0, \;\; g^Y_{22} = g_{22}(\cos\rho - k_2\sin\rho)^2,$$

where g_{ij} are the coefficients of the metric of X.

As an intermediate step in our computations, we have the following useful expressions.

$$\langle \Delta_\omega X, N \rangle = \left\langle \frac{1}{g^Y_{11}}\frac{\partial^2 X}{\partial x_1^2} + \frac{1}{g^Y_{22}}\frac{\partial^2 X}{\partial x_2^2}, N \right\rangle = \frac{k_1}{(\cos\rho - k_1\sin\rho)^2} + \frac{k_2}{(\cos\rho - k_2\sin\rho)^2},$$

$$\langle \Delta_\omega X, X \rangle = \left\langle \frac{1}{g^Y_{11}}\frac{\partial^2 X}{\partial x_1^2} + \frac{1}{g^Y_{22}}\frac{\partial^2 X}{\partial x_2^2}, X \right\rangle = -\frac{1}{(\cos\rho - k_1\sin\rho)^2} - \frac{1}{(\cos\rho - k_1\sin\rho)^2},$$

$$\langle \Delta_\omega N, N \rangle = \left\langle \frac{1}{g^Y_{11}}\frac{\partial^2 N}{\partial x_1^2} + \frac{1}{g^Y_{22}}\frac{\partial^2 N}{\partial x_2^2}, N \right\rangle = -\frac{k_1^2}{(\cos\rho - k_1\sin\rho)^2} - \frac{k_2^2}{(\cos\rho - k_2\sin\rho)^2},$$

$$\langle \Delta_\omega N, X \rangle = \left\langle \frac{1}{g^Y_{11}}\frac{\partial^2 N}{\partial x_1^2} + \frac{1}{g^Y_{22}}\frac{\partial^2 N}{\partial x_2^2}, X \right\rangle = \frac{k_1}{(\cos\rho - k_1\sin\rho)^2} + \frac{k_2}{(\cos\rho - k_1\sin\rho)^2}.$$

To obtain the above expressions, we have neglected the terms that are orthogonal either to N or to X in the expression for the Laplacians, and we have used the following relations.

$$\left\langle \frac{\partial^2 X}{\partial x_1^2}, X \right\rangle = -g_{11}, \;\; \left\langle \frac{\partial^2 X}{\partial x_2^2}, X \right\rangle = -g_{22},$$

$$\left\langle \frac{\partial^2 N}{\partial x_1^2}, N \right\rangle = -\left\langle \frac{\partial N}{\partial x_1}, \frac{\partial N}{\partial x_1} \right\rangle = -g_{11}k_1^2, \;\; \left\langle \frac{\partial^2 N}{\partial x_2^2}, N \right\rangle = -\left\langle \frac{\partial N}{\partial x_2}, \frac{\partial N}{\partial x_2} \right\rangle = -g_{22}k_2^2,$$

$$\left\langle \frac{\partial^2 N}{\partial x_1^2}, X \right\rangle = -\left\langle \frac{\partial N}{\partial x_1}, \frac{\partial X}{\partial x_1} \right\rangle = h_{11}, \ \left\langle \frac{\partial^2 N}{\partial x_2^2}, X \right\rangle = -\left\langle \frac{\partial N}{\partial x_2}, \frac{\partial X}{\partial x_2} \right\rangle = h_{22}.$$

A straightforward computation also shows that

$$\cos\rho \Delta_Y \sin\rho - \sin\rho \Delta_Y \cos\rho = \Delta_Y \rho.$$

Using the above relations to compute $\langle \Delta_Y Y, \eta \rangle$, and neglecting the terms that are orthogonal to X and N we conclude that $\langle \Delta_Y Y, \eta \rangle = 0$ if and only if,

$$-\sin\rho\cos\rho((k_2^2-1)d_1^2 + (k_1^2-1)d_2^2) + (\cos^2\rho - \sin^2\rho)(k_1 d_2^2 + k_2 d_1^2) = 0, \tag{5.12}$$

where $d_i = (\cos\rho - k_i \sin\rho)$, for $i = 1, 2$.

It turns out that the left hand side of (5.12) can be factored in a convenient manner. Using this fact, we rewrite (5.12) in the following form.

$$-d_1 d_2 (2\cos\rho\sin\rho(k_1k_2 - 1) - (\cos^2\rho - \sin^2\rho)(k_1 + k_2)) = 0$$

Now, using (5.7), we conclude that $\langle \Delta_\omega Y, \eta \rangle = 0$ if and only if $\rho = \frac{r_1 + r_2}{2}$.

□

We state the duality for Laguerre minimal surfaces in the following way.

Theorem 8. *Let X be an umbilic free Laguerre minimal surface in $\mathbb{S}^3$ or $\mathbb{H}^3$, then the other envelope $\tilde{X}$ of its middle sphere congruence is also a Laguerre minimal surface.*

Proof. Let Y be the minimal surface in the space of spheres that corresponds to the middle sphere congruence of X. Since the mean curvature of Y is zero, it is parallel to the lightlike field associated to $\tilde{X}$. By proposition 13, and its hyperbolic counterpart, it follows that Y is associated to the middle sphere congruence of $\tilde{X}$, and therefore $\tilde{X}$ is a Laguerre minimal surface. □

5.5 A relation between Bonnet surfaces in space forms and minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

As we have already mentioned, we define a Bonnet surface as a Laguerre minimal surface for which the centers of the spheres of its middle sphere congruence lie either in a great sphere or in a hyperbolic plane. In this section we establish a local correspondence between such surfaces and minimal surfaces in either $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$.

We have discovered such correspondence by considering a concrete geometric construction involving Gauss maps and geodesics. However, to harmonize with the other part of this thesis and for the sake of simplicity, we have chosen to expose this correspondence using the Calabi transformation for minimal surfaces in $\mathbb{E}(\kappa, 0)$ and $\mathbb{L}(\kappa, 0)$, where $\kappa = \pm 1$.

Let M be a simply connected non vertical minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ (or $\mathbb{H}^2 \times \mathbb{R}$). Using the Calabi correspondence we have a spacelike minimal surface $M^\star$ in the Lorentzian product $\mathbb{S}^2 \times \mathbb{R}_1$ ($\mathbb{H}^2 \times \mathbb{R}_1$). Now, $\mathbb{S}^2 \times \mathbb{R}$ ($\mathbb{H}^2 \times \mathbb{R}$) can be thought of a manifold embedded in $\mathbb{S}^3 \times \mathbb{R}_1$ ($\mathbb{H}^3 \times \mathbb{R}_1$) in a natural way. Thus, $M^\star$, as a surface immersed in the space of spheres, is minimal and the centers of the corresponding spheres lie in a great sphere (hyperbolic plane). Therefore, the envelopes of the congruence of spheres associated to $M^\star$ are Bonnet surfaces.

Conversely, given a simply connected Bonnet surface in a space form, its congruence of middle spheres defines a minimal surface $M^\star$ in the space of spheres that is contained in the product submanifold ($\mathbb{S}^2 \times \mathbb{R}_1$ or $\mathbb{H}^2 \times \mathbb{R}_1$). It is not difficult to see that $M^\star$ is also minimal as a surface in $\mathbb{S}^2 \times \mathbb{R}_1$ or $\mathbb{H}^2 \times \mathbb{R}_1$. Therefore, if this surface is non vertical, by the Calabi correspondence, we have a minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$.

Using the above remarks we will produce some examples of Bonnet surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$.

5.5.1 Examples

Example in $\mathbb{S}^3$

Let Σ be the minimal graph in $\mathbb{S}^2 \times \mathbb{R}$ with height function f. We will compute its minimal dual $\tilde{\Sigma}$ on $\mathbb{S}^2 \times \mathbb{R}_1$ using theorem 1.8 in [19]. The surface $\tilde{\Sigma}$ is a graph with height function g which satisfies the following relations.

$$g_x = -\frac{f_y}{\sqrt{1 + \delta^2 (f_x^2 + f_y^2)}},$$

$$g_y = \frac{f_x}{\sqrt{1 + \delta^2 (f_x^2 + f_y^2)}},$$

where $\delta = 1 + \frac{1}{4}(x^2 + y^2)$. For our particular choice for f, the solution g to the previous equations is given in terms of an elliptic integral. We note that g may be interpreted as the radius function of a congruence of spheres in $\mathbb{S}^3$ whose surface of corresponding centers $\sigma(x, y)$ lies on a great sphere of $\mathbb{S}^3$ and the corresponding surface in the space of spheres is given by $Y = (\sigma, g)$.

Let $X \subset \mathbb{S}^3$ be an envelope of the congruence given by Y and let η be its unitary normal vector field. Now, we consider $\mathbb{S}^3 \subset \mathbb{R}^4$ so that $\sigma(x, y) = \left(\frac{4x}{x^2 + y^2 + 4}, \frac{4y}{x^2 + y^2 + 4}, \frac{2x^2 + 2y^2}{x^2 + y^2 + 4} - 1, 0 \right)$. Then,

$$X = \cos(g)\sigma + \sin(g)\eta,$$

A simple computation shows that if we write η in the orthogonal frame $\{\sigma_x, \sigma_y, N\}$, where N is the unitary normal vector field defined over σ in $\mathbb{S}^3$, then we have the following expression for η.

$$\eta = -\nabla g \pm \sqrt{1 - |\nabla g|^2} N,$$

where ∇ denotes the gradient operator for the canonical metric on the great sphere parametrized by σ.

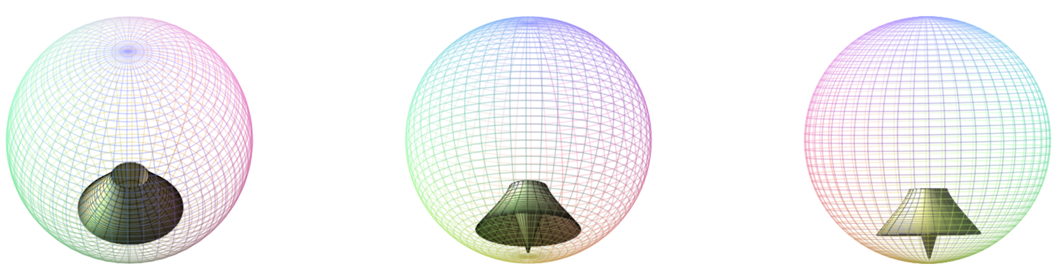
Notice that there are two possible expressions for η. This reflects the fact that a congruence of spheres has two envelopes: one for each choice of sign in the previous equation. Now we are in position to compute a parametrization of an envelope X associated to the minimal graph f in $\mathbb{S}^2 \times \mathbb{R}$.

Given the minimal graph $f(x, y) = \arctan \frac{y}{x}$ in $\mathbb{S}^2 \times \mathbb{R}$, we perform a Calabi transform to obtain a rotationally symmetric minimal graph g. Using polar coordinates $(x, y) \mapsto (r \cos t, r \sin t)$, we express $g(r)$ in terms of an elliptic integral as follows.

$$g(r) = \frac{2i}{\sqrt{2} - 1} F\left(\frac{ir(\sqrt{2} - 1)}{2}, 3 + 2\sqrt{2}\right),$$

where $i^2 = -1$ and $F\left(\frac{ir(\sqrt{2} - 1)}{2}, 3 + 2\sqrt{2}\right)$ is an incomplete elliptic integral of the first kind (see [24] for details).

Using g to compute η, we end up with a parametrization X of a Bonnet surface. Unfortunately, the expression of X is huge, so we will limit ourselves to exhibit a plot of the surface defined by X, using a stereographic projection of X onto $\mathbb{R}^3$. The sphere that appears in the pictures correspond to the great sphere parametrized by σ.



(a) (b) (c)

Figure 5.1: Views of the Bonnet example in $\mathbb{S}^3$

Example in $\mathbb{H}^3$

Proceeding in a similar fashion as in $\mathbb{S}^2 \times \mathbb{R}$, given a minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ with height function f, we use theorem 1.8 in [19] once more with $\delta = 1 - \frac{1}{4}(x^2 + y^2)$ and obtain g such that the graph with height g is minimal in $\mathbb{H}^2 \times \mathbb{R}_1$. Analogously to the stereographic projection on the spherical case, $\sigma(x, y)$ is given by a projection of the hyperbolic disk (of radius 2) onto the upper sheet of the hyperboloid $x^2 + y^2 + z^2 - t^2 = -1$

in $\mathbb{L}^4$ with signature $(+++-)$. In this case, $\sigma(x,y) = \left(\dfrac{4x}{4-x^2-y^2}, \dfrac{4y}{4-x^2-y^2}, 0, \dfrac{4+x^2+y^2}{4-x^2-y^2}\right)$ is viewed as $\mathbb{H}^2 \subset \mathbb{H}^3 \subset \mathbb{L}^4$. In this setting, the envelope we seek is given by $X = \cosh(g)\sigma + \sinh(g)\eta$. If we write η in the positive orthogonal frame $\{\sigma_x, \sigma_y, N\}$, where N is the unitary normal vector field defined over σ in $\mathbb{H}^3$, then we check that the following equation holds:

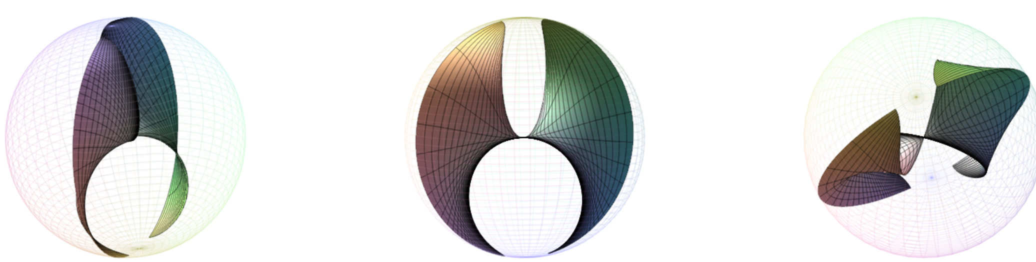
$$\eta = -\nabla g \pm \sqrt{1 - |\nabla g|^2} N$$

It is easy to check that $N = (0,0,1,0)$ is orthogonal to σ, σ_x and σ_y. This time, we set $f(x,y) = \operatorname{arcsinh} \frac{y}{x}$, which is minimal as a graph over the half-plane model. We use the Calabi correspondence to obtain g:

$$g(x,y) = \ln\left(\frac{\sqrt{(x^2+y^2)(x^2+2y^2)}}{x}\right),$$

which is minimal on $\mathbb{H}^2 \times \mathbb{R}_1$. Using g to compute η, we end up with a parametrization X of a Bonnet surface in $\mathbb{H}^3$. To visualize this surface, we map the hyperboloid that is a model of $\mathbb{H}^3$ in $\mathbb{L}^4$ to the Poincaré ball model. Using polar coordinates $(x,y) \mapsto (r\cos t, r\sin t)$ we end up with a parametrization $B(r,t)$ as follows:

$$B(r,t) = \frac{\left(2r(r^2+1)\cos t, r(r^2-1)\sin 2t, (r^4-1)\cos^2 t\right)}{(r^2-1)^2\cos^2 t + 4r^2(\sin t + 1)}$$



(a) (b) (c)

Figure 5.2: Views of the Bonnet example in $\mathbb{H}^3$ (Poincaré ball model)

Bibliography

[1] ABRESCH, U.; ROSENBERG, H. A Hopf differential for constant mean curvature surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. *Acta Mathematica*, v. 193, n. 2, p. 141–174, 2004.

[2] ABRESCH, U.; ROSENBERG, H. Generalized Hopf differentials. *Mat. Contemp*, v. 28, n. 1, p. 1–28, 2005.

[3] ALBUJER, A. L.; ALÍAS, L. J. Calabi–Bernstein results for maximal surfaces in Lorentzian product spaces. *Journal of Geometry and Physics*, v. 59, n. 5, p. 620–631, 2009.

[4] BATISTA DA SILVA, M. H. Simons Type Equation in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$ and Applications. *Annales de l'institut Fourier*, v. 61, n. 4, p. 1299–1322, 2011.

[5] BIANCHI, L. *Lezioni di Geometria Differenziale*. Enrico Spoerri, 1894.

[6] BLASCHKE, G.; THOMSEM, G. *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie III: Differentialgeometrie der Kreise und Kugeln*. Springer-Verlag, 2013. v. 29.

[7] CALABI, E. Examples of Bernstein problems for some non-linear equations. *Global Analysis*, p. 223–230, 1970.

[8] CHERN, S.; TERNG, C. An analogue of Bäcklund's theorem in affine geometry. *Rocky Mountain J. Math.*, v. 10, n. 1, p. 105–124, March 1980.

[9] CORRO, A.; FERREIRA, W.; TENENBLAT, K. On Ribaucour transformations for hypersurfaces. *Mat. Contemp*, v. 17, p. 137–160, 1999.

[10] CORRO, A.; FERREIRA, W.; TENENBLAT, K. Ribaucour transformations for constant mean curvature and linear Weingarten surfaces. *Pacific journal of mathematics*, v. 212, n. 2, p. 265–297, 2003.

[11] DANIEL, B. The Gauss map of minimal surfaces in the Heisenberg group. *International Mathematics Research Notices*, v. 2011, n. 3, p. 674–695, 2010.

[12] DANIEL, B. Minimal isometric immersions into $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. *Indiana Univ. Math. J*, v. 64, n. 5, p. 1425–1445, 2015.

[13] DANIEL, B.; FERNANDEZ, I.; MIRA, P. The Gauss map of surfaces in $\widetilde{PSL}(2,\mathbb{R})$. *Calculus of Variations and Partial Differential Equations*, v. 52, n. 3-4, p. 507–528, 2015.

[14] ESPINAR, J. M.; TREJOS, H. A. The Abresch-Rosenberg Shape Operator and applications. *arXiv preprint arXiv:1512.02099*, 2015.

[15] FERNÁNDEZ, I.; MIRA, P. A characterization of constant mean curvature surfaces in homogeneous 3-manifolds. *Differential geometry and its applications*, v. 25, n. 3, p. 281–289, 2007.

[16] FERNÁNDEZ, I.; MIRA, P. Harmonic maps and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *American Journal of Mathematics*, v. 129, n. 4, p. 1145–1181, 2007.

[17] FIGUEROA, C. On the Gauss map of a minimal surface in the Heisenberg group. *Mat. Contemp*, v. 33, p. 139–156, 2007.

[18] HOPF, H. *Differential geometry in the large: seminar lectures New York University 1946 and Stanford University 1956*. Springer, 2003.

[19] LEE, H. Extensions of the duality between minimal surfaces and maximal surfaces. *Geom. Dedicata*, v. 151, p. 373–386, 2011.

[20] MILNOR, T. K. Abstract Weingarten surfaces. *J. Differential Geom.*, v. 15, n. 3, p. 365–380, 1980.

[21] OLIVEIRA, D. C. *Superfícies de curvatura média ou ângulo constante em* Nil_3. Dissertação - Universidade de Brasília, 2018.

[22] PALMER, B. Hamiltonian minimality and Hamiltonian stability of Gauss maps. *Differential Geometry and its Applications*, v. 7, n. 1, p. 51–58, 1997.

[23] PARK, J. Ribaucour transformations on Lorentzian space forms in Lorentzian space forms. *Journal of the Korean Mathematical Society*, v. 45, n. 6, p. 1577–1590, 2008.

[24] PRASOLOV, V. V.; SOLOVYEV, I. P. *Elliptic functions and elliptic integrals*. American Mathematical Soc., 1997. v. 170.

[25] REILLY, R. C. Variational properties of functions of the mean curvatures for hypersurfaces in space forms. *J. Differential Geom.*, v. 8, n. 3, p. 465–477, 1973.

[26] TENENBLAT, K.; WANG, Q. Ribaucour transformations for hypersurfaces in space forms. *Annals of Global Analysis and Geometry*, v. 29, n. 2, p. 157, 2006.